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PAPERS

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PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE REPRESENTATIONS OF A NUMBER AS A SUM OF SQUARES AND PRODUCTS

(*Third paper.*)

By A. PAGE.

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1. In a recent paper* I have made a direct application of the Hardy-Littlewood method to obtain an asymptotic formula for the number of representations of an integer n as a sum of r squares and s products when $r+2s \geqslant 5$, $r > 0$, $s > 0$. I now consider the case $r+2s = 4$. In the case $s = 0$ the result is well known. The case $r = 0$ has been investigated by Estermann in his paper "On the representations of a number as the sum of two products"†. There remains for consideration the case $r = 2$, $s = 1$, and it is the object of this paper to prove the following theorem:

Let n be an integer greater than or equal to 2 and let $\nu(n)$ be the number of solutions of the equation $x_1^2+x_2^2+yz = n$ in integers x_1, x_2, y, and z, with $y > 0$, $z > 0$.

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* A. Page, "On the representations of a number as a sum of squares and products (*First paper*)", *Proc. London. Math. Soc.* (2), 36 (1934), 241–256.

† *Proc. London Math. Soc.* (2), 31 (1930), 123–133. This will hereafter be referred to as E1.

Let

$$E_r(n) = \sum_d (-1)^{\frac{1}{2}(d-1)} d^{-1} \log^r d \quad (r = 0, 1), \tag{1}$$

where d runs through the odd positive divisors of n;

$$E_r'(n) = \sum_d (-1)^{\frac{1}{2}(n/d-1)} d^{-1} \log^r d \quad (r = 0, 1), \tag{2}$$

where d runs through the positive divisors of n which are such that n/d is odd;

$$F_r(n) = 4E_r(n) + E_r'(n) \quad (r = 0, 1); \tag{3}$$

let

$$b_r = \sum_{\substack{q \\ q \text{ odd}}} (-1)^{\frac{1}{2}(q-1)} \mu(q) q^{-2} \log^r q \quad (r = 0, 1), \tag{4}$$

where $\mu(q)$ is Möbius's function, and let γ denote Euler's constant.

Then

$$\nu(n) = \tfrac{1}{4}\pi n \left[\{(\log n + 2\gamma - 1) b_0 - 2b_1\} F_0(n) - 2b_0 F_1(n) - 4b_0 \log 2\, E_0'(n)\right] + O(n^{\frac{1}{3}+\epsilon}) \tag{5}$$

for any positive ϵ.

The method of proof is an application of Kloosterman's refinements of the Hardy-Littlewood method, which have been simplified by Estermann in his paper "Vereinfachter Beweis eines Satzes von Kloosterman"*. Estermann's results have been improved in a recent paper by Salié.†

An elementary method leads to the less precise result‡

$$\nu(n) = \tfrac{1}{4}\pi n \left[\{(\log n + 2\gamma - 1) b_0 - 2b_1\} F_0(n) - 2b_0 F_1(n) - 4b_0 \log 2\, E_0'(n)\right] + O(n).$$

2. *Application of the Farey-dissection.*

In the usual notation of theta functions, let

$$\vartheta_3(\tau) = \sum_{m=-\infty}^{\infty} e^{\pi i m^2 \tau} \quad [I(\tau) > 0]; \tag{6}$$

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* *Hamburger Abh.*, 7 (1929), 82–98. This will hereafter be referred to as E 2.

† H. Salié, *Math. Zeitschrift*, 36 (1932), 263–278.

‡ A. Page, "On the representations of a number as a sum of squares and products (*Second paper*)", *Proc. London Math. Soc.* (2), 36 (1934), 471–484.

let

$$f(\tau) = \sum_{n=1}^{\infty} d(n)\, e^{\pi i n \tau} \quad [I(\tau) > 0], \tag{7}$$

where $d(n)$ is the number of positive divisors of n. Then it is obvious that

$$\vartheta_3^2(\tau) f(\tau) = \sum_{n=1}^{\infty} \nu(n)\, e^{\pi i n \tau},$$

and hence

$$\nu(n) = \tfrac{1}{4} \int_{\tau_0}^{\tau_0+4} \vartheta_3^2(\tau) f(\tau)\, e^{-\pi i n \tau}\, d\tau, \tag{8}$$

where τ_0 is any number with $I(\tau_0) > 0$.

Putting

$$M = [\sqrt{n}], \quad \tau_0 = \frac{1}{M+1} + \frac{i}{n}, \quad y = \frac{i}{n} + \theta, \tag{9}$$

and applying the Farey-dissection of order M, we obtain, from (8),

$$\nu(n) = \tfrac{1}{4} e^{\pi} \sum_{k=1}^{M} \sum_{\substack{0 < h \leqslant 4k \\ (h,\, k)=1}} \int_{-\theta_1}^{\theta_2} \vartheta_3^2 \left(\frac{h}{k} + y\right) f\left(\frac{h}{k} + y\right) \xi_{2k}^{-nh}\, e^{-\pi i n \vartheta}\, d\theta, \tag{10}$$

where $h/k - \theta_1$, $h/k + \theta_2$ are the medians between h/k and its neighbours in the Farey series, and ξ_k is an abbreviation for $e^{2\pi i/k}$. It is known that

$$(2kM)^{-1} \leqslant \theta_r \leqslant k^{-1}(M+1)^{-1} \quad (r = 1,\ 2). \tag{11}$$

The integral occurring in (10) may be written in the form

$$\int_{-k^{-1}n^{-\frac{1}{2}}}^{k^{-1}n^{-\frac{1}{2}}} g \vartheta_3^2 \left(\frac{h}{k} + y\right) f\left(\frac{h}{k} + y\right) \xi_{2k}^{-nh}\, e^{-\pi i n \vartheta}\, d\theta,$$

where g is an auxiliary function of n, h, k, and θ, defined by the equation

$$g = \begin{cases} 1 & \text{if} \quad -\theta_1 \leqslant \theta \leqslant \theta_2; \\ 0 & \text{otherwise.} \end{cases}$$

Equation (10) then becomes

$$\nu(n) = \tfrac{1}{4} e^{\pi} \sum_{k=1}^{M} c_{n,\,k}, \tag{12}$$

where

$$c_{n,\,k} = \int_{-k^{-1}n^{-\frac{1}{2}}}^{k^{-1}n^{-\frac{1}{2}}} e^{-\pi i n \theta}\, S\, d\theta, \tag{13}$$

$$S = \sum_{\substack{0 < h \leqslant 4k \\ (h,\, k)=1}} g \xi_{2k}^{-nh} \vartheta_3^2 \left(\frac{h}{k} + y\right) f\left(\frac{h}{k} + y\right). \tag{14}$$

Estermann investigates g as a function of h and obtains the following results*:

$$g = 1 \quad \text{if} \quad |\theta| \leqslant k^{-1}(k+M)^{-1}; \tag{15}$$

$$g = 0 \quad \text{if} \quad |\theta| > k^{-1}(1+M)^{-1}; \tag{16}$$

$$g = \sum_{r=1}^{k} b_r \xi_k^{r[h,k]}, \tag{17}$$

where b_r is independent of h and

$$\sum_{r=1}^{k} |b_r| < \log(4k). \tag{18}$$

In (17), $[h, k]$ is defined by

$$h[h, k] \equiv 1 \pmod{k}, \quad 0 < [h, k] \leqslant k. \tag{19}$$

3. *Investigation of* $\vartheta_3(h/k+y)$ *and* $f(h/k+y)$.

Henceforth we always take

$$(h, k) = 1. \tag{20}$$

Throughout paragraphs 3 and 4, let

$$k \leqslant \sqrt{n}, \quad |\theta| < k^{-1} n^{-\frac{1}{2}}. \tag{21}$$

3.1. The investigation of $\vartheta_3^2(h/k+y)$ depends upon the fact that $\vartheta_3^2(\tau)$ is a modular form of degree 4.

First it is clear that all modular substitutions are congruent mod 2 to one of the six substitutions

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us call these substitutions $S_1, S_2, \ldots, S_6$, and let

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad-bc = 1)$$

be any modular substitution.

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* E 2, 94–95.

On writing $$\tau_1 = \frac{a\tau+b}{c\tau+d},$$

we have, by the formulae for transformation of theta functions*,

$$(22)\quad \vartheta_3^2(\tau) = \begin{cases} (-1)^{\frac{1}{2}(d-1)}(c\tau+d)^{-1}\vartheta_3^2(\tau_1) & \text{if } S \equiv S_1 \pmod 2; \\ i(-1)^{\frac{1}{2}(c-1)}(c\tau+d)^{-1}\vartheta_3^2(\tau_1) & \text{if } S \equiv S_5 \pmod 2; \end{cases}$$

$$\vartheta_3^2(\tau) = \begin{cases} (-1)^{\frac{1}{2}(a-1)}(c\tau+d)^{-1}\vartheta_2^2(\tau_1) & \text{if } S \equiv S_3 \pmod 2; \\ i(-1)^{\frac{1}{2}(b+1)}(c\tau+d)^{-1}\vartheta_2^2(\tau_1) & \text{if } S \equiv S_4 \pmod 2; \end{cases}$$

$$\vartheta_3^2(\tau) = \begin{cases} (-1)^{\frac{1}{2}(d-1)}(c\tau+d)^{-1}\vartheta_0^2(\tau_1) & \text{if } S \equiv S_2 \pmod 2; \\ i(-1)^{\frac{1}{2}(c-1)}(c\tau+d)^{-1}\vartheta_0^2(\tau_1) & \text{if } S \equiv S_6 \pmod 2. \end{cases}$$

Further,

$$(23)\quad \vartheta_s^2(\tau) = \phi_s + \sum_{\nu=1}^{\infty} a_\nu^{(s)} e^{\pi i \nu \tau/2},$$

where

$$(24)\quad \phi_s = \begin{cases} 1 & \text{if } s = 0, 3; \\ 0 & \text{if } s = 2, \end{cases}$$

and $\sum_{\nu=1}^{\infty} a_\nu^{(s)} x^\nu$ converges absolutely if $|x| < 1$.

3.2. Henceforth let $A_1, A_2, \ldots$ denote sufficiently large numbers which may be absolute constants or may depend only on ϵ.

We have now to investigate the function $f(h/k+y)$, defined by (7). Estermann has found an asymptotic formula for $F(h/k+y)$, where $f(\tau) = F(\frac{1}{2}\tau)$ and $y = i/n+\theta$, $(h, k) = 1$, $k \leqslant \sqrt{n}$, $|\theta| \leqslant k^{-1}n^{-\frac{1}{2}}$; this result easily leads us to an expansion of $f(h/k+y)$. We shall use equations (22), (24), (49), and (50) of E1, and note that these equations hold if the two conditions $k \leqslant \sqrt{n}$, $|\theta| \leqslant k^{-1}n^{-\frac{1}{2}}$ are replaced by the conditions $k \leqslant a\sqrt{n}$, $|\theta| \leqslant bk^{-1}n^{-\frac{1}{2}}$, where a and b are any positive absolute constants.

Two cases arise according as h is even or odd. If h is even, we write

$$h_1 = \tfrac{1}{2}h, \quad k_1 = k, \quad y_1 = \tfrac{1}{2}y, \quad \theta_1 = \tfrac{1}{2}\theta, \quad n_1 = 2n,$$

and then $$f\left(\frac{h}{k}+y\right) = F\left(\frac{h_1}{k_1}+y_1\right);$$

* Tannery and Molk, *Théorie des fonctions elliptiques*, 1 (1893), 215, Table XX_7; 2 (1896), 28, Table $XXXVI_4$.

further, we easily see that

$$y_1 = \frac{i}{n_1} + \theta_1, \quad (h_1, k_1) = 1, \quad k_1 \leqslant \sqrt{(\tfrac{1}{2}n_1)}, \quad |\theta_1| \leqslant k_1^{-1}(2n_1)^{-\frac{1}{2}}.$$

If h is odd, we write

$$h_1 = h, \quad k_1 = 2k, \quad y_1 = \tfrac{1}{2}y, \quad \theta_1 = \tfrac{1}{2}\theta, \quad n_1 = 2n,$$

and then

$$f\left(\frac{h}{k} + y\right) = F\left(\frac{h_1}{k_1} + y_1\right),$$

where

$$y_1 = \frac{i}{n_1} + \theta_1, \quad (h_1, k_1) = 1, \quad k_1 \leqslant \sqrt{(2n_1)}, \quad |\theta_1| \leqslant \sqrt{2}\, k_1^{-1} n_1^{-\frac{1}{2}}.$$

Hence we have the following formula:

$$f\left(\frac{h}{k} + y\right) = Z + U, \tag{25}$$

where

$$Z = \begin{cases} (-\pi i k y)^{-1} [\gamma - 2\log k - \log(-\pi i y)] & \text{if } h \text{ is even;} \\ (-2\pi i k y)^{-1} [\gamma - 2\log(2k) - \log(-\pi i y)] & \text{if } h \text{ is odd;} \end{cases} \tag{26}$$

$$U = \begin{cases} \sum_{t=1}^{k} i_t \xi_k^{t[\frac{1}{2}h, k]} & \text{if } h \text{ is even;} \\ \sum_{t=1}^{2k} i_t \xi_{2k}^{t[h, 2k]} & \text{if } h \text{ is odd;} \end{cases} \tag{27}$$

in (27), $i_1, i_2, \ldots$ are certain numbers which depend upon n, k, θ, and the parity of h, and satisfy the inequality

$$\sum_{t=1}^{2k} |i_t| < A_1 |ky| n \log^2(4k) \tag{28}$$

(when h is even we take $i_t = 0$ if $k < t \leqslant 2k$).

Later we shall require an inequality for $|Z|$ and such an inequality is easily obtained from formulae (26), (9), and (21), which give

$$|Z| < A_2 |ky|^{-1} \log n \quad (n \geqslant 2). \tag{29}$$

4.

4.1. We apply the results of the last section to obtain an expansion for S, defined by (14), and then proceed to find an asymptotic formula for $c_{n,k}$, which is connected with S by (13).

First we may write

$$S = \sum_{\substack{0<l\leqslant 4 \\ (l,\,k,\,4)=1}} S_l, \tag{30}$$

where

$$S_l = \sum_{\substack{0<h\leqslant 4k \\ (h,\,k)=1 \\ h\equiv l \pmod 4}} g\xi_{2k}^{-nh}\,\vartheta_3{}^2\Big(\frac{h}{k}+y\Big)\,f\Big(\frac{h}{k}+y\Big). \tag{31}$$

So far the only restriction placed on h is that $(h, k) = 1$. Now suppose further that

$$h \equiv l \pmod 4, \quad (l, k, 4) = 1, \tag{32}$$

and put

$$(l, 4) = \delta, \quad l = \delta l', \quad 4 = \delta N', \quad h = \delta h'. \tag{33}$$

Then it is clear that

$$(\delta, k) = (l', N') = (h', N') = 1, \quad h' \equiv l' \pmod{N'}. \tag{34}$$

It follows from (31) that

$$S_l = \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} g\xi_{kN'}^{-2nh'}\,\vartheta_3{}^2\Big(\frac{\delta h'}{k}+y\Big)\,f\Big(\frac{\delta h'}{k}+y\Big). \tag{35}$$

4.2. We apply the formula (22) to obtain an expansion for $\vartheta_3{}^2(\delta h'/k+y)$, taking for S the modular substitution given by

$$c = k, \quad d = -h, \quad a = -\delta\,[\delta^2, k]\,[h', kN'], \quad ad-bc = 1 \tag{36}$$

[it follows from (19) that b is an integer]. Putting $\tau = \delta h'/k+y$ in (22) and using (23), we obtain the result

$$\vartheta_3{}^2\Big(\frac{\delta h'}{k}+y\Big) = (ky)^{-1}x\phi_s + (ky)^{-1}x\sum_{\nu=1}^{\infty} a_\nu^{(s)}\,\xi_{kN'}^{-\nu[\delta^2,\,k][h',\,kN']}\exp\Big(-\frac{\pi i\nu}{2k^2y}\Big), \tag{37}$$

where x is a fourth root of unity whose values are given by (22) when we take for S the modular substitution defined in (36). In this equation s and x depend upon k and l, but are independent of h'. For, from (36) and (32),

$$d \equiv -l \pmod 4; \tag{38}$$

from (19) and (34),

$$[h', kN'] \equiv [l', N'] \pmod{N'},$$

and hence, from (36) and (33),

$$a \equiv -\delta\,[\delta^2,\, k]\,[l',\, N'] \pmod 4;$$

finally we have, from (19) and (33),

$$\delta^2 h'\,[h',\, kN'] \equiv \delta^2 \pmod{4k},$$

and hence, by (36),

$$bk \equiv \delta^2\,[\delta^2,\, k] - 1 \pmod{4k},$$

which gives

$$b \equiv k^{-1}\{\delta^2\,[\delta^2,\, k] - 1\} \pmod 4.$$

We can simplify the right-hand side of (37) by writing

$$(39) \qquad \sum_{\nu=1}^{\infty} a_\nu^{(s)}\, \xi_{kN'}^{-\nu[\delta^2,\, k][h',\, kN']} \exp\left(-\frac{\pi i \nu}{2k^2 y}\right) = \sum_{u=1}^{kN'} e_u\, \xi_{kN'}^{u[h',\, kN']},$$

where

$$e_u = \sum_{\substack{\nu=1 \\ -\nu[\delta^2,\, k] \equiv u \pmod{kN'}}}^{\infty} a_\nu^{(s)} \exp\left(-\frac{\pi i \nu}{2k^2 y}\right).$$

It is clear that the numbers $e_1, e_2, \ldots$ are independent of h' and, on using (9), we prove that they satisfy the inequality

$$\sum_{u=1}^{kN'} |e_u| \leqslant \sum_{\nu=1}^{\infty} |a_\nu^{(s)}| \exp\left(-\frac{\pi \nu}{2n\,|ky|^2}\right).$$

The infinite series occurring in the last formula may be estimated by putting

$$\sum_{\nu=1}^{\infty} |a_\nu^{(s)}| \exp\left(-\frac{\pi \nu}{2n\,|ky|^2}\right) = \exp\left(-\frac{\pi}{2n\,|ky|^2}\right) \sum_{\nu=1}^{\infty} |a_\nu^{(s)}| \exp\left(-\frac{\pi(\nu-1)}{2n\,|ky|^2}\right).$$

Now, by (9) and (21),

$$\frac{\pi}{2n\,|ky|^2} \geqslant \tfrac{1}{4}\pi,$$

and hence

$$\sum_{\nu=1}^{\infty} |a_\nu^{(s)}| \exp\left(-\frac{\pi(\nu-1)}{2n\,|ky|^2}\right) < A_3.$$

It follows that

$$\sum_{\nu=1}^{\infty} |a_\nu^{(s)}| \exp\left(-\frac{\pi \nu}{2n\,|ky|^2}\right) < A_3 \exp\left(-\frac{\pi}{2n\,|ky|^2}\right) < A_3\, n\,|ky|^2,$$

and hence

$$\sum_{u=1}^{kN'} |e_u| < A_3\, n\,|ky|^2.$$

On using this inequality we obtain from (37) and (39) the required expansion for $\vartheta_3^2(\delta h'/k+y)$, viz.

$$\vartheta_3^2\left(\frac{\delta h'}{k}+y\right) = (ky)^{-1}x\phi_s + \sum_{u=1}^{kN'} j_u\, \xi_{kN'}^{u[h',\,kN']}, \tag{40}$$

where $j_1, j_2, \ldots, j_{kN'}$ are certain numbers independent of h' which satisfy the inequality

$$\sum_{u=1}^{kN'} |j_u| < A_3 n|ky|. \tag{41}$$

4.3. Turning our attention to $f(\delta h'/k+y)$, we note that, by (27), U may be written in the form

$$U = \sum_{t=1}^{2k} i_t\, \xi_{kN'}^{tv[h',\,kN']}, \tag{42}$$

where i_t and v depend upon k and l but are independent of h'. For, by (19), (20), (33), and (34), we have

$$\xi_k^{[\frac{1}{2}h,\,k]} = \xi_k^{[\frac{1}{2}\delta,\,k][h',\,k]} = \xi_{kN'}^{N'[\frac{1}{2}\delta,\,k][h',\,kN']} \quad \text{if} \quad h \text{ is even};$$

$$\xi_{2k}^{[h,\,2k]} = \xi_{4k}^{2[h,\,4k]} = \xi_{kN'}^{2[h',\,kN']} \quad \text{if} \quad h \text{ is odd}.$$

Also Z, defined by (26), is independent of h'.

By (25), (33), and (42),

$$f\left(\frac{\delta h'}{k}+y\right) = Z + \sum_{t=1}^{2k} i_t\, \xi_{kN'}^{tv[h',\,kN']},$$

and on combining this equation with (40) and using the inequalities (28), (29), and (41), we find by a simple calculation that

$$\vartheta_3^2\left(\frac{\delta h'}{k}+y\right) f\left(\frac{\delta h'}{k}+y\right) = (ky)^{-1}x\phi_s\, Z + \sum_{m=1}^{kN'} a_m\, \xi_{kN'}^{m[h',\,kN']}, \tag{43}$$

where $a_1, a_2, \ldots, a_{kN'}$ are certain numbers independent of h' which satisfy the inequality

$$\sum_{m=1}^{kN'} |a_m| < A_4\{n\log n + n\log^2(4k) + n^2|ky|^2\log^2(4k)\}.$$

The last inequality may be simplified by the use of formulae (9) and (21), and becomes

$$\sum_{m=1}^{kN'} |a_m| < A_5 n\log^2 n. \tag{44}$$

4.4. We substitute the expression for $\vartheta_3^2(\delta h'/k+y)f(\delta h'/k+y)$ given by (43) in equation (35) and obtain the result

$$S_l-(ky)^{-1}x\phi_s Z \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} g\xi_{kN'}^{-2nh'} = \sum_{m=1}^{kN'} a_m \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} g\xi_{kN'}^{-2nh'+m[h',\,kN']}. \tag{45}$$

This equation will lead to an asymptotic formula for S_l, the term on the right-hand side constituting the error term which must now be estimated.

First it is clear, by (19), (20), (33), and (34), that we may write (17) in the form

$$g = \sum_{r=1}^{k} b_r \xi_{kN'}^{rN'[\delta,\,k][h',\,kN']}. \tag{46}$$

On making this substitution in the right-hand side of (45), we may estimate the error term by using the fundamental theorem of Kloosterman's refinements*, which, for the purposes of this application, may be stated in the form

$$\left| \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} \xi_{kN'}^{-2nh'+m[h',\,kN']} \right| = O\{(kN')^{\frac{3}{4}+\epsilon}(2n,\,kN')^{\frac{1}{4}}\} < A_6 k^{\frac{3}{4}+\epsilon}(n,\,k)^{\frac{1}{4}} \tag{47}$$

(for any positive ϵ) if m is independent of h'.

Let us denote the dominant term in the expansion of S_l occurring in (45) by F_1, so that

$$F_1 = (ky)^{-1}x\phi_s Z \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} g\xi_{kN'}^{-2nh'}. \tag{48}$$

Then, by (45), (48), (46), and (47), it is seen that

$$|S_l-F_1| < A_6 k^{\frac{3}{4}+\epsilon}(n,\,k)^{\frac{1}{4}} \sum_{m=1}^{kN'} |a_m| \sum_{r=1}^{k} |b_r|,$$

and, by (18) and (44), this inequality becomes

$$\begin{aligned} |S_l-F_1| &< A_7 n \log^2 n\, k^{\frac{3}{4}+\epsilon}(n,\,k)^{\frac{1}{4}} \log(4k) \\ &< A_8 n \log^3 n\, k^{\frac{3}{4}+\epsilon}(n,\,k)^{\frac{1}{4}}. \end{aligned} \tag{49}$$

* H. Salié, *loc. cit.*, formulae (4), (5), and (7).

If now we define F_2 by the equation

$$F_2 = \sum_{\substack{0<l\leqslant 4\\(l,\,k,\,4)=1}} F_1, \tag{50}$$

it follows immediately from (30) and (49) that

$$|S-F_2| < A_9\, n \log^3 n\, k^{\frac{3}{2}+\epsilon}(n,\,k)^{\frac{1}{2}},$$

and hence, by (13),

$$\left| c_{n,k} - \int_{-k^{-1}n^{-\frac{1}{2}}}^{k^{-1}n^{-\frac{1}{2}}} F_2\, e^{-\pi i n\theta}\, d\theta \right| < A_{10}\, n^{\frac{1}{2}} \log^3 n\, k^{-\frac{1}{2}+\epsilon}(n,\,k)^{\frac{1}{2}}. \tag{51}$$

5. We now proceed to prove an important inequality for $\nu(n)$, defined in the introduction.

We abandon the condition $|\theta| < k^{-1}n^{-\frac{1}{2}}$, but still assume that $k \leqslant \sqrt{n}$.

From (9), (16), (50), and (48) it is clear that $F_2 = 0$ if $|\theta| > k^{-1}n^{-\frac{1}{2}}$, and hence we may write

$$\int_{-k^{-1}n^{-\frac{1}{2}}}^{k^{-1}n^{-\frac{1}{2}}} F_2\, e^{-\pi i n\theta}\, d\theta = \int_{-\infty}^{\infty} F_2\, e^{-\pi i n\theta}\, d\theta. \tag{52}$$

Now put

$$F_3 = (ky)^{-1} \sum_{\substack{0<l\leqslant 4\\(l,\,k,\,4)=1}} x\phi_s Z \sum_{\substack{0<h'\leqslant kN'\\(h',\,kN')=1\\h'\equiv l'\,(\mathrm{mod}\,N')}} \xi_{kN'}^{-2nh'}. \tag{53}$$

Then, from (9) and (15),

$$F_2 = F_3 \quad \text{if} \quad |\theta| < \tfrac{1}{2}k^{-1}n^{-\frac{1}{2}}. \tag{54}$$

In the case $|\theta| \geqslant \frac{1}{2}k^{-1}n^{-\frac{1}{2}}$, we have

$$|F_2 - F_3| \leqslant |ky|^{-1} \sum_{\substack{0<l\leqslant 4\\(l,\,k,\,4)=1}} |Z| \left| \sum_{\substack{0<h'\leqslant kN'\\(h',\,kN')=1\\h'\equiv l'\,(\mathrm{mod}\,N')}} (g-1)\, \xi_{kN'}^{-2nh'} \right|,$$

and this inequality, combined with (46), (47), and (18), gives

$$|F_2 - F_3| < 2A_6\, k^{\frac{3}{2}+\epsilon}(n,\,k)^{\frac{1}{2}} \log(4k)\, |ky|^{-1} \sum_{\substack{0<l\leqslant 4\\(l,\,k,\,4)=1}} |Z|.$$

Now, by (9) and (26),

$$|Z| < A_{11}\, |ky|^{-1} \log \max(n, |\theta|).$$

We have, therefore,

$$|F_2 - F_3| < A_{12}\, k^{\frac{3}{2}+\epsilon}(n,\,k)^{\frac{1}{2}} \log n\, |ky|^{-2} \log \max(n, |\theta|) \tag{55}$$

if $|\theta| \geqslant \frac{1}{2}k^{-1}n^{-\frac{1}{2}}$.

It follows immediately from (54) and (55) that

$$(56)\quad \left|\int_{-\infty}^{\infty}(F_2-F_3)\,e^{-\pi in\theta}\,d\theta\right| < A_{13}\,k^{-\frac{1}{3}+\epsilon}(n,k)^{\frac{1}{3}}\log n\int_{\frac{1}{2}k^{-1}n^{-\frac{1}{2}}}^{\infty}|y|^{-2}\log\max(n,|\theta|)\,d\theta$$

$$< A_{14}\,n^{\frac{1}{2}}\log^2 n\,k^{-\frac{1}{3}+\epsilon}(n,k)^{\frac{1}{3}}.$$

Let

$$(57)\qquad d_{n,k}=\int_{-\infty}^{\infty}F_3\,e^{-\pi in\theta}\,d\theta.$$

Then, by (51), (52), (56), and (57),

$$(58)\qquad |c_{n,k}-d_{n,k}|<A_{15}\,n^{\frac{1}{2}}\log^3 n\,k^{-\frac{1}{3}+\epsilon}(n,k)^{\frac{1}{3}}.$$

Finally,

$$\sum_{k=1}^{\sqrt{n}}k^{-\frac{1}{3}+\epsilon}(n,k)^{\frac{1}{3}}=\sum_{\substack{m\mid n\\ m\leqslant\sqrt{n}}}\ \sum_{\substack{l\leqslant\sqrt{n}/m\\ (l,\,n/m)=1}}(lm)^{-\frac{1}{3}+\epsilon}m^{\frac{1}{3}}$$

$$\leqslant\sum_{\substack{m\ n\\ m\leqslant\sqrt{n}}}m^{\epsilon}\sum_{l=1}^{\sqrt{n}/m}l^{-\frac{1}{3}+\epsilon}\leqslant A_{16}\sum_{\substack{m\ n\\ m\leqslant\sqrt{n}}}m^{\epsilon}\left(\frac{\sqrt{n}}{m}\right)^{\frac{2}{3}+\epsilon}$$

$$\leqslant A_{16}\,n^{\frac{1}{3}+\frac{1}{2}\epsilon}\sum_{m\ n}m^{-\frac{2}{3}}\leqslant A_{16}\,n^{\frac{1}{3}+\frac{1}{2}\epsilon}\,d(n),$$

where $d(n)$ denotes the number of divisors of n.

Hence we obtain the following inequality from (12), (9), and (58):

$$|\nu(n)-\tfrac{1}{4}e^{\pi}\sum_{k=1}^{\sqrt{n}}d_{n,k}|<A_{17}\,n^{\frac{5}{6}+\frac{1}{2}\epsilon}\log^3 n\,d(n),$$

and, on using the well-known results

$$d(n)=O(n^{\frac{1}{4}\epsilon})^{*},\quad \log n=O(n^{\frac{1}{12}\epsilon}),$$

we have

$$(59)\qquad \left|\nu(n)-\tfrac{1}{4}e^{\pi}\sum_{k=1}^{\sqrt{n}}d_{n,k}\right|<A_{18}\,n^{\frac{5}{6}+\epsilon}.$$

6. *The dominant term.*

We now consider F_3, defined in (53). In this equation, x is an abbreviation for a fourth root of unity and is defined in (37); the values of ϕ_s are

* Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), 250, Theorem 260.

given in (24). From these results it is clear that $x\phi_s$ vanishes if $S \equiv S_3$ or $S_4 \pmod 2$; in other cases,

$$x\phi_s = \begin{cases} (-1)^{\frac{1}{2}(d-1)} & \text{if} \quad S \equiv S_1 \text{ or } S_2 \pmod 2; \\ i(-1)^{\frac{1}{2}(c-1)} & \text{if} \quad S \equiv S_5 \text{ or } S_6 \pmod 2, \end{cases}$$

where c and d are given by (36). That is, on using (38),

$$x\phi_s = \begin{cases} (-1)^{\frac{1}{2}(l+1)} & \text{if} \quad S \equiv S_1 \text{ or } S_2 \pmod 2; \\ i(-1)^{\frac{1}{2}(k-1)} & \text{if} \quad S \equiv S_5 \text{ or } S_6 \pmod 2. \end{cases}$$

On considering the definitions of $S_1, S_2, \ldots, S_6$, and recalling the fact that $(h, k) = 1$, we may write the last equation in the form

$$x\phi_s = \begin{cases} (-1)^{\frac{1}{2}(l+1)} & \text{if} \quad k \text{ is even}; \\ i(-1)^{\frac{1}{2}(k-1)} & \text{if} \quad k \text{ is odd and } h \text{ is even}; \\ 0 & \text{if} \quad h \text{ and } k \text{ are both odd}. \end{cases}$$

Hence, by (53) and (26),

$$F_3 = i(ky)^{-1}(-\pi iky)^{-1}[\gamma - 2\log k - \log(-\pi iy)]V_1 \quad \text{if} \quad k \text{ is odd}, \tag{60}$$

where

$$V_1 = (-1)^{\frac{1}{2}(k-1)} \sum_{\substack{0<l\leqslant 4 \\ (l,\,k,\,4)=1 \\ 2\,l}} \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} \xi_{kN'}^{-2nh'}; \tag{61}$$

$$F_3 = (ky)^{-1}(-2\pi iky)^{-1}[\gamma - 2\log(2k) - \log(-\pi iy)]V_2 \quad \text{if} \quad k \text{ is even}, \tag{62}$$

where

$$V_2 = \sum_{\substack{0<l\leqslant 4 \\ 2+l}} (-1)^{\frac{1}{2}(l+1)} \sum_{\substack{0<h'\leqslant kN' \\ (h',\,kN')=1 \\ h'\equiv l' \pmod{N'}}} \xi_{kN'}^{-2nh'}. \tag{63}$$

We apply these results in (57) and consider first the case when k is odd. Then we have

$$d_{n,k} = iV_1 \int_{-\infty}^{\infty} (ky)^{-1}(-\pi iky)^{-1}[\gamma - 2\log k - \log(-\pi iy)]e^{-\pi in\theta}\,d\theta,$$

and, when we write $-\pi iny = w$, the integral becomes

$$2e^{-\pi}\pi n V_1 k^{-2}(2\pi i)^{-1} \int_{\pi - i\infty}^{\pi + i\infty} e^w w^{-2}[\gamma - 2\log k - \log w + \log n]\,dw.$$

Now it is well known that

$$(2\pi i)^{-1}\int_{\pi-i\infty}^{\pi+i\infty} e^w w^{-t}dw = \frac{1}{\Gamma(t)} \quad (t>1),$$

and hence

$$(2\pi i)^{-1}\int_{\pi-i\infty}^{\pi+i\infty} e^w w^{-2}(-\log w)\,dw = \left[\frac{d}{dt}\Gamma^{-1}(t)\right]_{t=2} = \gamma-1.$$

Hence

(64) $$d_{n,k} = 2e^{-\pi}\pi n V_1 k^{-2}[\log n + 2\gamma - 1 - 2\log k] \quad \text{if} \quad k \text{ is odd},$$

and similarly

(65) $$d_{n,k} = -ie^{-\pi}\pi n V_2 k^{-2}[\log n + 2\gamma - 1 - 2\log(2k)] \quad \text{if} \quad k \text{ is even}.$$

7. *Evaluation of* V_1 *and* V_2.

In this section we transform the sums occurring in equations (61) and (63).

On using (33) we may write (61) in the form

$$(-1)^{\frac{1}{2}(k-1)}V_1 = \sum_{\substack{0<h'\leqslant 2k\\(h',k)=1\\2\nmid h'}} \xi_k^{-nh'} + \sum_{\substack{0<h'\leqslant k\\(h',k)=1}} \xi_k^{-2nh'}$$

$$= \sum_{\substack{0<h'\leqslant 2k\\(h',k)=1}} \xi_k^{-nh'} = 2c_k(n),$$

where

$$c_k(n) = \sum_{\substack{0<h'\leqslant k\\(h',k)=1}} \xi_k^{-nh'}.$$

On using Ramanujan's formula

$$c_k(n) = \sum_{\substack{q,\,d\\k=qd,\ d\mid n}} \mu(q)\,d,$$

for which Landau* has given a simple proof, we obtain the result

(66) $$(-1)^{\frac{1}{2}(k-1)}V_1 = 2\sum_{\substack{q,\,d\\k=qd,\ d\mid n}} \mu(q)\,d.$$

Next we investigate V_2. From (63) and (33) we have

$$V_2 = \sum_{\substack{0<h'\leqslant 4k\\(h',k)=1}} (-1)^{\frac{1}{2}(h'+1)}\xi_{2k}^{-nh'} = 2\sum_{\substack{0<h'\leqslant 2k\\(h',k)=1}} (-1)^{\frac{1}{2}(h'+1)}\xi_{2k}^{-nh'}.$$

* Landau, *loc. cit.*, 1, 188, Theorem 220.

We can remove the condition $(h', k) = 1$ from the conditions of summation by using the well-known result

$$\sum_{\substack{q, r \\ qr=h',\ q|k}} \mu(q) = \begin{cases} 1 & \text{if} \quad (h', k) = 1; \\ 0 & \text{if} \quad (h', k) > 1. \end{cases}$$

This gives

$$V_2 = 2 \sum_{\substack{q, r \\ 0<qr\leqslant 2k \\ 2+qr,\ q|k}} (-1)^{\frac{1}{2}(qr+1)} \mu(q)\, \xi_{2k}^{-nqr}$$

$$= 2 \sum_{\substack{q \\ q|k,\ 2+q}} (-1)^{\frac{1}{2}(q+1)} \mu(q) \sum_{\substack{r \\ r\leqslant 2k/q,\ 2+r}} (-1)^{\frac{1}{2}(r-1)} \xi_{2k}^{-nqr}$$

$$= 2 \sum_{\substack{q \\ q|k,\ 2+q}} (-1)^{\frac{1}{2}(q+1)} \mu(q) \sum_{t=0}^{k/q-1} (-1)^t \xi_{2k}^{-nq(2t+1)}.$$

The inner sum vanishes unless $2nq/k$ is an odd integer, when its value is

$$\frac{k}{q}\, \xi_{2k}^{-nq} = \frac{k}{q}\, i(-1)^{\frac{1}{2}(2nq/k+1)}.$$

Hence

$$V_2 = 2i \sum_{\substack{q \\ q|k,\ k|2nq \\ 2+q,\ 2+(2nq/k)}} (-1)^{\frac{1}{2}(q+1)} \mu(q)\, \frac{k}{q}\, (-1)^{\frac{1}{2}(2nq/k+1)} \tag{67}$$

$$= 4i \sum_{\substack{q, d \\ k=2qd,\ d|n \\ 2+q,\ 2+(n/d)}} (-1)^{\frac{1}{2}(q-1)} \mu(q)\, d\, (-1)^{\frac{1}{2}(n/d-1)}.$$

8. It follows from (64), (65), (66), and (67) that

$$\sum_{k>\sqrt{n}} d_{n,k} < A_{19}\, n \sum_{\substack{q, d \\ 2qd>\sqrt{n},\ d|n}} q^{-2} d^{-1} (\log q + \log d)$$

$$< A_{20}\, n^{\frac{5}{6}},$$

and hence in (59) we may replace the finite series by the infinite series and write

$$\nu(n) = \tfrac{1}{4} e^{\pi} \sum_{k=1}^{\infty} d_{n,k} + O(n^{\frac{5}{6}+\epsilon}). \tag{68}$$

It only remains to express the dominant term on the right-hand side of (68) in terms of the functions introduced in the enunciation of the

theorem. We have

$$(69)\quad \tfrac{1}{4}e^{\pi}\sum_{k=1}^{\infty} d_{n,k}$$

$$= \pi n \sum_{\substack{q,\,d \\ 2+qd,\ d\mid n}} (-1)^{\frac{1}{2}(qd-1)}\mu(q)q^{-2}d^{-1}(\log n+2\gamma-1-2\log q-2\log d)$$

$$+\tfrac{1}{4}\pi n \sum_{\substack{q,\,d \\ d\mid n,\ 2+q,\ 2+(n/d)}} \{(-1)^{\frac{1}{2}(q-1)}\mu(q)q^{-2}d^{-1}(-1)^{\frac{1}{2}(n/d-1)}$$

$$\times(\log n+2\gamma-1-2\log q-2\log d-4\log 2)\}$$

$$= \pi n \sum_{\substack{d \\ d\mid n,\ 2+d}} (-1)^{\frac{1}{2}(d-1)}d^{-1}\sum_{\substack{q \\ 2+q}} \{(-1)^{\frac{1}{2}(q-1)}\mu(q)q^{-2}$$

$$\times(\log n+2\gamma-1-2\log q-2\log d)\}$$

$$+\tfrac{1}{4}\pi n \sum_{\substack{d \\ d\mid n,\ 2+(n/d)}} (-1)^{\frac{1}{2}(n/d-1)}d^{-1}\sum_{\substack{q \\ 2+q}} \{(-1)^{\frac{1}{2}(q-1)}\mu(q)q^{-2}$$

$$\times(\log n+2\gamma-1-2\log q-2\log d-4\log 2)\}$$

$$= \tfrac{1}{4}\pi n\,[\{(\log n+2\gamma-1)b_0-2b_1\}F_0(n)-2b_0F_1(n)-4b_0\log 2\,E_0'(n)]$$

on using (1), (2), (3), and (4). The result (5) follows immediately from (68) and (69).

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ON THE CESÀRO SUMMATION OF FOURIER SERIES AND ALLIED SERIES

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1. *Introduction.*

1.1. Several theorems are known connecting the Cesàro summability of a Fourier series at a point with the behaviour of the Cesàro means of the function near the point†. In this paper we obtain a necessary and sufficient condition for summability (C, β) in terms of the α-th mean of the function. From this it appears that, in many cases, a sufficient condition for the summability (C, β) of a Fourier series is given by a sufficient condition that the Fourier series of the α-th mean should be summable $(C, \beta-\alpha)$. Moreover, when α is an integer these two phenomena are actually equivalent‡. There are analogous results for allied series§.

1.2. We suppose throughout that $f(t)$ is a periodic function, with period 2π, and integrable L. We write

$$\left.\begin{aligned} f^+(t) &= f(x+t), \quad f^-(t) = f(x-t), \\ \phi(t) &= \tfrac{1}{2}\{f^+(t)+f^-(t)-2s\}, \\ \psi(t) &= \tfrac{1}{2}\{f^+(t)-f^-(t)\}, \\ \theta(t) &= \frac{2}{\pi}\int_t^\infty \frac{\psi(u)}{u}\,du\|, \end{aligned}\right\} \qquad (1.21)$$

† Hardy and Littlewood, **9**, **12**; Paley, **19**; Verblunsky, **23**, **25**; Wiener, **27**, **28**; Bosanquet, **3**; Bosanquet and Linfoot, **5**. See also Kogbetlianz, **17**.

‡ See Theorem 2. The analogous result for Hölder means was partly proved by Young, **29**, and completed by Hardy and Littlewood, **9**. This paper is the outcome of an attempt to extend their results to non-integral orders, and I am indebted to Prof. Hardy for drawing my attention to the problem in the first place. He also obtained some results of this character by employing a different definition of the α-th mean of a function, and it is still possible that the equivalence problem may admit of a complete solution if the "right" definition is used.

§ Hardy and Littlewood, **10**; Paley, **19**; Verblunsky, **25**.

‖ Here, and throughout the paper, $\int^\infty$ means $\lim_{N\to\infty}\int^N$. Since $\psi(t)$ is periodic and an odd function, $\theta(t)$ exists in this sense.

and suppose throughout that $t > 0$. We also write

$$\left.\begin{aligned} \Phi_\alpha(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du \quad (\alpha > 0), \\ \Phi_0(t) &= \phi(t), \\ \Phi_\alpha(t) &= \Phi'_{\alpha+1}(t) \quad (-1 < \alpha < 0), \end{aligned}\right\} \tag{1.22}$$

and

$$\phi_\alpha(t) = \Gamma(\alpha+1)\,t^{-\alpha}\,\Phi_\alpha(t) \quad (\alpha > -1). \tag{1.23}$$

We employ the functions $\Psi_\alpha(t)$, $\psi_\alpha(t)$, $\Theta_\alpha(t)$, $\theta_\alpha(t)$, $f_\alpha^+(t)$, $f_\alpha^-(t)$ with similar meanings. We then have

$$\phi_\alpha(t) = \tfrac{1}{2}\{f_\alpha^+(t)+f_\alpha^-(t)-2s\}$$

and

$$\psi_\alpha(t) = \tfrac{1}{2}\{f_\alpha^+(t)-f_\alpha^-(t)\}.$$

If $\alpha > -1$ and $\phi_\alpha(t) \to l$ as $t \to 0$ we say that $\phi(t) \to l\,(C, \alpha)$†. If α is unspecified we say that $\phi(t) \to l\,(C)$. There are similar definitions with boundedness in place of convergence.

If we write the Fourier series of $f(t)$ as

$$\tfrac{1}{2}a_0 + \Sigma\,(a_n \cos nt + b_n \sin nt), \tag{1.24}$$

then that of $\phi(t)$ is

$$(\tfrac{1}{2}a_0 - s) + \Sigma(a_n \cos nx + b_n \sin nx)\cos nt = \tfrac{1}{2}A_0 + \Sigma\,A_n \cos nt. \tag{1.25}$$

The allied series of $f(t)$ is

$$\Sigma(b_n \cos nt - a_n \sin nt), \tag{1.26}$$

and that of $\psi(t)$ is

$$\Sigma(b_n \cos nx - a_n \sin nx)\cos nt = \Sigma\,B_n \cos nt. \tag{1.27}$$

1.3. It is convenient to give here the properties of certain functions which will be employed in the sequel.

We require the functions $\gamma_\alpha(x)$, $\bar{\gamma}_\alpha(x)$, where

$$\kappa_\alpha(x) = \gamma_\alpha(x) + i\bar{\gamma}_\alpha(x) = \int_0^1 (1-t)^{\alpha-1} e^{itx}\,dt \quad (\alpha > 0). \tag{1.31}$$

† It should be remarked that, since a derivative is measurable, if $\phi_\alpha(t) = O(1)$ in $(0, \eta)$, *i.e.* $\Phi_\alpha(t) = O(t^\alpha)$, then the function $\Phi_\alpha(t)$ is integrable L in $(0, \eta)$ for $\alpha > -1$. Hence the consistency theorems for mean continuity and boundedness are valid for $\alpha > -1$. *Cf.* Bosanquet, 2.

It is known that $\gamma_\alpha(x)$, $\bar{\gamma}_\alpha(x)$ and their derivatives are bounded for $x \geqslant 0$, and that, for large x,

$$\kappa_\alpha(x) = \frac{i}{x} + \Gamma(\alpha)\frac{e^{i(x-\frac{1}{2}\pi\alpha)}}{x^\alpha} + O\left(\frac{1}{x^{\alpha+1}}\right) + O\left(\frac{1}{x^2}\right), \tag{1.32}$$

while asymptotic formulae for their derivatives are obtained by formally differentiating this formula. Moreover, it is easily verified† that, for $n < \alpha$,

$$\kappa_\alpha^{(n)}(x) = \frac{1}{x^n}\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r\,\Gamma(\alpha+r)}{\Gamma(\alpha-n+r)}\,\kappa_{\alpha-n+r}(x). \tag{1.33}$$

We require also the functions $J_\beta{}^\alpha(x)$, $\bar{J}_\beta{}^\alpha(x)$, where

$$K_\beta{}^\alpha(x) = J_\beta{}^\alpha(x) + i\,\bar{J}_\beta{}^\alpha(x)$$

$$= \frac{(-1)^{h+1}\,x^{h+1}}{\Gamma(1+h-\alpha)\,\Gamma(\alpha+1)}\int_1^\infty (t-1)^{h-\alpha}\,\kappa_{1+\beta}^{(h+1)}(xt)\,dt \quad (\beta \geqslant \alpha \geqslant 0), \tag{1.34}$$

and h is the greatest integer not greater than α. It is easily seen that these functions and their derivatives are bounded for $x \geqslant 0$. Moreover,

$$K_\beta{}^\alpha(x) = \frac{i}{x} + \frac{\Gamma(1+\beta)}{\Gamma(\alpha+1)}\frac{e^{i\{x-\frac{1}{2}\pi(1+\beta+\alpha)\}}}{x^{1+\beta-\alpha}} + O\left(\frac{1}{x^{2+\beta-\alpha}}\right) + O\left(\frac{1}{x^2}\right) \tag{1.35}$$

for large x, and this formula may also be differentiated. To obtain (1.35) we substitute the asymptotic formula for $\kappa_{1+\beta}^{(h+1)}(xt)$ in (1.34) and observe that, by Cauchy's theorem and Jordan's lemma‡,

$$x^{h+1}\int_1^\infty (t-1)^{h-\alpha}\,\Gamma(1+\beta)(xt)^{-1-\beta}\,e^{i\{xt-\frac{1}{2}\pi(\beta-h)\}}\,dt$$

$$= \Gamma(1+\beta)\,i^{1+h-\alpha}\,x^{h-\beta}\,e^{i\{x-\frac{1}{2}\pi(\beta-h)\}}\int_0^\infty y^{h-\alpha}(1+iy)^{-1-\beta}\,e^{-xy}\,dy$$

$$= \Gamma(1+\beta)\,\Gamma(1+h-\alpha)\,\frac{(-1)^{h+1}\,e^{i\{x-\frac{1}{2}\pi(1+\beta+\alpha)\}}}{x^{1+\beta-\alpha}} + O\left(\frac{1}{x^{2+\beta-\alpha}}\right),$$

while

$$x^{h+1}\int_1^\infty (t-1)^{h-\alpha}\,\frac{(-1)^{h+1}\,\Gamma(h+2)}{(xt)^{h+2}}\,dt = \frac{(-1)^{h+1}\,\Gamma(\alpha+1)\,\Gamma(1+h-\alpha)}{x},$$

† For instance, by Leibniz's theorem, writing $\kappa_\alpha(x) = \Gamma(\alpha)\,t^{-\alpha}E_\alpha(x)$ and using the fact that $E_\alpha'(x) = E_{\alpha-1}(x)$.

‡ Substituting $1+iy$ for t.

and the last two terms of (1.35) are obtained in a similar way. It should also be remarked that, when $\alpha = h$ is an integer, we have†

$$K_\beta^h(x) = \frac{(-1)^h}{\Gamma(h+1)} \cdot x^h \kappa_{1+\beta}^{(h)}(x). \tag{1.36}$$

2.

2.1. In this section we give some preliminary lemmas.

LEMMA 1. *If* $\alpha > 0$ *and* $\phi(t) = o(1)\,(C)$, *then* $\phi_\alpha(t) = o(1)\,(C)$, *and conversely.*

The first part has been proved elsewhere‡. To prove the converse, suppose that $\phi_\alpha(t) = o(1)\,(C, \beta)$, where $\beta > 0$. Then

$$\begin{aligned}
\Phi_{\alpha+\beta}(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} \Phi_\alpha(u)\, du \\
&= \frac{1}{\Gamma(\beta)\,\Gamma(\alpha+1)} \int_0^t (t-u)^{\beta-1} u^\alpha \phi_\alpha(u)\, du \\
&= \frac{t^\alpha}{\Gamma(\beta)\,\Gamma(\alpha+1)} \int_{\theta t}^t (t-u)^{\beta-1} \phi_\alpha(u)\, du \quad (0 < \theta < 1), \\
&= t^\alpha\, o(t^\beta) = o(t^{\alpha+\beta}),
\end{aligned}$$

by Riesz's theorem of the mean§. Thus $\phi(t) = o(1)\ (C, \alpha+\beta)$.

LEMMA 2‖. *If* $\alpha > 0$, *then*

$$\theta_\alpha(t) = \frac{2}{\pi} \int_t^\infty \frac{\psi_\alpha(u)}{u}\, du. \tag{2.11}$$

Starting from the identity

$$\Psi_1(u) = \tfrac{1}{2}\pi \{\Theta_1(u) + (t-u)\,\Theta_0(u) - t\,\Theta_0(u)\}, \tag{2.12}$$

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† The formula (1.35) would perhaps be made more intelligible by writing the second term in the form

$$\frac{e^{-i\pi\alpha}\,\Gamma(1+\beta)}{\Gamma(\alpha+1)} \cdot \frac{e^{i\{x - \frac{1}{2}\pi(1+\beta-\alpha)\}}}{x^{1+\beta-\alpha}}.$$

The factor $e^{-i\pi\alpha}$ is *real* only when α is an integer. This fact may be regarded as the reason why h is restricted to be an integer in Theorem 2.

‡ Bosanquet, **4**, 101, where a more precise result is given.

§ Riesz, **22**, 115. See also Verblunsky, **24**, 173, Bosanquet and Linfoot, **5**, 210.

‖ This is analogous to the identity (2.16).

we obtain, for $\alpha \geqslant 1$,

$$\Psi_\alpha(t) = \tfrac{1}{2}\pi\{\alpha\Theta_\alpha(t) - t\Theta_{\alpha-1}(t)\}. \qquad (2.13)$$

By differentiation this holds for $\alpha > 0$. Hence

$$\frac{\psi_\alpha(t)}{t} = \tfrac{1}{2}\pi\Gamma(\alpha+1)\left\{\alpha\frac{\Theta_\alpha(t)}{t^{\alpha+1}} - \frac{\Theta_{\alpha-1}(t)}{t^\alpha}\right\}$$

$$= -\frac{d}{dt}\{\tfrac{1}{2}\pi\Gamma(\alpha+1)t^{-\alpha}\Theta_\alpha(t)\}$$

$$= -\tfrac{1}{2}\pi\phi_\alpha'(t).$$

The result follows by integration since, as $t \to \infty$, $\theta(t) \to 0$, and so $\theta_\alpha(t) \to 0$.

LEMMA 3†. *If $\alpha > -1$, then a necessary and sufficient condition that $\theta(t) \to s$ (C, α) is that $\psi(t) = o(1)$ $(C, \alpha+1)$ and $\theta(t) \to s$ (C).*

From (2.13), with $\alpha+1$ in place of α, we obtain

$$\psi_{\alpha+1}(t) = \tfrac{1}{2}\pi(\alpha+1)\{\theta_{\alpha+1}(t) - \theta_\alpha(t)\}. \qquad (2.14)$$

The result follows by induction and the consistency theorem for Cesàro limits.

LEMMA 4. *A necessary and sufficient condition that the Fourier series of $f(t)$ should be summable (C) for $t = x$ to the sum s is that $\phi(t) = o(1)$ (C).*

LEMMA 5. *A necessary and sufficient condition that the allied series of $f(t)$ should be summable (C) for $t = x$ to the sum s is that $\theta(t) \to s$ (C).*

The last two lemmas are due to Hardy and Littlewood‡.

LEMMA 6§. *If $\alpha > -1$, then a necessary and sufficient condition that Σa_n should be summable (C, α) is that it should be summable (C) [or (A)] and $na_n = o(1)$ $(C, \alpha+1)$.*

There is an analogous result with boundedness in place of summability and O in place of o.

The (C) version follows by induction from the identity‖

$$\tau_n^{\alpha+1} = (\alpha+1)(s_n{}^\alpha - s_n^{\alpha+1}), \qquad (2.15)$$

† This was given by Paley, **19**, 176, for the case $\alpha \geqslant 0$. It is analogous to the (C) version of Lemma 6.

‡ Hardy and Littlewood, **9** and **10**.

§ This was given by Hardy and Littlewood, **14**, 283, and Kogbetliantz, **17**, 38. The case $\alpha = 0$ is due to Tauber.

‖ See Kogbetliantz, **17**, 30.

where τ_n^{α}, s_n^{α} are the n-th Cesàro means of order α of the sequences na_n and $s_n = a_0 + \ldots + a_n$ respectively.

The (A) version follows from the identity†

$$s_n^{\alpha} = a_0 + \sum_{m=1}^{n} \frac{\tau_m^{\alpha}}{m} \tag{2.16}$$

and Tauber's theorem‡ when $\alpha \geqslant 0$, and from Tauber's theorem and the (C) version when $-1 < \alpha < 0$.

LEMMA 7§. *If Σa_n is bounded (C, α) and is summable (A), then the series is summable $(C, \alpha+\delta)$ for every $\delta > 0$.*

3.

3.1. THEOREM 1. *A necessary and sufficient condition that the Fourier series of $f(t)$ should be summable (C, β) for $t = x$ to the sum s is that*

$$\omega \int_0^{\infty} \phi_{\alpha}(t)\, J_{\beta}^{\alpha}(\omega t)\, dt = o(1) \tag{3.11}$$

as $\omega \to \infty$, where $\beta \geqslant \alpha \geqslant 0$.

THEOREM 1*a*. *A necessary and sufficient condition that the allied series of $f(t)$ should be summable (C, β) for $t = x$ to the sum s is that*

$$\omega \int_0^{\infty} \psi_{\alpha}(t)\, \bar{J}_{\beta}^{\alpha}(\omega t)\, dt \to \tfrac{1}{2}\pi s \tag{3.12}$$

as $\omega \to \infty$, where $\beta \geqslant \alpha \geqslant 0$.

THEOREM 1*b*. *The condition* (3.12) *may be replaced by the conditions*

$$\omega \int_0^{\infty} \psi_{\alpha}(t) \{\bar{J}_{\beta}^{\alpha}(\omega t) - \bar{J}_{\beta+\lambda}^{\alpha}(\omega t)\}\, dt = o(1) \tag{3.13}$$

as $\omega \to \infty$, and $\theta(t) \to s$ (C), where $\lambda > 0$ and $\beta \geqslant \alpha \geqslant 0$.

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† Kogbetliantz, **16**, and **17**, 23.

‡ *I.e.*, the case $\alpha = 0$. See Dienes, **6**, 466. We also use the fact that, if $s_n \to s\,(A)$, then $s_n^{\alpha} \to s\,(A)$ for $\alpha > 0$. See Kogbetliantz, **17**, 38.

§ This was given by Littlewood, **18**, for the case α = integer, $\delta = 1$, and was completed by Andersen, **1**. See also Hardy and Littlewood, **13**, Kogbetliantz, **17**.

THEOREM 1c. *If* $\Sigma B_n \cos nt$ *is the allied series of* $\psi(t)$, *a necessary and sufficient condition that*

$$nB_n = o(1) \quad (C, \beta+1)\dagger$$

is that

$$\omega \int_0^\infty \psi_\alpha(t)\{\bar{J}_\beta{}^\alpha(\omega t) - \bar{J}^\alpha_{\beta+1}(\omega t)\}\,dt = o(1)\ddagger \tag{3.14}$$

as $\omega \to \infty$, *where* $\beta \geqslant \alpha \geqslant 0$.

There are analogous theorems with boundedness in place of convergence and O in place of o.

Proof of Theorem 1. Suppose first that α is not an integer. After $h+1$ integrations by parts, where h is the greatest integer not greater than α, we find, since $\Phi_\rho(t) = O(t^\rho)$ for large t, and $\beta \geqslant \alpha > h$, that

$$\begin{aligned}
J(\phi, \omega, \beta) &= \omega \int_0^\infty \phi(t)\,\gamma_{1+\beta}(\omega t)\,dt \\
&= \Big[\sum_{\rho=1}^{h+1} (-1)^{\rho-1}\omega^\rho \Phi_\rho(t)\,\gamma^{(\rho-1)}_{1+\beta}(\omega t)\Big]_0^\infty \\
&\qquad + (-1)^{h+1}\omega^{h+2}\int_0^\infty \Phi_{h+1}(t)\,\gamma^{(h+1)}_{\beta+1}(\omega t) \\
&= (-1)^{h+1}\omega^{h+2}\int_0^\infty \Phi_{h+1}(t)\,\gamma^{(h+1)}_{1+\beta}(\omega t)\,dt \\
&= \frac{(-1)^{h+1}}{\Gamma(1+h-\alpha)}\,\omega^{h+2}\int_0^\infty \gamma^{(h+1)}_{1+\beta}(\omega t)\,dt \int_0^t (t-u)^{h-\alpha}\Phi_\alpha(u)\,du \\
&= \frac{(-1)^{h+1}}{\Gamma(1+h-\alpha)}\,\omega^{h+2}\int_0^\infty \Phi_\alpha(u)\,du \int_u^\infty (t-u)^{h-\alpha}\gamma^{(h+1)}_{1+\beta}(\omega t)\,dt \\
&= \omega \int_0^\infty \phi_\alpha(u)\,J_\beta{}^\alpha(\omega u)\,du,
\end{aligned}$$

provided that the inversion of the repeated integral is justified. To justify this inversion, we must show that, for $\omega > 0$,

$$\lim_{N\to\infty} \int_0^N \Phi_\alpha(u)\,du \int_N^\infty (t-u)^{h-\alpha}\gamma^{(h+1)}_{1+\beta}(\omega t)\,dt = 0.$$

† *I.e.*, $\sum_{\nu=0}^{n} \nu B_\nu = o(n)$ (C, β), which is equivalent to $\sum_{\nu<\omega} \nu B_\nu = o(\omega)$ (R, β). *Cf.* Hobson, **15**, 90–98.

‡ This may also be written $\int_0^\infty \psi_\alpha(t)\,\frac{d}{dt}\,J_\beta{}^\alpha(\omega t)\,dt = o(1)$.

Now, for $u < N$, and a fixed $\omega > 0$,

$$\left|\int_N^\infty (t-u)^{h-\alpha}\gamma_{1+\beta}^{(h+1)}(\omega t)\,dt\right| \leqslant (N-u)^{h-\alpha}\max_{N'>N}\left|\int_N^{N'}\gamma_{1+\beta}^{(h+1)}(\omega t)\,dt\right|$$
$$= (N-u)^{h-\alpha}\,O(N^{-1-\kappa}),$$

where $\kappa = \min(\beta, h+2)$. Also

$$N^{-1-\kappa}\int_0^N (N-u)^{h-\alpha}|\Phi_\alpha(u)|\,du \leqslant \frac{N^{-1-\kappa}}{\Gamma(\alpha)}\int_0^N (N-u)^{h-\alpha}\,du\int_0^u (u-v)^{\alpha-1}|\phi(v)|\,dv$$
$$= \frac{N^{-1-\kappa}}{\Gamma(\alpha)}\int_0^N |\phi(v)|\,dv\int_v^N (N-u)^{h-\alpha}(u-v)^{\alpha-1}\,du$$
$$= \frac{\Gamma(1+h-\alpha)}{\Gamma(h+1)}\,N^{-1-\kappa}\int_0^N (N-v)^h|\phi(v)|\,dv$$
$$= N^{-1-\kappa}\,O(N^{h+1})$$
$$= o(1),$$

since $|\phi(v)|$ is periodic and $\beta \geqslant \alpha > h$. If $\alpha = h$ is an integer, we integrate by parts h times and use (1.36).

Finally, a necessary and sufficient condition for summability (C, β) is that $J(\phi, \omega, \beta) = o(1)$ as $\omega \to \infty$†. The theorem is therefore proved.

To prove Theorem 1*a*, we replace $\phi(t)$ by $\psi(t)$, $\gamma_{1+\beta}(t)$ by $\bar{\gamma}_{1+\beta}(t)$, and take $\kappa = \min(\beta, h+1)$, and proceed as before.

Theorem 1*b* follows by induction from Theorem 1*a* and Lemma 3.

Theorem 1*c* follows from the identity

$$\sum_{\nu<\omega}\left\{\left(1-\frac{\nu}{\omega}\right)^\beta - \left(1-\frac{\nu}{\omega}\right)^{\beta+1}\right\}B_\nu = \frac{1}{\omega}\sum_{\nu<\omega}\left(1-\frac{\nu}{\omega}\right)^\beta \nu B_\nu.$$

3.2. When $\alpha = h$ is an integer, we can prove more precise results.

THEOREM 2. *A necessary and sufficient condition that the Fourier‡ series of $f(t)$ should be summable (C, β) for $t = x$ to the sum s is that the Fourier series‡ of $\frac{1}{2}\{f_h^+(t)+f_h^-(t)\}$ should be summable $(C, \beta-h)$ to s for $t = 0$, where h is an integer such that $0 \leqslant h \leqslant \beta$.*

THEOREM 2*a*. *A necessary and sufficient condition that the allied series of $f(t)$ should be summable (C, β) for $t = x$ to the sum s is that the allied series of $\frac{1}{2}\{f_h^+(t)-f_h^-(t)\}$ should be summable $(C, \beta-h)$ to s for $t = 0$, where h is an integer such that $0 \leqslant h \leqslant \beta$.*

† See Hobson, **15**, 2 (1926), 567.

‡ By the Fourier series of a non-periodic function near a point, we mean the Fourier series of a periodic function which coincides with the given function near the point considered.

Proof of Theorem 2. We have, after h integrations by parts,

$$J(\phi, \omega, \beta) = \omega \int_0^\infty \phi(t)\, \gamma_{1+\beta}(\omega t)\, dt$$

$$= (-1)^h \omega^{h+1} \int_0^\infty \Phi_h(t)\, \gamma_{1+\beta}^{(h)}(\omega t)\, dt$$

$$= (-1)^h \omega^{h+1} \int_0^\infty \frac{\Phi_h(t)}{(\omega t)^h} \sum_{r=0}^{h} \binom{h}{r} \frac{(-1)^r \Gamma(1+\beta+r)}{\Gamma(1+\beta-h+r)} \gamma_{1+\beta-h+r}(\omega t)\, dt$$

$$= \sum_{r=0}^{h} \binom{h}{r} \frac{(-1)^{h+r} \Gamma(1+\beta+r)}{\Gamma(h+1)\Gamma(1+\beta-h+r)} \omega \int_0^\infty \phi_h(t)\, \gamma_{1+\beta-h+r}(\omega t)\, dt$$

$$= \sum_{r=0}^{h} \binom{h}{r} \frac{(-1)^{h+r} \Gamma(1+\beta+r)}{\Gamma(h+1)\Gamma(1+\beta-h+r)} J(\phi_h, \omega, \beta-h+r).$$

The condition is sufficient. For, if the Fourier series of $\frac{1}{2}\{f_h^+(t)+f_h^-(t)\}$ is summable $(C, \beta-h)$ to s for $t=x$, then $J(\phi_h, \omega, \beta-h+r) = o(1)$ as $\omega \to \infty$, for $r \geqslant 0$. It follows that $J(\phi, \omega, \beta) = o(1)$ as $\omega \to \infty$.

The condition is necessary. For suppose the Fourier series of $f(t)$ to be summable (C, β) to s for $t=x$. Then, by Lemma 4, $\phi(t) = o(1)$ (C), and hence, by Lemma 1, $\phi_h(t) = o(1)$ (C). Thus, by Lemma 4, there is an integer λ such that $J(\phi_h, \omega, \beta-h+\kappa) = o(1)$ as $\omega \to \infty$, for $\kappa \geqslant \lambda$. Again, by hypothesis, if $\lambda \geqslant 1$, $J(\phi, \omega, \beta+\lambda-1) = o(1)$ as $\omega \to \infty$, and also

$$J(\phi, \omega, \beta+\lambda-1) = \sum_{r=0}^{h} \frac{(-1)^{h+r} \Gamma(\beta+\lambda+r)}{\Gamma(h+1)\Gamma(\beta+\lambda-h+r)} J(\phi_h, \omega, \beta-h+\lambda-1+r).$$

Now we have just remarked that every term in this sum after the first is $o(1)$ as $\omega \to \infty$, while the left-hand side is also $o(1)$. Hence

$$J(\phi_h, \omega, \beta-h+\lambda-1) = o(1),$$

and thus, by induction,

$$J(\phi_h, \omega, \beta-h) = o(1),$$

which completes the proof.

To prove Theorem 2*a*, we replace ϕ by ψ and J by $\bar{J}$, and use Lemmas 2, 3, and 5 instead of 4.

3.3. We next give the Riemann reductions corresponding to Theorems 1 and 1*a*.

THEOREM 3. *If* $\eta > 0$, *then*

$$\omega \int_{\eta}^{\infty} \phi_{\alpha}(t) J_{\beta}^{\alpha}(\omega t)\, dt = o(1) \tag{3.31}$$

as $\omega \to \infty$, *where* $\beta \geqslant \alpha \geqslant 0$.

THEOREM 3*a*. *If* $\eta > 0$, *then*

$$\omega \int_{\eta}^{\infty} \psi_{\alpha}(t) \left\{ \bar{J}_{\beta}^{\alpha}(\omega t) - \frac{1}{\omega t} \right\} dt = o(1) \tag{3.32}$$

as $\omega \to \infty$, *where* $\beta \geqslant \alpha \geqslant 0$.

These theorems are proved by standard arguments, and the proofs are therefore omitted†.

4.

4.1. In this section we give some applications of the above results. They include some well-known theorems and, in the cases where α is an integer, are immediate consequences of Theorems 2 and 2*a* and the special case $\alpha = 0$.

THEOREM 4‡. *If* $\alpha \geqslant 0$ *and*

$$\lim_{\kappa \to \infty} \overline{\lim_{x \to +0}} \int_{\kappa x}^{\eta} \frac{|\phi_{\alpha}(x+t) - \phi_{\alpha}(t)|}{t}\, dt = 0, \tag{4.11}$$

then the Fourier series of $f(t)$ *is either summable* (C, α) *for* $t = x$ *or by no Cesàro mean. A necessary and sufficient condition for summability to* s *is that* $\phi(t) = o(1)$ $(C, \alpha+1)$.

THEOREM 4*a*. *If* $\alpha \geqslant 0$ *and*

$$\lim_{\kappa \to \infty} \overline{\lim_{x \to +0}} \int_{\kappa x}^{\eta} \frac{|\psi_{\alpha}(x+t) - \psi_{\alpha}(t)|}{t}\, dt = 0, \tag{4.12}$$

† *Cf.* Pollard, **20**, 216, where the case $\alpha = 0$ is given.

‡ The case $\alpha = 0$ was given by Gergen, **7**, 256. Gergen's conditions include all the classical conditions. Thus it follows, for instance, from Theorem 4 that the Fourier series is summable (C, α) if (a) $|\phi_{\alpha}(t)|/t$ is integrable (*cf.* Dini's condition); (b) $\frac{1}{t}\int_0^t \phi_{\alpha}(u)\, du$ is of bounded variation in $(0, \eta)$ (*cf.* de la Vallée Poussin's condition); (c) the series is summable (C) and $\frac{1}{t}\int_0^t |d\{u\phi_{\alpha}(u)\}| = O(1)$ (*cf.* the Young-Hardy-Littlewood condition); and so on. Condition (b) is equivalent to (d) $\phi_{\alpha+1}(t)$ is of bounded variation. For a full discussion of the case $\alpha = 0$ see Gergen, **7**, where references are given.

then the allied series of $f(t)$ is either summable (C, α) for $t = x$ or by no Cesàro mean. A necessary and sufficient condition for summability to s is that $\theta(t) \to s$ (C, α).

Proof of Theorem 4. We have to consider the integral

$$J(\omega) = \omega \int_0^\eta \phi_\alpha(t) J_\alpha{}^\alpha(\omega t)\, dt, \tag{4.13}$$

where $J_\alpha{}^\alpha(t)$ and its derivatives are bounded for $t \geqslant 0$, and for large t

$$J_\alpha{}^\alpha(t) = \frac{\sin(t - \pi\alpha)}{t} + O\left(\frac{1}{t^2}\right). \tag{4.14}$$

It is necessary for summability to s that $\phi(t) = o(1)$ (C). On the other hand, if this condition is satisfied, we have also, by Lemma 1, $\phi_\alpha(t) = o(1)\,(C)$, and hence, from (4.11), as in Gergen's theorem, $\phi_\alpha(t) = o(1)$ $(C, 1)$, and so $\phi(t) = o(1)$ $(C, \alpha+1)$†. It follows from the fact that $\phi_\alpha(t) = o(1)$ $(C, 1)$, as in Gergen's theorem, that

$$a(\omega, \kappa) = \left\{ \int_0^{\kappa/\omega} + 2\int_0^{(\kappa+1)/\omega} + \int_0^{(\kappa+2)/\omega} - 2\int_\eta^{\eta+1/\omega} - \int_\eta^{\eta+2/\omega} \right\} \omega \phi_\alpha(t) J_\alpha{}^\alpha(\omega t)\, dt = o(1).$$

Next, as in Gergen's theorem,

$$\begin{aligned} 4J(\omega) - a(\omega, \kappa) &= \left\{ \int_{\kappa/\omega}^{\eta} + 2\int_{(\kappa+1)/\omega}^{\eta+1/\omega} + \int_{(\kappa+2)/\omega}^{\eta+2/\omega} \right\} \phi_\alpha(t)\, \frac{\sin(\omega t - \pi\alpha)}{t}\, dt \\ &\quad + \left\{ \int_{\kappa/\omega}^{\eta} + 2\int_{(\kappa+1)/\omega}^{\eta+1/\omega} + \int_{(\kappa+2)/\omega}^{\eta+2/\omega} \right\} O\left(\frac{1}{\omega t^2}\right) dt \\ &= o(1) + O\left(\frac{1}{\kappa}\right), \end{aligned}$$

where the O is uniform in κ, as $\omega \to \infty$. The last term is annihilated by the operation $\lim_{\omega\to\infty} \overline{\lim_{\kappa\to\infty}}$, and so $J(\omega) = o(1)$ as $\omega \to \infty$. This completes the proof.

Theorem 4*a* may be deduced from Theorems 1*a* and 3*a* in a similar way. Here the condition $\theta(t) \to s$ (C) implies $\psi(t) = o(1)$ (C), by Lemma 3, and this when combined with (4.12) implies $\psi(t) = o(1)$ $(C, \alpha+1)$, and thus,

† The last step is obtained by expressing $\phi_{\alpha+1}(t)$ in terms of $\frac{1}{t}\int_0^t \phi_\alpha(u)\, du$.

by Lemma 3, $\theta(t) \to s$ (C, α). There is an additional term†,

$$\tfrac{1}{4}\left\{\int_{\kappa/\omega}^{\infty} + 2\int_{(\kappa+1)/\omega}^{\infty} + \int_{(\kappa+2)/\omega}^{\infty}\right\} \frac{\psi_\alpha(t)}{t}\, dt,$$

which tends to $\frac{1}{2}\pi s$, by Lemma 2, as $\omega \to \infty$.

4.2. THEOREM 5‡. *If* $\alpha \geqslant 0$ *and*

$$\frac{1}{t}\int_0^t |\phi_\alpha(u)|\, du = O(1), \tag{4.21}$$

then the Fourier series of $f(t)$ *is either summable* $(C, \alpha+\delta)$ *for* $t = x$ *and* $\delta > 0$ *or is not summable* (A). *A necessary and sufficient condition for summability to* s *is that* $\phi(t) = o(1)$ (C, β) *for* $\beta > \alpha+1$.

THEOREM 5a§. *If* $\alpha \geqslant 0$ *and*

$$\frac{1}{t}\int_0^t |\psi_\alpha(u)|\, du = O(1), \tag{4.22}$$

then the allied series of $f(t)$ *is either summable* $(C, \alpha+\delta)$ *for* $t = x$ *and* $\delta > 0$ *or is not summable* (A). *A necessary and sufficient condition for summability to* s *is that* $\theta(t) \to s$ (C, β) *for* $\beta > \alpha$.

Proof of Theorem 5. We have to consider the integral

$$J(\omega) = \omega \int_0^\eta \phi_\alpha(t)\, J^\alpha_{\alpha+\delta}(\omega t)\, dt,$$

where $J^\alpha_{\alpha+\delta}(t)$ is bounded for all $t \geqslant 0$, and for large t,

$$J^\alpha_{\alpha+\delta}(t) = O\left(\frac{1}{t^{1+\delta}}\right).$$

† *Cf.* Bosanquet and Linfoot, **5**, 227.

‡ An analogous theorem, equivalent when $\alpha \leqslant 1$, has been given by Verblunsky, **25**. The case $\alpha = 0$ was given by Hardy and Littlewood, **11**, but the corresponding o result is due to Hardy, **8**. The sufficiency part, for the case where α is an integer, $\delta = 1$, and $\beta = \alpha+1$, was given by Pollard, **20**, 220.

§ An analogous theorem, equivalent when $\alpha \leqslant 1$, has been given by Verblunsky, **25**. The case $\alpha = 0$ is due to Hardy and Littlewood, **13**.

The hypothesis (4.22) might be replaced by

$$\frac{1}{t}\int_0^t |\theta_{\alpha-1}(u)|\, du = O(1) \quad (\alpha > 0)$$

(*cf.* Verblunsky, **25**). But since, from (2.14),

$$|\psi_\alpha(u)| \leqslant \tfrac{1}{2}\pi\alpha\left\{|\theta_{\alpha-1}(u)| + |\theta_\alpha(u)|\right\},$$

we see that the new condition would imply (4.22) and also $\theta(t) = O(1)$ (C). Thus the theorem would be less general.

Writing $$J(\omega) = \omega\left\{\int_0^{1/\omega} + \int_{1/\omega}^{\eta}\right\} = J_1 + J_2,$$

we have, by (4.21),

$$|J_1| \leqslant A\omega \int_0^{1/\omega} |\phi_\alpha(t)|\, dt = O(1),$$

while, if $$\Phi_\alpha{}^*(t) = \int_0^t |\phi_\alpha(u)|\, du,$$

$$\begin{aligned} |J_2| &\leqslant \frac{A}{\omega^\delta} \int_{1/\omega}^{\eta} |\phi_\alpha(t)| \frac{dt}{t^{1+\delta}} \\ &\leqslant A\omega\Phi_\alpha{}^*\left(\frac{1}{\omega}\right) + \frac{(1+\delta)A}{\omega^\delta} \int_{1/\omega}^{\eta} \Phi_\alpha{}^*(t) \frac{dt}{t^{2+\delta}} \\ &= \omega O\left(\frac{1}{\omega}\right) + \frac{1}{\omega^\delta} \int_{1/\omega}^{\eta} O(t) \frac{dt}{t^{2+\delta}} \\ &= O(1) + O(1). \end{aligned}$$

Thus the Fourier series is bounded $(C, \alpha+\delta)$ for every $\delta > 0$, and so, by Lemma 7, is either summable $(C, \alpha+\delta)$ or not summable (A). Finally, the condition $\phi(t) = o(1)$ (C, β), $\beta > \alpha+1$, is, by Lemma 4, sufficient for summability to s, and it is also known to be necessary for summability $(C, \alpha+\delta)$ for $\delta < \beta-\alpha-1$†.

To prove Theorem 5*a* it is convenient to employ the O version of Theorem 1*c*‡. We first prove that (4.22) implies $nB_n = O(1)$ $(C, \alpha+1+\delta)$, from which the first part follows by the O version of Lemma 6 and Lemma 7. Finally, the condition $\theta(t) \to s$ (C, β), $\beta > \alpha$, is sufficient for summability and known to be necessary for summability $(C, \alpha+\delta)$ to s for $\delta < \beta-\alpha$§.

4.3. Theorem 5 is known to be false with $\delta = 0$. In fact, it has been shown‖ in certain cases that there are functions for which $\phi_\alpha(t) = o(1)$ whose Fourier series are not summable (C, α) for $t = x$. These examples show, in the light of the next lemma, that the hypotheses of Theorem 3 are not sufficient for summability $(C, \alpha-\delta)$ for any $\delta > 0$.

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† Paley, **19**; Verblunsky, **23**; Wiener, **27, 28**; Bosanquet, **3**.

‡ Lemma 6 then plays the part of Lemma 2.

§ Paley, **19**.

‖ For references to the cases $\alpha = 0$ or 1, see Bosanquet, **3**. The general case is stated by Kogbetliantz and attributed by him to Wiener, **26**. The analogous problem for allied series has been settled in the case $\alpha = 0$. Paley, **19**.

LEMMA 8. *If* $\phi_\alpha(t) = O(1)$, *then*

$$\lim_{\kappa\to\infty}\overline{\lim_{x\to+0}}\int_{\kappa x}^{\eta}\frac{|\phi_\beta(x+t)-\phi_\beta(t)|}{t}\,dt = 0,$$

where $\beta > \alpha \geqslant 0$.

We may suppose without loss of generality that $\beta < \alpha+1$. Then we have

$$\begin{aligned}
&\beta^{-1}\Gamma(\beta-\alpha)\,|\phi_\beta(t)-\phi_\beta(x+t)| \\
&\quad = \left|\frac{1}{t^\beta}\int_0^t (t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du - \frac{1}{(x+t)^\beta}\int_0^{x+t}(x+t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du\right| \\
&\quad = \left|\int_0^t\left\{\frac{(t-u)^{\beta-\alpha-1}}{t^\beta}-\frac{(x+t-u)^{\beta-\alpha-1}}{(x+t)^\beta}\right\}\Phi_\alpha(u)\,du\right. \\
&\qquad\qquad \left. -\frac{1}{(x+t)^\beta}\int_t^{x+t}(x+t-u)^{\beta-\alpha-1}\Phi_\alpha(u)\,du\right| \\
&\quad \leqslant At^\alpha\int_0^t\left\{\frac{(t-u)^{\beta-\alpha-1}}{t^\beta}-\frac{(x+t-u)^{\beta-\alpha-1}}{(x+t)^\beta}\right\}du\dagger \\
&\qquad\qquad + \frac{A}{(x+t)^{\beta-\alpha}}\int_t^{x+t}(x+t-u)^{\beta-\alpha-1}\,du \\
&\quad = A\{1-t^\alpha(x+t)^{-\alpha}+x^{\beta-\alpha}t^\alpha(x+t)^{-\beta}\}+Ax^{\beta-\alpha}(x+t)^{-\beta+\alpha}.
\end{aligned}$$

Now

$$\int_{\kappa x}^{\eta}\left\{1-\frac{t^\alpha}{(x+t)^\alpha}\right\}\frac{dt}{t} = \int_\kappa^{\eta/x}\left\{1-\frac{u^\alpha}{(1+u)^\alpha}\right\}\frac{du}{u} = \int_\kappa^{\eta/x} O\left(\frac{1}{u^2}\right)du,$$

$$\int_{\kappa x}^{\eta}\frac{x^{\beta-\alpha}t^\alpha}{(x+t)^\beta}\frac{dt}{t} = \int_\kappa^{\eta/x}\frac{u^\alpha}{(1+u)^\beta}\frac{du}{u},$$

$$\int_{\kappa x}^{\eta}\frac{x^{\beta-\alpha}}{(x+t)^{\beta-\alpha}}\frac{dt}{t} = \int_\kappa^{\eta/x}\frac{1}{(1+u)^{\beta-\alpha}}\frac{du}{u},$$

and all these integrals are annihilated by the operation $\lim_{\kappa\to\infty}\overline{\lim_{x\to+0}}$. The lemma is therefore proved.

4.4. Finally we state some theorems which are partly included in the above.

† It is here that we use the fact that $\beta < \alpha+1$.

THEOREM 6†. *If $\alpha \geqslant 0$ and $\phi(t) = O(1)$ (C, α), then the Fourier series is either summable $(C, \alpha+\delta)$ or not summable (A). A necessary and sufficient condition for summability to s is $\phi(t) = o(1)$ (C, β) for $\beta > \alpha$.*

THEOREM 6a‡. *If $\alpha \geqslant 0$ and $\psi(t) = O(1)$ (C, α), then the allied series is either summable $(C, \alpha+\delta)$ or not summable (A). A necessary and sufficient condition for summability to s is $\theta(t) \to s$ (C, β) for $\beta > \alpha - 1$.*

THEOREM 7§. *If $\alpha \geqslant 0$ and $\phi_\alpha(t) > -\kappa$ in $(0, \eta)$, then the Fourier series is either summable $(C, \alpha+\delta)$ or not summable (C). A necessary and sufficient condition for summability to s is $\phi(t) = o(1)$ $(C, \alpha+1)$.*

THEOREM 7a. *If $\alpha \geqslant 0$ and $\psi_\alpha(t) > -\kappa$ in $(0, \eta)$, then the Fourier series is either summable $(C, \alpha+\delta)$ or not summable (C). A necessary and sufficient condition for summability to s is $\theta(t) \to s$ (C, α).*

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† The case $0 < \alpha < 1$ of the o version was given by Hardy and Littlewood, **12**. For the general case see Paley, **19**; Verblunsky, **23**; Wiener, **27, 28**; Bosanquet, **3**.

‡ The case $\alpha = 0$, $\beta = 0$ was given by Prasad, **21**.

§ The case $\alpha = 0$ was given by Hardy and Littlewood, **11**.

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SOME THEOREMS ON GENERALIZED FOURIER CONSTANTS

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1. Given a sequence of functions $\phi_n(x)$ defined in the interval (a, b) and a function $f(x)$ defined in that interval, we call the sequence

$$\int_a^b f\phi_n\,dx$$

the sequence of generalized Fourier constants of f. Most of the theorems in this paper are concerned with the generalized Fourier constants of a bounded function. We shall not give a summary of the results, but only mention a few consequences. In virtue of Theorem V, if ϕ_n is a uniformly bounded sequence of normal orthogonal functions, there exist sequences n_ν with the following property. Given any sequence a_ν with $\Sigma a_\nu^2 < \infty$, there is a bounded f such that

$$a_\nu = \int_a^b f\phi_{n_\nu}dx \quad (\nu = 1, 2, \ldots).$$

This property does not depend on the orthogonality. For Theorem IV tells us that, if $|\phi(x)| \leqslant 1$ in $(0, 1)$, and

$$\int_0^1 \phi dx = 0,$$

then, ϕ being defined outside $(0, 1)$ by the condition of periodicity, the sequence $\phi_n = \phi(nx)$ has the same property.

If we are given two sequences γ_j, α_j, we can define a sequence ϵ_j satisfying $\epsilon_j^2 = 1$ such that, if

$$\Sigma \epsilon_j \gamma_j \sin(jx+\alpha_j)/j$$

is of bounded variation, then $\Sigma \gamma_j^2$ converges. The sequence ϵ_j may be defined inductively by $\epsilon_1 = 1$,

$$\epsilon_j = -\operatorname{sgn}\int_0^{2\pi}\left[\sum_1^{j-1}\epsilon_k\gamma_k\cos(kx+\alpha_k)\right]^3\gamma_j\cos(jx+\alpha_j)\,dx \quad (j = 2, 3, \ldots),$$

if the second member is not zero, and $\epsilon_j = \pm 1$ otherwise. Thus ϵ_j depends only on the γ's and α's of index not exceeding j.

If we are given two sequences a_n, b_n such that $\Sigma {a_n}^2 + {b_n}^2$ converges, we can define a sequence ϵ_n satisfying ${\epsilon_n}^2 = 1$ such that

$$\Sigma \epsilon_n (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function which belongs to $L(p)$ for an assigned p. The inductive formula for ϵ_n is similar to the preceding one.

2. THEOREM I. *Let $\phi_j(x)$ $(j = 1, \ldots, n)$ be a set of absolutely integrable functions defined in the finite or infinite interval (a, b). Let a_j be a set of constants such that for every set of numbers μ_j we have*

$$\left| \sum_1^n \mu_j a_j \right| \leqslant \int_a^b \left| \sum_1^n \mu_j \phi_j(x) \right| dx. \tag{1}$$

Then there is a function $f(x)$, defined in (a, b), satisfying $|f| \leqslant 1$, and such that

$$a_j = \int_a^b f(x) \phi_j(x)\, dx \quad (j = 1, \ldots, n).$$

Consider the set E of points $\xi \equiv (\xi_1, \ldots, \xi_n)$ in n dimensional space, such that to each ξ there corresponds an $f_\xi(x)$ with $|f_\xi(x)| \leqslant 1$ which satisfies

$$\xi_j = \int_a^b f_\xi(x) \phi_j(x)\, dx \quad (j = 1, \ldots, n).$$

The set E is closed. For let $\xi^\nu \equiv (\xi_1{}^\nu, \ldots, \xi_n{}^\nu)$ be a sequence of points of E which tends to $\xi \equiv (\xi_1, \ldots, \xi_n)$ as $\nu \to \infty$. Let f_ν be the corresponding sequence of functions. Write

$$F_\nu(x) = \int_a^x f_\nu(t)\, dt, \tag{2}$$

where a is a fixed point interior to (a, b). We can find a subsequence ν_k such that

$$\lim_{k \to \infty} F_{\nu_k}(x) = F(x) \quad (a < x < b), \tag{3}$$

where

$$F(x) = \int_a^x f(t)\, dt \tag{4}$$

and $|f| \leqslant 1$. By a classical theorem of Lebesgue*, it follows that

$$\xi_j^{\nu_k} = \int_a^b f_{\nu_k} \phi_j\, dx \to \int_a^b f \phi_j\, dx \quad (j = 1, \ldots, n).$$

* Hobson, *Functions of a real variable*, 2 (1926), 423.

By hypothesis, $\xi_j^{\nu_k} \to \xi_j$. Hence

$$\xi_j = \int_a^b f\phi_j\, dx \quad (j = 1, \ldots, n),$$

and so ξ belongs to E.

Further, if $(\xi_1', \ldots, \xi_n')$ and $(\xi_1'', \ldots, \xi_n'')$ belong to E, then so does $(\lambda\xi_1'+\mu\xi_1'', \ldots, \lambda\xi_n'+\mu\xi_n'')$ for all $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda+\mu = 1$.

The condition (1) means that to each set of μ_j there corresponds a θ satisfying $\theta^2 \leqslant 1$ such that

$$\sum_1^n \mu_j a_j = \theta \int_a^b \left| \sum_1^n \mu_j \phi_j \right| dx = \sum_1^n \mu_j c_j$$

with

$$c_j = \int_a^b \theta \operatorname{sgn}(\Sigma \mu_j \phi_j)\, \phi_j\, dx \quad (j = 1, \ldots, n).$$

Thus $C \equiv (c_1, \ldots, c_n)$ belongs to E. Hence every plane

$$\sum_1^n \mu_j (x_j - a_j) = 0$$

through $A \equiv (a_1, \ldots, a_n)$ has a point of E on it. If A does not belong to E, then, since E is closed, there is a point $P \equiv (\mu_1, \ldots, \mu_n)$ of E whose distance from A is positive and an absolute minimum. Consider the plane

$$\Sigma(p_j - a_j)(x_j - a_j) = 0.$$

By what has been proved, there is a point $Q \equiv (q_1, \ldots, q_n)$ of E which lies on this plane. Hence

$$\Sigma(p_j - a_j)(q_j - a_j) = 0. \tag{5}$$

Further, the point $R(\lambda) \equiv \{\lambda p_1 + (1-\lambda) q_1, \ldots, \lambda p_n + (1-\lambda) q_n\}$ belongs to E for $0 < \lambda < 1$. The distance $d(\lambda)$ of $R(\lambda)$ from A is given by

$$\begin{aligned} d^2(\lambda) &= \Sigma\, [\lambda(p_j - a_j) + (1-\lambda)(q_j - a_j)]^2 \\ &= \lambda^2 \Sigma (p_j - a_j)^2 + (1-\lambda)^2 \Sigma (q_j - a_j)^2 \end{aligned} \tag{6}$$

by (5), and, if

$$0 < 1 - \lambda < \Sigma(p_j - a_j)^2 / \Sigma(q_j - a_j)^2,$$

the expression (6) is less than $\Sigma(p_j - a_j)^2$. This contradicts the fact that there is no point of E nearer to A than P. Hence A belongs to E. By the definition of E, this is the required result.

The noteworthy feature of this theorem is that the only restriction on the ϕ_j is that of absolute integrability.

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* *Cf.* Hobson, *Functions of a real variable*, 2 (1926), 430.

THEOREM II. *Let* $\phi_j(x)$ $(j = 1, 2, \ldots)$ *be a sequence of absolutely integrable functions defined in the finite or infinite interval* (a, b). *Let* a_j *be a sequence of constants such that, for every finite set of numbers* $\mu_1, \ldots, \mu_n$, *the condition* (1) *is satisfied. Then there is a function* $f(x)$, *defined in* (a, b), *satisfying* $|f| \leqslant 1$, *and such that*

$$a_j = \int_a^b f\phi_j\, dx \quad (j = 1, 2, \ldots).$$

For every integer ν, the conditions of the preceding theorem are satisfied. Hence there is a function f_ν with $|f_\nu| \leqslant 1$ such that

$$a_j = \int_a^b f_\nu \phi_j\, dx \quad (j = 1, \ldots, \nu).$$

Let $F_\nu(x)$ be defined by (2). We can find a sequence ν_k such that (3) and (4) are satisfied. By the theorem of Lebesgue,

$$\lim_{k\to\infty} \int_a^b f_{\nu_k}\phi_j\, dx = \int_a^b f\phi_j\, dx$$

for each j. But $$a_j = \int_a^b f_{\nu_k}\phi_j\, dx$$

for $\nu_k \geqslant j$. Hence the result.

3. The condition (1) has been shown to be sufficient in order that the conclusion of Theorem I may hold; and it is obviously a necessary condition. A similar remark is true of Theorem II. As an example of Theorem II, we take (0, 1) for the fundamental interval, and

$$\phi_n(t) = \operatorname{sgn}(\sin 2^n \pi t).$$

Consider the expression

$$\int_0^1 \left| \sum_1^n \mu_j \phi_j(t) \right| dt. \tag{7}$$

Let $(\epsilon_1, \ldots, \epsilon_n)$ be a complex of numbers each of which satisfies $\epsilon_j{}^2 = 1$. It is clear that we can divide (0, 1) into 2^n equal sub-intervals δ_ν, such that for each ν there is a complex $(\epsilon_1^{(\nu)}, \ldots, \epsilon_n^{(\nu)})$ with the property

$$\phi_1(t) = \epsilon_1^{(\nu)}, \quad \ldots, \quad \phi_n(t) = \epsilon_n^{(\nu)} \quad (t \subset \delta_\nu). \tag{8}$$

Conversely, to each complex $(\epsilon_1, \ldots, \epsilon_n)$ there corresponds one of the δ_ν for which a relation analogous to (8) is satisfied. Thus the value of (7) is

$$\frac{1}{2^n} \sum_n \left| \sum_1^n \mu_j \epsilon_j^{(\nu)} \right|, \tag{9}$$

where the symbol $\sum_n$ denotes a summation over the 2^n complexes

$(\epsilon_1^{(\nu)}, \ldots, \epsilon_n^{(\nu)})$. The expression (9) makes it obvious that (7) is unaltered if we replace μ_j by $\epsilon_j \mu_j$. We now need the following lemma.

LEMMA 1. *If* η_j $(j = 1, \ldots, n)$ *are* n *numbers, then*

$$\frac{1}{2^n} \sum_n \left| \sum_1^n \eta_j \epsilon_j^{(\nu)} \right| > \frac{1}{\sqrt{2}} \left(\sum_1^n \eta_j^2 \right)^{\frac{1}{2}}. \tag{10}$$

We have

$$\int_0^1 \left| \sum_1^n \mu_j \phi_j(t) \right| dt \geqslant \left[\int_0^1 |\Sigma|^2 dt \right]^{\frac{3}{2}} \Big/ \left[\int_0^1 |\Sigma|^4 dt \right]^{\frac{1}{2}}.$$

We may suppose that $\Sigma \mu_j^2 = 1$. Then

$$\int_0^1 |\Sigma|^2 dt = 1. \tag{11}$$

Further,

$$\begin{aligned} \int_0^1 |\Sigma|^4 dt &= \Sigma \mu_j^4 + 6 \sum_{j \neq k} \mu_j^2 \mu_k^2 \\ &< 3(\Sigma \mu_j^2)^2 \\ &< 3. \end{aligned} \tag{12}$$

By (11) and (12),

$$\int_0^1 \left| \sum_1^n \right| dt > \frac{1}{\sqrt{3}}.$$

If now a_n is a sequence such that $\Sigma a_n^2 = 1$, we have, for every n,

$$\left| \sum_1^n \mu_j a_j \right| \leqslant \left(\sum_1^n \mu_j^2 \right)^{\frac{1}{2}} \leqslant \sqrt{3} \int_0^1 \left| \sum_1^n \mu_j \phi_j \right| dt.$$

By Theorem II, there is an $|f| \leqslant \sqrt{3}$ such that

$$a_j = \int_0^1 f \phi_j dt \quad (j = 1, 2, \ldots).$$

The number $\sqrt{3}$ cannot be replaced by a number less than $\sqrt{2}$. For if $a_1 = a_2 = 1/\sqrt{2}$, $a_3 = a_4 = \ldots = 0$, then, taking $\mu_1 = \mu_2 = 1$, $\mu_3 = \mu_4 = \ldots = 0$, we have

$$a_1 + a_2 = \sqrt{2} \int_0^1 |\phi_1 + \phi_2| dt.$$

4. THEOREM III. *Let* ϕ_j $(j = 1, \ldots, n)$ *be a set of functions which are absolutely integrable in the finite or infinite interval* (a, b). *Let* a_j *be a set*

of non-negative constants. Then, if

$$\sum_1^n \mu_j a_j \leqslant \int_a^b \left| \sum_1^n \mu_j \phi_j \right| dx, \tag{14}$$

either for (i) *every set of* $\mu_j \geqslant 0$,

or for (ii) *every set of* $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0$,

or for (iii) *every set of*

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0, \quad \mu_1 - \mu_2 \geqslant \mu_2 - \mu_3 \geqslant \ldots \geqslant \mu_{n-1} - \mu_n \geqslant 0,$$

then there is an f *such that* $|f| \leqslant 1$ *and*

in case (i) $$a_j \leqslant \int_a^b f\phi_j\, dx \qquad (j = 1, \ldots, n),$$

in case (ii) $$\sum_1^k a_j \leqslant \sum_1^k \int_a^b f\phi_j\, dx \qquad (k = 1, \ldots, n),$$

in case (iii) $$\sum_1^k (k+1-j)\, a_j \leqslant \sum_1^k (k+1-j) \int_a^b f\phi_j\, dx \quad (k = 1, \ldots, n).$$

Let E be the set of points $\xi \equiv (\xi_1, \ldots, \xi_n)$ whose coordinates are such that there is an f with $|f| \leqslant 1$ for which

$$\xi_j \leqslant \int_a^b f_\xi \phi_j\, dx \quad (j = 1, \ldots, n).$$

Then, as in the proof of Theorem I, E is closed and convex. Suppose that (i) holds. Then (14) means that for every set of non-negative μ_j there is a non-negative $\theta < 1$ such that

$$\Sigma \mu_j a_j = \theta \int_a^b \operatorname{sgn}(\Sigma \mu_j \phi_j)\, \Sigma \mu_j \phi_j\, dx$$

$$= \Sigma \mu_j c_j$$

with $$c_j = \int_a^b \theta \operatorname{sgn}(\Sigma \mu_j \phi_j)\, \phi_j\, dx.$$

Accordingly, $C \equiv (c_1, \ldots, c_n)$ belongs to E. Thus every plane of the form

$$\Sigma \mu_j (x_j - a_j) = 0 \quad (\mu_j \geqslant 0)$$

through $A \equiv (a_1, \ldots, a_n)$ has a point of E on it. If now A does not belong to E, then there is a point $P \equiv (p_1, \ldots, p_n)$ whose distance from A is positive and an absolute minimum. We must have $a_j \geqslant p_j$ for $j = 1, \ldots, n$. For if not, then, with a suitable notation of coordinates, we should have

$$a_1 \geqslant p_1, \quad \ldots, \quad a_k \geqslant p_k, \quad a_{k+1} < p_{k+1}, \quad \ldots, \quad a_n < p_n \quad (k < n). \tag{15}$$

Consider the point $S \equiv (s_1, \ldots, s_n)$, where $s_j = p_j$ for $j = 1, \ldots, k$, and $s_j = a_j$ for $j = k+1, \ldots, n$. Then $s_j \leqslant p_j$ for all j; and, since P belongs to E, it follows from the definition of E that S belongs to E. The distance of S from A is given by

$$\begin{aligned} d^2(S, A) &= \sum_1^k (s_j - a_j)^2 + \sum_{k+1}^n (s_j - a_j)^2 \\ &= \sum_1^k (p_j - a_j)^2 \\ &< \sum_1^n (p_j - a_j)^2 = d^2(P, A) \end{aligned}$$

by (15). This contradicts the definition of P, and proves our assertion. We now consider the plane

$$\sum_1^n (a_j - p_j)(x_j - a_j) = 0 \quad (a_j - p_j \geqslant 0).$$

Then, by what we have proved, this plane has a point $Q \equiv (q_1, \ldots, q_n)$ of E on it. We can now use the argument of Theorem I to show that A belongs to E.

Suppose now that (ii) holds. Let

$$\begin{gathered} \alpha_1 = a_1, \quad \alpha_2 = a_1 + a_2, \quad \ldots, \quad \alpha_n = a_1 + a_2 + \ldots + a_n, \\ \psi_1 = \phi_1, \quad \psi_2 = \phi_1 + \phi_2, \quad \ldots, \quad \psi_n = \phi_1 + \ldots + \phi_n, \\ \lambda_1 = \mu_1 - \mu_2, \quad \lambda_2 = \mu_2 - \mu_3, \quad \ldots, \quad \lambda_{n-1} = \mu_{n-1} - \mu_n, \quad \lambda_n = \mu_n. \end{gathered}$$

Then condition (14) is now the same as: for all $\lambda_j \geqslant 0$,

$$\sum_1^n a_j \lambda_j \leqslant \int_a^b |\Sigma \lambda_j \psi_j| \, dx. \tag{16}$$

The result therefore follows from (i). Suppose, finally, that (iii) holds. With the preceding notation, condition (14) is now the same as: the relation (16) holds for all $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$. Hence the result follows from (ii).

The corresponding theorem for a sequence of functions ϕ_j and a sequence of non-negative a_j may be deduced by the same argument as Theorem II is deduced from Theorem I.

5. THEOREM IV. *Let $\phi(t)$ be a function of period 1 which satisfies the conditions*

$$\int_0^1 \phi \, dt = 0, \quad \int_0^1 \phi^2 \, dt = 1, \quad \int_0^1 \phi^4 \, dt = \kappa^2.$$

Then there exist sequences n_ν of integers with the following property. Given any sequence a_ν such that $\Sigma a_\nu^2 = 1$, there is an f such that $|f| \leqslant 4\kappa$ and

$$a_\nu = \int_0^1 f\phi(n_\nu t)\,dt \quad (\nu = 1, 2, \ldots).$$

LEMMA 2. *If $\psi(t)$ is of $L(4/3)$ in $(0, 1)$, then*

$$\lim_{n\to\infty} \int_0^1 \psi(t)\,\phi(nt)\,dt = 0.$$

We observe that, for all n,

$$\int_0^1 \phi^4(nt)\,dt = \kappa^2,$$

and that, if $0 \leqslant \alpha < \beta \leqslant 1$, then

$$\int_\alpha^\beta \phi(nt)\,dt = \frac{1}{n}\int_{n\alpha}^{n\beta} \phi(t)\,dt$$

tends to zero as $n \to \infty$. The lemma is thus a consequence of the well-known theorem of Lebesgue*.

LEMMA 3. *If $\psi(t)$ is of $L(4)$ in $(0, 1)$, and*

$$\int_0^1 \psi(t)\,dt = 0,$$

then

$$\lim_{n\to\infty} \int_0^1 \psi(t)\,\phi^3(nt)\,dt = 0.$$

If $0 \leqslant \alpha < \beta \leqslant 1$, then, writing

$$\int_0^1 \phi^3(t)\,dt = C,$$

we have

$$\int_\alpha^\beta \phi^3(nt)\,dt = \frac{1}{n}\int_{n\alpha}^{n\beta} \phi^3(t)\,dt \to (\beta - \alpha)\,C$$

as $n \to \infty$, so that

$$\int_\alpha^\beta \{\phi^3(nt) - C\}\,dt \to 0.$$

Further,

$$\int_0^1 |\phi^3(nt) - C|^{\frac{4}{3}}\,dt \leqslant \left[\left(\int_0^1 \phi^4\,dt\right)^{\frac{3}{4}} + C\right]^{\frac{4}{3}}$$

* *Cf.* Hobson, *Functions of a real variable*, 2 (1926), 430.

for all n. Hence, by the theorem of Lebesgue,

$$\lim_{n\to\infty}\int_0^1 \psi(t)\{\phi^3(nt)-C\}\,dt = 0,$$

which proves the lemma.

Let $0<\theta<1$ satisfy

$$\frac{3\kappa+\theta}{(1-\theta)^{\frac{3}{4}}} \leqslant 4\kappa.$$

Let α_ν be a sequence of positive numbers such that $\Sigma\alpha_\nu<\theta$. The sequences (n_ν) of the theorem are defined inductively. The number n_1 is arbitrarily chosen. When $n_2, \ldots, n_\nu$ have been determined, we consider the expressions

$$I_1(\nu, n) = 2\sum_1^\nu \left|\int_0^1 \phi(n_i t)\,\phi(nt)\,dt\right|, \quad I_2(\nu, n) = 4\sum_1^\nu \left|\int_0^1 \phi(n_i t)\,\phi^3(nt)\,dt\right|,$$

$$I_3(\nu, n) = 4\Big[\sum_{i=1}^\nu \left|\int_0^1 \phi^3(n_i t)\,\phi(nt)\,dt\right| + 3\sum_{i,j=1}^\nu \left|\int_0^1 \phi^2(n_i t)\,\phi(n_j t)\,\phi(nt)\,dt\right|$$

$$+ 9\sum_{i,j,k=1}^\nu \left|\int_0^1 \phi(n_i t)\,\phi(n_j t)\,\phi(n_k t)\,\phi(nt)\,dt\right|\Big],$$

where i, j, k denote distinct integers. By Lemma 2, I_1 and I_3 tend to zero as $n\to\infty$. By Lemma 3, so does I_2. We choose $n_{\nu+1}$ so that

$$I_1(\nu, n_{\nu+1}) < \alpha_{\nu+1}, \quad I_2(\nu, n_{\nu+1}) + I_3(\nu, n_{\nu+1}) < \alpha_{\nu+1}^2.$$

We now have

LEMMA 4. *For a sequence n_ν defined as above, we have, for any ν numbers γ_i, the relation*

$$\int_0^1 \left|\sum_{i=1}^\nu \gamma_i\phi(n_i t)\right| dt \geqslant \frac{1}{4\kappa}\left(\sum_1^\nu \gamma_i^2\right)^{\frac{1}{2}}.$$

We may suppose that

$$\sum_1^\nu \gamma_i^2 = 1.$$

The integral is not less than J_ν^3/I_ν^2, where

$$J_\nu^2 = \int_0^1 U_\nu^2\,dt, \quad I_\nu^4 = \int_0^1 U_\nu^4\,dt, \quad U_\nu = \sum_{i=1}^\nu \gamma_i\phi(n_i t).$$

We have

$$J_\nu^2 = \int_0^1 \{U_{\nu-1}^2 + 2U_{\nu-1}\gamma_\nu\phi(n_\nu t) + \gamma_\nu^2\phi^2(n_\nu t)\}\,dt.$$

Since $\gamma_i^2 \leqslant 1$, we have

$$\left| \int_0^1 2U_{\nu-1}\gamma_\nu \phi(n_\nu t)\,dt \right| \leqslant 2 \sum_1^{\nu-1} \left| \int_0^1 \phi(n_i t)\,\phi(n_\nu t)\,dt \right| < a_\nu'.$$

Thus $J_\nu^2 - J_{\nu-1}^2 \geqslant \gamma_\nu^2 - a_\nu$, and so, by addition, $J_\nu^2 \geqslant 1-\theta$. Further,

$$\begin{aligned} I_\nu^4 &= \int_0^1 \{U_{\nu-1}^4 + 6U_{\nu-1}^2\gamma_\nu^2\phi^2(n_\nu t) + \gamma_\nu^4\phi^4(n_\nu t)\}\,dt \\ &\qquad + 4\int_0^1 U_{\nu-1}^3\gamma_\nu\phi(n_\nu t)\,dt + 4\int_0^1 U_{\nu-1}\gamma_\nu^3\phi^3(n_\nu t)\,dt \\ &\leqslant I_{\nu-1}^4 + 6\kappa\gamma_\nu^2 I_{\nu-1}^2 + \gamma_\nu^4\kappa^2 + a_\nu^2 \\ &\leqslant (I_{\nu-1}^2 + 3\kappa\gamma_\nu^2)^2 + a_\nu^2. \end{aligned}$$

Thus $I_\nu^2 < I_{\nu-1}^2 + 3\kappa\gamma_\nu^2 + a_\nu$, and so, by addition, $I_\nu^2 \leqslant 3\kappa + \theta$. This proves the lemma.

We can now prove the main theorem. Let N be an arbitrarily assigned integer and n_ν a sequence constructed as above. Then

$$\left| \sum_1^N a_i\gamma_i \right| \Big/ \int_0^1 \left| \sum_1^N \gamma_i\phi(n_i t) \right| dt \leqslant 4\kappa$$

by Lemma 4; and the result follows from Theorem II.

It may be observed that, if ψ is a function which is absolutely integrable in $(-\infty, \infty)$ and which satisfies

$$\int_{-\infty}^{\infty} \psi\,dt = 0, \quad \int_{-\infty}^{\infty} \psi^2\,dt = 1, \quad \int_{\infty}^{\infty} \psi^4\,dt = \kappa^2,$$

then trivial changes in the above proof will establish the corresponding theorem in this case. Further, n_ν need not be an integer.

THEOREM V. *Let $\phi_n(x)$ be a sequence of functions which are normal and orthogonal in $(0, 1)$. Let*

$$\int_0^1 \phi_n^4\,dx \leqslant \kappa^2 \quad (n = 1, 2, \ldots).$$

Then there exist sequences (n_ν) with the following property. Given any sequence a_ν with $\Sigma a_\nu^2 = 1$, there is an f such that $|f| \leqslant 6\kappa$ and

$$a_\nu = \int_0^1 f\phi_{n_\nu}\,dx \quad (\nu = 1, 2, \ldots).$$

LEMMA 5. *If $\psi(x)$ is of $L(\frac{4}{3})$ in $(0, 1)$, then*

$$\lim_{n\to\infty} \int_0^1 \psi\,\phi_n\,dx = 0.$$

For, if $0 \leqslant \alpha < \beta \leqslant 1$, then

$$\lim_{n\to\infty} \int_\alpha^\beta \phi_n \, dx = 0$$

is a consequence of Bessel's inequality.

Let α_ν be a sequence of positive numbers such that $\Sigma \alpha_\nu < 1$. The sequences n_ν are defined inductively. The number n_1 having been arbitrarily chosen, and $n_2, \ldots, n_\nu$ having been determined, we consider the expression

$$I(\nu, n) = 4\Big[\sum_{i=1}^{\nu} \Big| \int_0^1 \phi_{n_i}^3 \phi_n \, dx \Big| + 3 \sum_{i,j=1}^{\nu} \Big| \int_0^1 \phi_{n_i}^2 \phi_{n_j} \phi_n \, dx \Big| + 9 \sum_{i,j,k=1}^{\nu} \Big| \int_0^1 \phi_{n_i} \phi_{n_j} \phi_{n_k} \phi_n \, dx \Big| \Big],$$

where i, j, k are distinct integers. By Lemma 5, I tends to zero as $n \to \infty$. We choose $n_{\nu+1}$ so that $I(\nu, n_{\nu+1}) < \alpha_{\nu+1}^2$.

LEMMA 6. *For the sequences (n_ν) defined above, we have*

$$\int_0^1 \Big| \sum_{i=1}^{\nu} \gamma_i \phi_{n_i} \Big| \, dx \geqslant \frac{1}{6\kappa} \Big(\sum_1^\nu \gamma_i^2 \Big)^{\frac{3}{2}}.$$

We may suppose that $\Sigma \gamma_i^2 = 1$. Then the integral is not less than J_ν^3 / I_ν^2, where

$$J_\nu^2 = \int_0^1 U_\nu^2 \, dx, \quad I_\nu^4 = \int_0^1 U_\nu^4 \, dx, \quad U_\nu = \sum_1^\nu \gamma_i \phi_{n_i}.$$

In virtue of the orthogonality, $J_\nu = 1$. Further,

$$I_\nu^4 = \int_0^1 U_{\nu-1}^4 \, dx + 6 \int_0^1 U_{\nu-1}^2 \gamma_\nu^2 \phi_{n_\nu}^2 \, dx + \int_0^1 \gamma_\nu^4 \phi_{n_\nu}^4 \, dx + 4 \int_0^1 U_{\nu-1}^3 \gamma_n \phi_{n_\nu} \, dx + 4 \int_0^1 U_{\nu-1} \gamma_\nu^3 \phi_{n_\nu}^3 \, dx.$$

If now $I_{\nu-1}^2 < \kappa \gamma_\nu^2$, we find, on applying Hölder's inequality to the last two terms, that

$$I_\nu^4 \leqslant \gamma_\nu^4 \kappa^2 + 6\gamma_\nu^4 \kappa^2 + \gamma_\nu^4 \kappa^2 + 4\gamma_\nu^4 \kappa^2 + 4\gamma_\nu^4 \kappa^2.$$

Thus $I_\nu^2 \leqslant 4\kappa\gamma_\nu^2$. If, however, $I_{\nu-1}^2 \geqslant \kappa \gamma_\nu^2$, we have

$$I_\nu^4 \leqslant I_{\nu-1}^4 + 6\kappa\gamma_\nu^2 I_{\nu-1}^2 + \kappa^2 \gamma_\nu^4 + \alpha_\nu^2 + A_\nu,$$

where

$$A_\nu = \Big| 4 \int_0^1 U_{\nu-1} \gamma_\nu^3 \phi_{n_\nu}^3 \, dx \Big|$$

$$\leqslant 4\gamma_\nu^2 I_{\nu-1} \kappa^{-\frac{1}{2}} \Big(\int_0^1 U_{\nu-1}^4 \Big)^{\frac{1}{4}} \Big(\int \phi_{n_\nu}^2 \Big)^{\frac{3}{4}}$$

$$\leqslant 4\gamma_\nu^2 \kappa I_{\nu-1}^2.$$

Thus
$$I_\nu^4 \leqslant I_{\nu-1}^4 + 10\kappa\gamma_\nu^2 I_{\nu-1}^2 + \kappa^2\gamma_\nu^4 + a_\nu^2$$
$$\leqslant (I_{\nu-1}^2 + 5\kappa\gamma_\nu^2)^2 + a_\nu^2.$$

Hence, in either case,
$$I_\nu^2 \leqslant I_{\nu-1}^2 + 5\kappa\gamma_\nu^2 + a_\nu,$$
and so, by addition,
$$I_\nu^2 \leqslant 5\kappa + 1 \leqslant 6\kappa.$$

To complete the proof of the theorem, we observe that
$$\left| \sum_1^N a_i\gamma_i \right| \Big/ \int_0^1 \left| \sum_1^N \gamma_i \phi_{n_i} \right| dx \leqslant 6\kappa,$$
so that the result follows from Theorem II.

6. THEOREM VI. *Let γ_n, a_n be two sequences. Let*
$$\eta_j = -\operatorname{sgn} \int_0^{2\pi} \left[\sum_1^{j-1} \epsilon_k \gamma_k \cos(kx+a_k) \right]^3 \gamma_j \cos(jx+a_j)\, dx \quad (j = 2, 3, \ldots),$$
where the ϵ_k are defined inductively by
$$\epsilon_1 = 1, \quad \epsilon_j = \eta_j \text{ if } \eta_j^2 = 1, \quad \epsilon_j = \pm 1 \text{ if } \eta_j = 0.$$

If
$$\sum_1^\infty \epsilon_j \gamma_j \sin(jx+a_j)/j$$
is a function of bounded variation, then $\Sigma\gamma_j^2$ converges.

We require a particular case of the following lemma.

LEMMA 7. *Let ϕ_j $(j = 1, \ldots, n)$ be a set of absolutely integrable functions which are defined in the finite or infinite interval (a, b), and which are normal and orthogonal in that interval. Let κ be a positive number such that*
$$\int_a^b \phi_j^4\, dx \leqslant \kappa^2 \quad (j = 1, \ldots, n).$$
Then, given n numbers γ_j, there is a set of ϵ_j satisfying $\epsilon_j^2 = 1$ such that
$$\int_a^b \left| \sum_1^n \epsilon_j \gamma_j \phi_j \right|^\alpha dx \geqslant \frac{1}{(3\kappa)^{2-\alpha}} \left(\sum_1^n \gamma_j^2 \right)^{\frac{1}{2}\alpha} \quad (0 < \alpha \leqslant 2).$$
With the notation
$$\eta_j = -\operatorname{sgn} \int_a^b \left[\left(\sum_1^{j-1} \epsilon_k \gamma_k \phi_k \right)^3 \gamma_j \phi_j + \left(\sum_1^{j-1} \epsilon_k \gamma_k \phi_k \right) \gamma_j^3 \phi_j^3 \right] dx,$$

the set ϵ_j may be defined inductively by

$$\epsilon_1 = 1, \quad \epsilon_j = \eta_j \text{ if } \eta_j^2 = 1, \quad \epsilon_j = \pm 1 \text{ if } \eta_j = 0.$$

Write $$U_n = \sum_1^n \epsilon_j \gamma_j \phi_j.$$

Then $$\int_a^b |U_n|^\alpha dx \geqslant J_n^{4-\alpha}/I_n^{4-2\alpha},$$

where $$J_n^2 = \int_a^b U_n^2 dx, \qquad I_n^4 = \int_a^b U_n^4 dx.$$

Now $J_n^2 = \sum_1^n \gamma_j^2$, and

$$I_n^4 = \int_a^b \{U_{n-1}^4 + 6U_{n-1}^2 \gamma_n^2 \phi_n^2 + \gamma_n^4 \phi_n^4\} dx + 4\epsilon_n \int_a^b \{U_{n-1}^3 \gamma_n \phi_n + U_{n-1} \gamma_n^3 \phi_n^3\} dx$$
$$\leqslant I_{n-1}^4 + 6\kappa \gamma_n^2 I_{n-1}^2 + \kappa^2 \gamma_n^4$$
$$\leqslant (I_{n-1}^2 + 3\kappa \gamma_n^2)^2.$$

The reduction formula $I_n^2 - I_{n-1}^2 \leqslant 3\kappa \gamma_n^2$ gives $I_n^2 \leqslant 3\kappa \Sigma \gamma_j^2$ and proves the lemma.

Turning to the proof of Theorem VI, we see that, since

$$\int_0^{2\pi} \left[\sum_1^{j-1} \epsilon_k \gamma_k \cos(kx + \alpha_k)\right] \gamma_j^3 \cos^3(jx + \alpha_j) dx = 0,$$

then, by Lemma 7,

$$\int_0^{2\pi} \left| \sum_1^n \epsilon_j \gamma_j \cos(jx + \alpha_j) \right|^\theta dx \geqslant \kappa(\theta) \left(\sum_1^n \gamma_j^2\right)^{\frac{1}{2}\theta} \quad (0 < \theta < 1).$$

The theorem will follow if we can prove that the first member is bounded in n for a fixed θ. This, however, is a consequence of the following theorem.

7. THEOREM VII. *If*

$$A(x) = \sum_1^\infty \gamma_n \sin(nx + \alpha_n)/n$$

is of bounded variation, then, with the notation

$$S_n(x) = \sum_1^n \gamma_j \cos(jx + \alpha_j),$$

we have $$\int_{-\pi}^{\pi} |S_n(x)|^\theta dx < \frac{C}{1-\theta} \int_{-\pi}^{\pi} |dA(x)| \quad (0 < \theta < 1),$$

where C is an absolute constant.

We require a series of lemmas. In what follows, integrals are taken over $(-\pi, \pi)$ unless otherwise indicated.

LEMMA 8*. *If $g(x)$ is defined in $(-\pi, \pi)$, and, for all $R > 0$,*

$$Rm(|g| > R) < \kappa,$$

then

$$\int |g|^\theta dx < 2\kappa\pi/(1-\theta).$$

The symbol $m(|g| > R)$ denotes the measure of the set in $(-\pi, \pi)$ at which the inequality holds. Turning to the proof of the theorem, we have

$$S_n(x) = \frac{1}{\pi}\int \frac{\sin n(x-t)}{\tan \frac{1}{2}(x-t)}\, dA(t) + \frac{1}{2\pi}\int \cos n(x-t)\, dA(t),$$

and so

$$\left| S_n(x) - \frac{1}{2\pi}\int \frac{\sin n(x-t)}{x-t}\, dA(t) \right| < \kappa \int |dA|,$$

where here and in what follows κ denotes an absolute constant which may be different at different occurrences. It is sufficient, by Lemma 8, to prove that, if

$$a(x) = \frac{1}{2\pi}\int \frac{\sin n(x-t)}{x-t}\, dA, \tag{17}$$

then

$$Rm(|a| > R) < \kappa \int |dA| \tag{18}$$

for all $R > 0$.

LEMMA 9. *Let O be an open set, $u_n \equiv (a_n, b_n)$ its constituent intervals, $\Sigma \omega_n$ a convergent series of positive terms. If O' is the open set formed by the intervals $(\alpha_n = a_n - u_n, \beta_n = b_n + u_n)$, and*

$$\phi(t) = \sum_1^\infty \frac{u_n \omega_n}{(t-a_n)(t-b_n)},$$

then the set in CO' at which $|\phi| > R$ has a measure less than $2\Sigma\omega_n/R$.

For in that set, ϕ is the sum of positive terms and

$$\int_{CO'} \phi\, dt = \Sigma \int_{CO'} \frac{u_n \omega_n}{(t-a_n)(t-b_n)}\, dt$$

$$< \Sigma \int_{-\infty}^{a_n} + \int_{\beta_n}^{\infty}$$

$$< \Sigma(2 \log 2)\, \omega_n < 2\Sigma\omega_n.$$

* Kolmogoroff, *Fundamenta Math.*, 7 (1925), 24–29 (27).

LEMMA 10*. *If $G(x)$ is a continuous function of bounded variation, then*

$$g(x) = \int \frac{dG}{x-t}$$

exists as a principal value for almost all x.

LEMMA 11. *With the notation of Lemma 10, we have, for all $R > 0$,*

$$Rm(|g| > R) < \kappa \int |dG|. \tag{19}$$

We may suppose that G is non-decreasing. Let

$$\rho > \frac{G(\pi) - G(-\pi)}{2\pi} > 0.$$

By Denjoy's grading principle†, there is a perfect set P such that, if $O \equiv \Sigma u_n$ is its complement in $(-\pi, \pi)$, then $G(b_n) - G(a_n) = \rho u_n$, and G increases on P not faster than ρx. We have

$$mO < [G(\pi) - G(-\pi)]/\rho.$$

Let $f_1 = G'$ on P, and $f_1 = \rho$ on O. Let $G_1 = \int_{-\pi}^{x} f_1 dt$. Write $G = G_1 + G_2$. Then $|f_1| \leqslant \rho$, and so

$$g_1(x) = \int \frac{dG_1}{x-t} = \int \frac{f_1}{x-t}\, dt.$$

By familiar results in the theory of Fourier series,

$$\int g_1^2 dx \leqslant \kappa \int f_1^2 dx \leqslant \kappa\rho \int f_1 dx.$$

Thus
$$\rho^2 m(|g_1| > \rho) \leqslant \kappa\rho \int f_1 dx$$

and
$$\rho m(|g_1| > \rho) < \kappa \int f_1 dx < \kappa [G(\pi) - G(-\pi)]. \tag{20}$$

Consider now
$$g_2(x) = \int \frac{dG_2}{x-t}.$$

By Lemma 10, it exists for almost all x. Hence, for such x in CO', we have

$$|g_2| < \Sigma \left| \int_{u_n} \frac{dG_2}{x-t} \right|$$

$$< \Sigma \left| \frac{1}{x-a_n} \int_{a_n}^{\xi} dG_2 + \frac{1}{x-b_n} \int_{\xi}^{b_n} dG_2 \right| \quad (a_n \leqslant \xi \leqslant b_n).$$

* Pollard, *Journal London Math. Soc.*, 1 (1926), 37–41.

† Denjoy, *Ann. Sci. de l'Ecole Normale Sup.*, 33 (1916), 127–222 (133–140).

But $$\int_{a_n}^{b_n} dG_2 = 0, \quad \int_{a_n}^{\xi} dG_2 < 2\rho u_n.$$

Hence $$|g_2| < \Sigma \left| \left(\frac{1}{x-a_n} - \frac{1}{x-b_n} \right) 2\rho u_n \right|$$

$$< \Sigma \frac{u_n \cdot 2\rho u_n}{(x-a_n)(x-b_n)}.$$

By Lemma 9, the set in CO' at which $|g_2| > \rho$ has measure less than $4\rho \Sigma u_n/\rho = 4mO$. Hence the set in $(-\pi, \pi)$ at which $|g_2| > \rho$ is of measure less than $7mO$. Hence

$$\rho m(|g_2| > \rho) < 7\rho mO < 7[G(\pi) - G(-\pi)]. \tag{21}$$

By (20) and (21), we have

$$\rho m(|g| > 2\rho) < \kappa \int |dG|,$$

and so $$Rm(|g| > R) < \kappa \int |dG|$$

for $R = 2\rho > \frac{1}{\pi} \int |dG|$. For $R \leqslant \frac{1}{\pi} \int |dG|$, we have

$$Rm(|g| > R) \leqslant 2\pi R \leqslant 2 \int |dG|.$$

LEMMA 12. *If $G(x)$ is a continuous function of bounded variation, and*

$$g(x) = \int \begin{matrix} \sin \\ \cos \end{matrix} nt \, \frac{dG}{x-t},$$

then (19) *holds.*

For, writing $$G_1 = \int_{-\pi}^{t} \begin{matrix} \sin \\ \cos \end{matrix} n\xi \, dG(\xi),$$

we have $$g(x) = \int \frac{dG_1}{x-t}$$

and $$\int |dG_1| \leqslant \int |dG|.$$

A function $G(x)$ of bounded variation will be said to be *merely discontinuous* if its total variation is the sum of its discontinuities. We are concerned only with functions which satisfy $G(\xi-0) \leqslant G(\xi) \leqslant G(\xi+0)$, or the corresponding relation with the inequalities reversed, and then, for a merely discontinuous function $G(x)$, we have

$$\int |dG| = \Sigma |G(\xi+0) - G(\xi-0)|.$$

LEMMA 13*. *If $G(x)$ is a merely discontinuous function of bounded variation, then*

$$g(x) = \int \frac{dG}{x-t}$$

exists for almost all x as a principal value.

LEMMA 14. *With the notation of Lemma 13, we have, for all $R > 0$,*

$$Rm(|g| > R) < \kappa \int |dG|.$$

Let $O \equiv \Sigma(a_n, b_n)$ be an open set containing the discontinuities of G, which we may suppose to be non-diminishing. Then, writing

$$\omega_n = G(b_n) - G(a_n),$$

we have $\Sigma \omega_n = G(\pi) - G(-\pi)$. Let $g_1 = \omega_n/u_n$ in u_n and $g_1 = 0$ in CO. Write

$$G_1(x) = \int_{-\pi}^{x} g_1 \, dt.$$

Then $G = G_1 + G_2$ and

$$g(x) = \int \frac{dG_1}{x-t} + \int \frac{dG_2}{x-t}$$

$$= g_1(x) + g_2(x).$$

The function G_1 is a continuous function of bounded variation. By Lemma 11 we have, for all $\rho > 0$,

$$\rho m(|g_1| > \rho) < \kappa \int |dG_1| \leqslant \kappa \int |dG|. \tag{22}$$

The function G_2 is of bounded variation. It is therefore the sum of a merely discontinuous and a continuous function of bounded variation. By Lemmas 10 and 13, g_2 exists as a principal value for almost all x. For such x in CO' we have

$$|g_2| < \Sigma \left| \int_{u_n} \frac{dG_2}{x-t} \right|.$$

Now $$\int_{u_n} dG_2 = 0, \quad \int_{a_n}^{\xi} dG_2 < 2\omega_n \quad (a_n \leqslant \xi \leqslant b_n).$$

Hence, as in the proof of Lemma 11, for almost all x in CO',

$$|g_2| < \Sigma \frac{u_n \cdot 2\omega_n}{(x-a_n)(x-b_n)}.$$

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* R. C. Young and S. Pollard, *Proc. London Math. Soc* (2), 28 (1929), 293–300.

By Lemma 9, the set in CO' at which $|g_2| > \rho$ is of measure less than $4\Sigma\omega_n/\rho$. Hence, for all $\rho > 0$,

$$\rho m(|g_2| > \rho) < 3\rho mO + 4\int |dG|,$$

and, since mO may be taken arbitrarily small, we get

$$\rho m(|g_2| > \rho) \leqslant 4\int |dG|. \tag{23}$$

The lemma follows from (22) and (23).

LEMMA 15. *If $G(x)$ is a merely discontinuous function of bounded variation and*

$$g(x) = \int \begin{matrix}\sin \\ \cos\end{matrix} nt \frac{dG}{x-t},$$

then, for all $R > 0$,

$$Rm(|g| > R) < \kappa \int |dG|.$$

For, writing

$$G_1 = \int_{-\pi}^{t} \begin{matrix}\sin \\ \cos\end{matrix} n\xi \, dG(\xi),$$

we have

$$g(x) = \int \frac{dG_1}{x-t},$$

and G_1 is a merely discontinuous function of bounded variation for which

$$\int |dG_1| \leqslant \int |dG|.$$

We can now prove (18). We can write $A = A_1 + A_2$, where A_1 is a merely discontinuous and A_2 is a continuous function of bounded variation, and

$$\int |dA| = \int |dA_1| + \int |dA_2|.$$

Then

$$a(x) = \frac{\sin nx}{2\pi}\left[\int \frac{\cos t \, dA_1}{x-t} + \int \frac{\cos nt \, dA_2}{x-t}\right] - \frac{\cos nx}{2\pi}\left[\int \frac{\sin nt \, dA_1}{x-t} + \int \frac{\sin nt \, dA_2}{x-t}\right]$$

$$= \frac{\sin nx}{2\pi}(g_1 + g_2) - \frac{\cos nx}{2\pi}(g_3 + g_4).$$

Now

$$m\left(\left|\frac{\sin nx}{2\pi} g_1\right| > \rho\right) < m(|g_1| > \rho) < \frac{\kappa}{\rho}\int |dA_1|$$

by Lemma 15; and

$$m\left(\left|\frac{\sin nx}{2\pi}\, g_2\right| > \rho\right) < m(|g_2| > \rho) < \frac{\kappa}{\rho}\int |dA_2|$$

by Lemma 12. The other two terms are treated similarly. Hence $m(|a| > 4\rho) < 2\kappa\int |dA|/\rho$, which completes the proof.

8. LEMMA 16. *Let ϕ_j $(j = 1, \ldots, n)$ be a set of functions defined in* (0, 1) *which satisfy*

$$\int_0^1 \phi_j^{2m}\, dx \leqslant \kappa^m \quad (j = 1, \ldots, n),$$

where m is a positive integer. If γ_j are n numbers, there is a set of ϵ_j satisfying $\epsilon_j^2 = 1$, such that

$$\left[\int_0^1 \left|\sum_1^n \epsilon_j \gamma_j \phi_j\right|^{2m} dx\right]^{1/m} \leqslant 2\kappa m(\Sigma \gamma_j^2).$$

With the notation

$$\eta_j = -\operatorname{sgn} \sum_{r=1}^{m} \frac{2m!}{(2r-1)!\,(2m-2r+1)!} \int_0^1 \left(\sum_1^{j-1} \epsilon_k \gamma_k \phi_k\right)^{2m-2r+1} \gamma_j^{2r-1} \phi_j^{2r-1}\, dx \qquad (j = 2, 3, \ldots)$$

the numbers ϵ_j may be defined inductively by

$$\epsilon_1 = 1, \quad \epsilon_j = \eta_j \text{ if } \eta_j^2 = 1, \quad \epsilon_j = \pm 1 \text{ if } \eta_j = 0.$$

We write $$U_n = \sum_1^n \epsilon_j \gamma_j \phi_j.$$

Then

$$\begin{aligned} I_n^{2m} &= \int_0^1 |U_{n-1} + \epsilon_n \gamma_n \phi_n|^{2m}\, dx \\ &= I_{n-1}^{2m} + \sum_{r=1}^{m} \frac{2m!}{2r!\,(2m-2r)!} \int_0^1 U_{n-1}^{2m-2r} \gamma_n^{2r} \phi_n^{2r}\, dx \\ &\qquad + \epsilon_n \sum_{r=1}^{m} \frac{2m!}{(2r-1)!\,(2m-2r+1)!} \int_0^1 U_{n-1}^{2m-2r+1} \gamma_n^{2r-1} \phi_n^{2r-1}\, dx \qquad (24) \\ &\leqslant I_{n-1}^{2m} + \sum_{r=1}^{m} \frac{2m!}{2r!\,(2m-2r)!} \int_0^1 U_{n-1}^{2m-2r} \gamma_n^{2r} \phi_n^{2r}\, dx. \end{aligned}$$

We have $$\left|\int_0^1 U_{n-1}^{2m-2r} \phi_n^{2r}\, dx\right| \leqslant I_{n-1}^{2m-2r} \kappa^r,$$

and a superficial examination shows that

$$\frac{2m!}{2r!\,(2m-2r)!}\Big/\frac{m!}{r!\,(m-r)!}<(2m)^r.$$

Hence
$$I_n^{2m}\leqslant I_{n-1}^{2m}+\sum_{r=1}^{m}\frac{m!}{r!\,(m-r)!}\,I_{n-1}^{2m-2r}(2\kappa m\gamma_n{}^2)^r$$
$$\leqslant(I_{n-1}^2+2\kappa_m\gamma_n{}^2)^m.$$

The reduction formula $I_n{}^2\leqslant I_{n-1}^2+2\kappa m\gamma_n{}^2$ establishes the lemma.

The following lemma gives a less precise inequality, but a simpler choice for the ϵ_j.

LEMMA 17. *Let the ϕ_j be defined as in Lemma* 16. *If γ_j are n numbers, there is a set of ϵ_j satisfying $\epsilon_j{}^2=1$ such that*

$$\left[\int_0^1\left|\sum_1^n\epsilon_j\gamma_j\phi_j\right|^{2m}dx\right]^{1/m}\leqslant 4\kappa m^2(\Sigma\gamma_j{}^2).$$

With the notation

$$\eta_j=-\operatorname{sgn}\int_0^1\left(\sum_1^{j-1}\epsilon_k\gamma_k\phi_k\right)^{2m-1}\gamma_j\phi_j\,dx\quad(j=2,\,3,\,\ldots),$$

the numbers ϵ_j may be defined inductively as before.

We may suppose that $\Sigma\gamma_j{}^2=1$. Then, if $I_{n-1}^2<4\kappa$, the equation (24) gives

$$I_n^{2m}\leqslant\kappa^m\Big(2^{2m}+\sum_{r=1}^{m}\frac{2m!}{2r!\,(2m-2r)!}\,2^{2m-2r}$$
$$+\sum_{r=1}^{m}\frac{2m!}{(2r-1)!\,(2m-2r+1)!}\,2^{2m-2r+1}\Big)$$
$$\leqslant\kappa^m 2^{4m},$$

so that $I_n{}^2\leqslant\kappa 2^4\leqslant 4\kappa m^2$ for $m>1$, while for $m=1$ the lemma is included in the preceding one. If, however, $I_{n-1}^2\geqslant 4\kappa$, we examine the last term in (24). For $r=2,\,\ldots,\,m$, we have

$$\frac{2m!}{(2r-1)!\,(2m-2r+1)!}<2m\,\frac{2m!}{(2r-2)!\,(2m-2r+2)!}$$

and

$$\left|\int_0^1 U_{n-1}^{2m-2r+1}\gamma_n^{2r-1}\phi_n^{2r-1}dx\right|\leqslant I_{n-1}^{2m-2r+2}\gamma_n^{2r-2}\kappa^{\frac{1}{2}(r-1)}/I_{n-1}$$
$$\leqslant\tfrac{1}{2}I_{n-1}^{2m-2r+2}\gamma_n^{2r-2}\kappa^{r-1}.$$

Thus (24) gives

$$I_n^{2m} \leqslant I_{n-1}^{2m} + \sum_{r=1}^{m} \frac{2m!}{2r!\,(2m-2r)!}\,(m+1)\,I_{n-1}^{2m-2r}(\kappa\gamma_n{}^2)^r + \epsilon_n . 2m \int_0^1 U_{n-1}^{2m-1}\gamma_n\phi_n\,dx$$

$$\leqslant I_{n-1}^{2m} + \sum_{r=1}^{m} \frac{m!}{r!\,(m-r)!}\,(2m)^{r+1}\,I_{n-1}^{2m-2r}(\kappa\gamma_n{}^2)^r$$

$$\leqslant (I_{n-1}^2 + 4\kappa m^2\gamma_n{}^2)^m.$$

An immediate consequence is

THEOREM VIII. *Let ϕ_n be a complete set of normal orthogonal functions in $(0, 1)$ which satisfy*

$$\int_0^1 \phi_n^{2m}\,dx \leqslant \kappa^m.$$

Let γ_n be a sequence such that $\Sigma\gamma_n{}^2 = 1$. If

$$\eta_j = -\operatorname{sgn}\int_0^1 \Big(\sum_1^{j-1} \epsilon_k\gamma_k\phi_k\Big)^{2m-1}\gamma_j\phi_j\,dx \quad (j = 2, 3, \ldots),$$

where $\epsilon_1 = 1, \quad \epsilon_j = \eta_j$ *if* $\eta_j{}^2 = 1, \quad \epsilon_j = \pm 1$ *if* $\eta_j = 0,$

then there is an $f(x)$ satisfying

$$\Big[\int_0^1 |f|^{2m}\,dx\Big]^{1/m} \leqslant 4m^2\kappa$$

such that $$f(x) \sim \sum_1^\infty \epsilon_j\gamma_j\phi_j.$$

For by the preceding lemma,

$$\Big[\int_0^1 \Big|\sum_1^n \epsilon_j\gamma_j\phi_j\Big|^{2m} dx\Big]^{\frac{1}{2}m} \leqslant 2m\kappa^{\frac{1}{2}}$$

for all n, and the result follows from a theorem of F. Riesz*. A similar theorem can be deduced from Lemma 16.

9. The following lemma was proved in a special case by Littlewood†.

LEMMA 18. *Let ϕ_j be defined as in Lemma 16. Then, if γ_j are n numbers,*

$$\frac{1}{2^n}\,\Sigma\int_0^1 \Big|\sum_1^n \epsilon_j\gamma_j\phi_j\Big|^{2m} dx \leqslant (2\kappa m)^m\,(\Sigma\gamma_j{}^2)^m,$$

where $\epsilon_j{}^2 = 1$, and the sum is taken over the 2^n complexes $(\epsilon_1, \ldots, \epsilon_n)$.

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* *Cf.* Hobson, *Functions of a real variable*, 2 (1926), 608.

† Littlewood, *Journal London Math. Soc.*, 5 (1930), 179–182.

To each set of $\epsilon_1, \ldots, \epsilon_{n-1}$ correspond the two complexes $(\epsilon_1, \ldots, \epsilon_{n-1}, 1)$, $(\epsilon_1, \ldots, \epsilon_{n-1}, -1)$. Hence, by (24),

$$\sum_n I_n^{2m} = 2 \sum_{n-1} \left[I_n^{2m} + \sum_{r=1}^{m} \frac{2m!}{2r!\,(2m-2r)!}\, I_{n-1}^{2m-2r} \kappa^r \gamma_n^{2r} \right],$$

where $\sum_n$ denotes a summation over the 2^n complexes, and $\sum_{n-1}$ denotes a summation over the 2^{n-1} complexes of $n-1$ elements. Then

$$\sum_{n-1} I_n^{2m-2r} \leqslant 2^{(n-1)r/m} \left(\sum_{n-1} I_{n-1}^{2m} \right)^{(2m-2r)/2m},$$

so that, with the notation

$$G_n = \left(\sum_n I_n^{2m} \right)^{\frac{1}{2}m},$$

we have

$$G_n^{2m} \leqslant 2 \left[G_{n-1}^{2m} + \sum_{r=1}^{m} \frac{m!}{r!\,(m-r)!} (2m)^r G_{n-1}^{2m-2r} (\kappa \gamma_n{}^2 2^{(n-1)/m})^r \right]$$

$$\leqslant 2 \left[G_{n-1}^2 + 2\kappa m \gamma_n{}^2 2^{(n-1)/m} \right]^m.$$

We can complete the proof by induction; *i.e.* taking

$$G_{n-1}^2 \leqslant 2^{(n-1)/m} 2\kappa m \left(\sum_1^{n-1} \gamma_j{}^2 \right),$$

we obtain the desired result.

Another interesting mean value is given by the following lemma.

LEMMA 19. *Let ϕ_j $(j = 1, \ldots, n)$ be normal and orthogonal in the finite or infinite interval (a, b). Let*

$$\int_a^b \phi_j{}^4 dx \leqslant \kappa^2 \quad (j = 1, \ldots, n).$$

Then, given n numbers γ_j, we have

$$\frac{1}{2^n} \sum \int_a^b \left| \sum_1^n \epsilon_j \gamma_j \phi_j \right|^\alpha dx \geqslant \frac{1}{(3\kappa)^{2-\alpha}} (\Sigma \gamma_j{}^2)^{\frac{1}{2}\alpha} \quad (0 < \alpha \leqslant 2).$$

LEMMA. *If $a_1, \ldots, a_k$ are positive numbers, then*

$$\sum_{i=1}^{k} a_i^{\frac{1}{2}\alpha - 1} \geqslant k^{2 - \frac{1}{2}\alpha} (\Sigma a_i)^{\frac{1}{2}\alpha - 1}.$$

Consider the function

$$\Sigma a_i^{\frac{1}{2}\alpha - 1} / (\Sigma a_i)^{\frac{1}{2}\alpha - 1}$$

on the closed set $S \equiv [(\Sigma a_i = 1),\ a_i \geqslant 0]$. The function attains its minimum M on S, and, on grounds of homogeneity, M is an absolute minimum of the function in the space $a_i \geqslant 0$. At the point $(a_1^0, \ldots, a_k^0)$ at which the minimum is attained, none of the a_i vanishes, the function possesses partial derivatives with respect to the a_i, and we have the relations

$$a_i^{\frac{1}{2}\alpha-2} = M(\Sigma a_i)^{\frac{1}{2}\alpha-2}.$$

Hence $a_1^0 = \ldots = a_k^0$, and so $M = k^{2-\frac{1}{2}\alpha}$.

To prove Lemma 19, we use the notation of Lemma 7, and observe that, in virtue of the orthogonality, J_n has the same value for each of the 2^n complexes. Hence

$$\frac{\Sigma_n \int |U_n|^\alpha dx}{J_n^{4-\alpha}} \geqslant \Sigma_n \frac{1}{I_n^{4-2\alpha}} \geqslant \frac{2^{n(2-\frac{1}{2}\alpha)}}{\left(\Sigma_n I_n^4\right)^{1-\frac{1}{2}\alpha}}$$

It remains to prove that

$$\frac{1}{2^n} \Sigma_n I_n^4 \leqslant 9\kappa^2 (\Sigma \gamma_j^2)^2.$$

We have

$$I_n^4 = \int \{U_{n-1}^4 + 6U_{n-1}^2 \gamma_n^2 \phi_n^2 + \gamma_n^4 \phi_n^4\} dx + 4\epsilon_n \int \{U_{n-1}^3 \gamma_n \phi_n + U_{n-1} \gamma_n^3 \phi_n^3\} dx. \tag{25}$$

To each complex $(\epsilon_1, \ldots, \epsilon_{n-1})$ correspond the two complexes $(\epsilon_1, \ldots, \epsilon_{n-1}, 1)$, $(\epsilon_1, \ldots, \epsilon_{n-1}, -1)$. Hence, by (25),

$$\begin{aligned} \Sigma_n I_n^4 &\leqslant \Sigma_n [I_{n-1}^4 + 6\gamma_n^2 \kappa I_{n-1}^2 + \kappa^2 \gamma_n^4] \\ &\leqslant 2 \Sigma_{n-1} I_{n-1}^4 + 12\kappa \gamma_n^2 \left(\Sigma_{n-1} I_n^4\right)^{\frac{1}{2}} 2^{\frac{1}{2}(n-1)} + 2^n \kappa^2 \gamma_n^4, \end{aligned}$$

or

$$\frac{1}{2^n} \Sigma_n I_n^4 \leqslant \frac{1}{2^{n-1}} \Sigma_{n-1} I_{n-1}^4 + 6\kappa \gamma_n^2 \left(\frac{1}{2^{n-1}} \Sigma_{n-1} I_{n-1}^4\right)^{\frac{1}{2}} + \kappa^2 \gamma_n^4,$$

whence

$$\left(\frac{1}{2^n} \Sigma_n I_n^4\right)^{\frac{1}{2}} \leqslant \left(\frac{1}{2^{n-1}} \Sigma_{n-1} I_{n-1}^4\right)^{\frac{1}{2}} + 3\kappa \gamma_n^2.$$

This reduction formula establishes the desired result.

10. LEMMA 20. *If $(\lambda) \equiv (\lambda_k)$ are an increasing sequence of positive integers such that, for any α_k, β_k,*

$$\int_0^{2\pi} \left| \Sigma_1^n \alpha_k \cos \lambda_k x + \beta_k \sin \lambda_k x \right| dx \geqslant C(\lambda) \left(\Sigma_1^n \alpha_k^2 + \beta_k^2\right)^{\frac{1}{2}}$$

for all n, then, given two sequences a_k, b_k with $\Sigma a_k^2+b_k^2 = 1$, there is a function f such that $|f| \leqslant \pi/C(\lambda)$ and

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f\cos\lambda_k x\,dx, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f\sin\lambda_k x\,dx \quad (k=1, 2, \ldots). \tag{26}$$

This is an immediate consequence of Theorem II.

Definition. An increasing sequence (λ_k) of positive integers will be said to possess the property (A) if there are two numbers M, N such that, for $n > M$, the equations

$$\lambda_j+\lambda_k\pm\lambda_l = \lambda_n \quad (n > j \geqslant k \geqslant l) \tag{27}$$

have, for given n, only such solutions for which j differs from n by less than N, and k has less than N possible values. The pair of numbers (M, N) will be called the index of the sequence.

A sequence (λ_k) of positive integers for which $\lambda_{k+1}/\lambda_k \geqslant q > 1$ possesses the property (A). For consider the equation

$$\lambda_j+\lambda_k+\lambda_l = \lambda_n \quad (n > j \geqslant k \geqslant l). \tag{28}$$

Then $3\lambda_j \geqslant \lambda_n$, and $\lambda_n/\lambda_j \geqslant q^{n-j}$, so that

$$n-j \leqslant \log 3/\log q.$$

Further, (27) gives

$$2\lambda_k \geqslant \lambda_k+\lambda_l = \lambda_n-\lambda_j \geqslant \lambda_n(1-1/q),$$

so that

$$n-k \leqslant \log\frac{2q}{q-1}\Big/\log q.$$

Again, the equation

$$\lambda_j+\lambda_k-\lambda_l = \lambda_n \quad (n > j \geqslant k \geqslant l)$$

gives $2\lambda_j > \lambda_n$, so that

$$n-j < \log 2/\log q,$$

while

$$\lambda_k > \lambda_k-\lambda_l = \lambda_n-\lambda_j \geqslant \lambda_n(1-1/q),$$

so that

$$n-k \leqslant \log\frac{q}{q-1}\Big/\log q.$$

We may therefore take

$$N = \text{Max}\left\{1+[\log 3/\log q], \quad 1+\left[\log\frac{2q}{q-1}\Big/\log q\right]\right\}$$

and, say, $M = 3$.

LEMMA 21. *If* (λ_k) *are an increasing sequence of positive integers which possesses the property* (A), *and if* (M, N) *is its index, then*

$$\frac{1}{\pi}\int_0^{2\pi}\left|\sum_1^n(\alpha_j\cos\lambda_j x+\beta_j\sin\lambda_j x)\right|^\alpha dx \geqslant \left[\frac{\pi}{C(M,N)}\right]^{1-\frac{1}{2}\alpha}\left(\sum_1^n \alpha_j{}^2+\beta_j{}^2\right)^{\frac{1}{2}\alpha} \qquad (0<\alpha\leqslant 2),$$

where $C(M, N)$ *is a constant which depends only on* M *and* N.

We can write

$$\alpha_j\cos\lambda_j x+\beta_j\sin\lambda_j x = c_j\cos(\lambda_j x+\mu_j),$$

where c_j is non-negative and $c_j{}^2=\alpha_j{}^2+\beta_j{}^2$. Then, if

$$J_n{}^2=\int U_n{}^2\,dx, \quad I_n{}^4=\int U_n{}^4\,dx, \quad U_n=\sum_1^n c_j\cos(\lambda_j x+\mu_j),$$

we have

$$\int|U_n|^\alpha \geqslant J_n^{4-\alpha}/I_n^{4-2\alpha}.$$

We may suppose that

$$\sum_1^n c_j{}^2=1. \tag{29}$$

Then $J_n=\pi^{\frac{1}{2}}$. We may suppose that $n>M$. Then

$$\begin{aligned} I_n{}^4 &= I_{n-1}^4+6c_n{}^2\int U_{n-1}^2\cos^2(\lambda_n x+\mu_n)\,dx+c_n{}^4\int\cos^4(\lambda_n x+\mu_n)\,dx \\ &\quad +4\int U_{n-1}^3 c_n\cos(\lambda_n x+\mu_n)\,dx+4\int U_{n-1}c_n{}^3\cos^3(\lambda_n x+\mu_n)\,dx \\ &= I_{n-1}^4+3\pi c_n{}^2\sum_1^{n-1} c_j{}^2+\frac{3\pi}{4}c_n{}^4+4\int U_{n-1}^3 c_n\cos(\lambda_n x+\mu_n)\,dx. \end{aligned} \tag{30}$$

The sum of the second and third terms does not exceed

$$3\pi c_n{}^2\sum_1^n c_j{}^2 = 3\pi c_n{}^2$$

by (29). The fourth term is non-negative and equal to

$$c_n\int\Sigma c_r c_s c_t\cos(\lambda_r\pm\lambda_s\pm\lambda_t)x\cos\lambda_n x\,dx, \tag{31}$$

where r, s, t are three integers less than n which satisfy any one of the relations $\lambda_r\pm\lambda_s\pm\lambda_t=\pm\lambda_n$, and the sum is taken for all such triads. The expression (31) does not exceed

$$2c_n\int\Sigma c_j c_k c_l\cos(\lambda_j\pm\lambda_k\pm\lambda_l)x\cos\lambda_n x\,dx,$$

where $n>j\geqslant k\geqslant l$ and one of the relations $\lambda_j\pm\lambda_k\pm\lambda_l=\lambda_n$ is satisfied.

The relations $\lambda_j - \lambda_k \pm \lambda_l = \lambda_n$ cannot, however, be satisfied, so that we have to consider the expression

$$2\pi \Sigma c_j c_k c_l c_n,$$

where j, k, l satisfy (27). In virtue of (29), the preceding expression does not exceed

$$2\Sigma c_j c_n \leqslant \Sigma(c_j^2 + c_n^2). \tag{32}$$

In virtue of the property (A), there are at most $N-1$ choices for each of j and k. Further, when j and k have given values, only one of the equations

$$\lambda_j + \lambda_k + \lambda_l = \lambda_n, \quad \lambda_j + \lambda_k - \lambda_l = \lambda_n \quad (n > j \geqslant k \geqslant l)$$

can have a solution, and that equation cannot have more than one solution. Hence the number of terms in (32) does not exceed $(N-1)^2$. Further, since each permissible j lies between $n-N$ and n, the second member in (32) does not exceed

$$\pi \Sigma(c_{n-N+1}^2 + \ldots + c_n^2) \leqslant \pi(N-1)^2(c_{n-N+1}^2 + \ldots + c_n^2).$$

Hence (30) gives

$$I_n^4 \leqslant I_{n-1}^4 + 3\pi c_n^2 + \pi(N-1)^2(c_{n-N+1}^2 + \ldots + c_n^2). \tag{33}$$

Further,

$$I_M^4 \leqslant \left[2\pi \operatorname{Max} \left| \sum_1^M c_j \cos \lambda_j x + \mu_j \right| \right]^4$$

$$\leqslant \left[2\pi \sum_1^M c_j \right]^4 \leqslant 16\pi^4 M^2. \tag{34}$$

Hence, applying (33) until we arrive at I_M^4, and then using (34), we get

$$I_n^4 \leqslant 16\pi^4 M^2 + 3\pi + \pi N^3 = C(M, N).$$

Taking $\alpha = 1$ in Lemma 21, we infer, from Lemma 20,

THEOREM IX. *If (λ_k) is an increasing sequence of positive integers which possesses the property* (A), *and if (M, N) is its index, then, given the sequences a_k, b_k with $\Sigma a_k^2 + b_k^2 = 1$, there is an f such that $|f| \leqslant \kappa(M, N)$ and such that* (26) *holds.*

Taking $\alpha < 1$ in Lemma 21 and using Theorem VII, we deduce

THEOREM X. *If (λ_k) is an increasing sequence of positive integers which possesses the property* (A), *and if*

$$\sum_1^\infty c_k \sin(\lambda_k x + \mu_k)/\lambda_k$$

is a function of bounded variation, then Σc_k^2 converges.

11. The following general theorem is an interesting application of the ideas used in the proof of Theorem I.

THEOREM XI. *Let $\phi(x)$, defined for $x \geqslant 0$, be continuous and increasing, and let it tend to ∞ as $x \to \infty$. Let ψ be the function inverse to ϕ, and let $\Phi(x) = x\phi(x)$, $\Psi(x) = x\psi(x)$. Let $\Phi(x)$ be convex, and possess the property that*

$$\int_0^1 \Phi|f|\,dx < \infty \quad \textit{implies} \quad \int_0^1 \Phi|\lambda f|\,dx < \infty \tag{35}$$

for all $\lambda > 0$. Let $h_j(x)$ be a sequence of bounded functions defined in $(0, 1)$. Then, given a sequence a_j of constants, a necessary and sufficient condition for a function $f(x)$ to exist such that $\int \Phi|f|\,dx < \infty$ and such that

$$a_j = \int_0^1 f(x)\,h_j(x)\,dx \quad (j = 1, 2, \ldots)$$

is that a constant C exists such that, for all n,

$$\left|\sum_1^n \mu_j a_j\right| \leqslant C \tag{36}$$

for all μ_j which satisfy

$$\int_0^1 \Psi\left|\sum_1^n \mu_j h_j\right| dx = 1. \tag{37}$$

We first observe that, if x and y are non-negative, then

$$xy \leqslant \Phi(x) + \Psi(y).$$

For

$$xy \leqslant \int_0^x \phi\,dt + \int_0^y \psi\,dt,$$

so that the result follows by the mean value theorem. The necessity of the condition follows immediately. For, if the μ_j satisfy (37), then

$$\begin{aligned}
\left|\sum_1^n \mu_j a_j\right| &= \left|\int_0^1 f \sum_1^n h_j \mu_j\,dx\right| \\
&\leqslant \int_0^1 |f| \left|\sum_1^n h_j \mu_j\right| dx \\
&\leqslant \int_0^1 \Phi|f|\,dx + \int_0^1 \Psi\left|\sum_1^n h_j \mu_j\right| dx \\
&\leqslant \int_0^1 \Phi|f|\,dx + 1 = C.
\end{aligned}$$

To prove the sufficiency we require the following lemma.

LEMMA 22. *If f_n is a sequence of functions such that*

$$\int_0^1 \Phi|f_n|\,dx \leqslant A,$$

then the sequence

$$F_n(x) = \int_0^x f_n(t)\,dt \quad (0 \leqslant x \leqslant 1)$$

is uniformly absolutely continuous; and there is a subsequence n_r and a function f such that

$$\int_0^1 \Phi|f|\,dx \leqslant A \tag{38}$$

and

$$\lim_{r\to\infty} \int_\alpha^\beta f_{n_r}\,dx = \int_\alpha^\beta f\,dx \tag{39}$$

for every α, β satisfying $0 \leqslant \alpha < \beta \leqslant 1$.

Let e be a measurable set in (0, 1) and let n be arbitrarily chosen. If $M > 0$, and $e = e_1 + e_2$, where e_1 is the part of e in which $|f_n| \leqslant M$, then

$$\int_{e_1} |f_n|\,dx \leqslant Mm(e), \quad \int_{e_2} |f_n|\,dx = \int_{e_2} \Phi|f_n|/\phi|f_n|\,dx < A/\phi(M).$$

Given $\epsilon > 0$, we choose M so that $A/\phi(M) < \frac{1}{2}\epsilon$. Then, if $m(e) < \frac{1}{2}\epsilon/M$, we have

$$\int_e |f_n|\,dx < \epsilon.$$

Thus the sequence $F_n(x)$ is uniformly absolutely continuous, and, since $F_n(0) = 0$, it is uniformly bounded. Hence there is a sequence f_{n_r} and a function f such that

$$F_{n_r}(x) \to F(x) = \int_0^x f(t)\,dt.$$

In particular, (39) holds. The relation (38) then follows by a theorem of W. H. Young*.

Turning to the main theorem, let n be arbitrarily assigned. Let E denote the set of points $\xi \equiv (\xi_1, \ldots, \xi_n)$ such that to each ξ there corresponds an f_ξ which satisfies

$$\int_0^1 \Phi|f_\xi|\,dx \leqslant 1$$

and

$$\xi_j = \int_0^1 f_\xi h_j\,dx \quad (j = 1, \ldots, n).$$

* W. H. Young, *Proc. London Math. Soc.* (2), 24 (1925), 1–20 (3). Trivial changes in the proof there given will establish the result which we require.

In virtue of the convexity of Φ, if ξ and η belong to E, so does $\{\lambda\xi_1+(1-\lambda)\eta_1, \ldots\}$ for all $0<\lambda<1$. The set E is closed. For, if ξ^ν is a sequence of points of E which tends to ξ, and if f_ν is the corresponding sequence of functions, then

$$\int_0^1 \Phi|f_\nu|dx \leqslant 1,$$

$$\xi_j^\nu = \int_0^1 f_\nu h_j dx \quad (j=1, \ldots, n).$$

By Lemma 22, there is a subsequence f_{ν_k} and a function f with the properties specified by that lemma. Since h_j is bounded,

$$\lim_{k\to\infty} \int_0^1 f_{\nu_k} h_j dx = \int_0^1 f h_j dx,$$

so that ξ belongs to E.

The first member of (37) is manifestly a continuous function of the μ_j which can take all positive values. Let $\mu_1, \ldots, \mu_n$ be assigned. Then there is a positive number c such that

$$\int_0^1 \Psi \left| \sum_1^n h_j c\mu_j \right| dx = 1. \tag{40}$$

By (36),
$$\left| \sum_1^n a_j c\mu_j \right| \leqslant C.$$

Writing $\alpha_j = a_j/C$, we have

$$\left| \sum_1^n \alpha_j c\mu_j \right| \leqslant \int_0^1 \Psi \left| \sum_1^n h_j c\mu_j \right| dx. \tag{41}$$

Now
$$\int_0^1 \Psi \left| \sum_1^n h_j c\mu_j \right| dx = \int_0^1 \psi|\Sigma| \operatorname{sgn}(\Sigma) \sum_1^n h_j c\mu_j dx.$$

Writing
$$f_\mu = \psi|\Sigma| \operatorname{sgn}(\Sigma),$$

there is, by (41), a θ satisfying $\theta^2 \leqslant 1$ such that

$$\sum_1^n \alpha_j c\mu_j = \theta \int_0^1 f_\mu \sum_1^n \mu_j c h_j dx = \sum_1^n \beta_j \mu_j c \tag{42}$$

with
$$\beta_j = \int_0^1 \theta f_\mu h_j dx \quad (j=1, \ldots, n).$$

But
$$\int_0^1 \Phi|\theta f_\mu|dx \leqslant \int_0^1 \Phi|f_\mu|dx = \int_0^1 \Phi \left| \psi|\Sigma| \right| dx$$

$$= \int_0^1 \Psi|\Sigma|dx$$

$$= 1.$$

Hence $\beta \equiv (\beta_1, \ldots, \beta_n)$ belongs to E. The equation (42) tells us that every plane $\Sigma \mu_j (x_j - \alpha_j) = 0$ through α has a point of E on it. Hence, as in Theorem I, α belongs to E. Hence there is a function f_n such that

$$\int_0^1 \Phi |f_n| dx \leqslant 1,$$

$$a_j = \int_0^1 f_n h_j dx \quad (j = 1, \ldots, n).$$

This holds for every n. By another application of Lemma 22, we infer the existence of an f such that

$$\int_0^1 \Phi |f| dx \leqslant 1$$

and

$$a_j = \int_0^1 f h_j dx \quad (j = 1, 2, \ldots).$$

Finally,

$$a_j = \int_0^1 C f h_j dx,$$

and

$$\int_0^1 \Phi |Cf| dx < \infty$$

by (35).

[*Added* 3 *January*, 1934. I gather from a statement made by Banach, *Opérations linéaires* (1932), 75, that Theorem I of this paper has been proved, at least for functions defined in a finite interval, by F. Riesz.]

THE MINIMA OF QUATERNARY QUADRATIC FORMS OF SIGNATURE ZERO

By A. OPPENHEIM.

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1. In a previous paper* I showed that if Ψ is an indefinite quaternary quadratic form with real coefficients of signature zero and determinant Δ, necessarily positive, L^4/Δ must have the values $\frac{4}{9}$ or $\frac{4}{17}$ or else be at most $\frac{64}{375}$, where $L(\Psi) = L$ denotes the lower bound of $|\Psi|$ for integral values of the variables, the zero set excluded.

I show here that the next five possible values of L^4/Δ are $\frac{4}{25}$, $\frac{16}{117}$, $\frac{4}{33}$, $\frac{1}{9}$, and $\frac{64}{625}$. I prove, in fact, the following

THEOREM. *Let Ψ be an (indefinite) quaternary quadratic form of signature zero so that its determinant $\Delta > 0$, and of absolute lower bound L. If $10L^4 \geqslant \Delta$, then necessarily*

$$L^4/\Delta = \tfrac{4}{9},\ \tfrac{4}{17},\ \tfrac{4}{25},\ \tfrac{16}{117},\ \tfrac{4}{33},\ \tfrac{1}{9},\ \text{or}\ \tfrac{64}{625}.$$

In the respective cases, $\pm\Psi/L$ is necessarily equivalent to

$$t^2-x^2+y^2-z^2+xt+zt-xy+xz, \qquad 4\Delta = 9L^4;$$

$$t^2-x^2+y^2-z^2+xt+zt-xy+xz+3yz, \qquad 4\Delta = 17L^4;$$

$$t^2+tx-x^2+2y^2+2yz-2z^2, \qquad 4\Delta = 25L^4;$$

$$t^2-x^2+y^2-z^2+xt+zt-xy+4yz, \qquad 16\Delta = 117L^4;$$

$$t^2-x^2-2y^2-z^2+xt+zt+6yz, \qquad 4\Delta = 33L^4;$$

$$t^2-x^2+y^2-z^2+xt+zt-xy+xz+5yz, \qquad \Delta = 9L^4;$$

or

$$t^2+x^2-3y^2-3z^2, \qquad \Delta = 9L^4;$$

$$t^2+tx-x^2+\tfrac{5}{2}(y^2+yz-z^2), \qquad 64\Delta = 625L^4.$$

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* *Annals of Math.*, 32 (1931), 271–298, Theorem A (272). This paper is referred to as Q. I take this opportunity of repairing two omissions in Q. In Q, 283, Case B_2C_2 leads also to the form

$$t^2 + tx - x^2 - 2y^2 - 2yz + 2z^2, \quad 4\Delta = 25d^4.$$

This omission did not affect the proof of Theorem A. Since Q was published, I have discovered that the first number and form in Theorem A and in Theorem B are due to Markoff in a paper published in Russian about 1900.

The first term is equivalent to

$$t^2+tx+x^2-2(y^2+yz+z^2).$$

Each form is improperly equivalent to itself.

Each form, with the possible exception of the form with $4\Delta = 33L^4$, is equivalent to its negative.

This theorem has enabled me to obtain some results in the analogous problem for binary Hermitian quadratic forms in any quadratic field.

2. The method of proof is similar to that in Q. There I used the notation:

$$\Psi = \Psi(x, y, z, t) = (a, b, c, f, g, h, u, v, w)(x, y, z, t)^2 \tag{1}$$

is an indefinite quaternary quadratic form of absolute lower bound $L > 0$ with positive determinant $\Delta = \Delta(\Psi)$. We may take* $L = d$, the coefficient of t^2, and write

$$d\Psi = (ux+vy+wz+dt)^2+\Phi(x, y, z), \tag{2}$$

$$\Phi = (\alpha, \beta, \gamma, \rho, \sigma, \tau)(x, y, z)^2, \tag{3}$$

$$\alpha = ad-u^2, \ldots, \quad \rho = fd-vw, \ldots, \tag{4}$$

$$n = L(\Phi) \quad (n > 0). \tag{5}$$

We take

$$\alpha = -n \quad \text{if } \Phi \text{ represents* } -n, \tag{6}$$

$$\alpha = n \quad \text{if } \Phi \text{ represents } n \text{ but not } -n. \tag{7}$$

Write

$$\alpha\Phi = (\alpha x+\tau y+\sigma z)^2+d\phi(y, z), \quad \phi = Cy^2-2Fyz+Bz^2, \tag{8}$$

where B, C, ... are the cofactors of b, c, ... in Δ. Then

$$\Delta_1 = \Delta(\Phi) = d^2\Delta, \quad \Delta_2 = \Delta(\phi) = BC-F^2 = \alpha\Delta. \tag{9}$$

It is convenient to use another representation of Φ. Let n' be the lower bound of the positive values represented by $-\Phi$. We consider also the representation

$$\alpha = -n'. \tag{10}$$

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* We assume throughout that L, n, n', ... are all attained. I have shown in Q and elsewhere how these restrictions may be removed.

Plainly

$$n' \geqslant n. \tag{11}$$

Note also that Φ is indefinite of signature -1, and that ϕ is negative definite if $a > 0$, indefinite if $a < 0$. We may suppose in either case that ϕ is reduced.

Call the representation (10) N, the representation (6) P_1, and the representation (7) P_2.

3. N, $a = -n'$. The binary form (a, u, d) has positive discriminant $-4a = 4n'$ and represents d, the minimum of $|\Psi|$. Since all its values are values of Ψ, its own minimum must be d, so that, by Markoff's theorem (Q, Theorem 1),

$$d^2/n' = \tfrac{4}{5},\ \tfrac{1}{2},\ \text{or}\ \leqslant \tfrac{1}{2}\tfrac{0}{2}\tfrac{0}{1}. \tag{1}$$

Note that this result holds also for *any* negative value of Φ.

4–6. *The case* $5d^2 = 4n'$. We use the representation $N(2, 10)$ and the machinery of Q, 6, etc. The form $\phi(y, z)$ is indefinite of determinant $a\Delta$. We may suppose it to be reduced and identify it with any member ϕ_i of the chain of reduced forms to which it belongs (Q, § 2),

$$\phi_i = [(-)^i A_i,\ B_i,\ (-)^{i+1} A_{i+1}]^* \quad (A_i > 0,\ B_i > 0). \tag{1}$$

The indefinite form $q = [a, b, c]$ of discriminant $R^2 = b^2 - 4ac > 0$ $(R > 0)$ is reduced if

$$0 < R - b < 2|a| < R + b. \tag{2}$$

These inequalities imply a like set with c in place of a. In a reduced form a and c have opposite signs.

Properties of a chain of reduced forms (ϕ_i) (Q, 2.3):

$$B_{i-1} + B_i = 2g_{i-1} A_i \quad (g_i > 0, \text{ integral}); \tag{3}$$

$$F_i = (g_i, g_{i+1}, g_{i+2}, \ldots), \quad S_i = (0, g_{i-1}, g_{i-2}, \ldots) \quad \text{Df.}, \tag{4}$$

$$K_i = F_i + S_i = R/A_{i+1}, \quad F_i - S_i = B_i/A_{i+1}, \quad F_i S_i = A_i/A_{i+1}. \tag{5}$$

$$R^2 = B_i^2 + 4A_i A_{i+1} \quad (R > 0). \tag{6}$$

The roots of $\phi_i(t, 1) = 0$ are $(-)^i/F_i$ and $(-)^{i+1}/S_i$. The transformation $x = Y$, $y = -X + k_i Y$, $k_i = (-)^i g_i$, carries ϕ_i into ϕ_{i+1}.

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* The notation $[a, b, c]$ indicates $ax^2 + bxy + cy^2$. The notation (a, b, c) indicates $ax^2 + 2bxy + cy^2$.

When we put $z=0$ in Ψ we obtain an indefinite ternary quadratic form of determinant $(-)^i A_i$ and minimum d. By the theorems of Markoff and Dickson (Q, Theorem 2), we have therefore one of the following possibilities:

$$(7) \qquad \mu_i = \frac{A_i}{d^3} = \tfrac{3}{2},\ \tfrac{5}{2},\ \tfrac{2\,5}{8} \quad \text{or} \quad > \frac{1}{\lambda}, \quad \lambda = 0{\cdot}30033,$$

since the possibility $d^3/A_i = \frac{1}{3}$ is incompatible with $a = -n' = -5d^2/4$, by Q, Lemma 15.

The discussion falls under the cases below:

I. For *some* i,

$$\mu_i = \tfrac{3}{2},\ \tfrac{5}{2}, \quad \mu_{i+1} = \tfrac{3}{2},\ \tfrac{5}{2}.$$

II. For each i, one of μ_i, μ_{i+1} exceeds $\frac{5}{2}$, and therefore

$$\text{Max}\,(\mu_i,\ \mu_{i+1}) \geqslant \tfrac{2\,5}{8} \quad (\text{all } i).$$

5. *Case* I. We may take the i in question to be 0. Write

$$-2F = B_0 = 2\nu d^3 > 0.$$

Then

$$(1) \qquad \frac{\Delta}{d^4} = \tfrac{4}{5}(\mu_0\mu_1 + \nu^2).$$

$$(2) \qquad \frac{\Phi}{d} = (ux+vy+wz+t)^2 - \tfrac{5}{4}(x+\tau y+\sigma z)^2 + \tfrac{4}{5}(\mu_0 y^2 + 2\nu yz - \mu_1 z^2),$$

the change of notation being obvious.

A transformation of the type

$$x = X+ky+lz, \quad t = T+aX+by+cz$$

(k, l, ... integral) enables us to suppose that

$$(3) \qquad -\tfrac{1}{2} < u,\ v,\ w,\ \tau,\ \sigma \leqslant \tfrac{1}{2}.$$

Case I_1. $\mu_0 = \mu_1 = \frac{5}{2}$. The ternary form obtained from Ψ/d by putting $z=0$ has minimum 1 and determinant $-\frac{5}{2}$. Its coefficients therefore are integers (Q, Theorem 2). We find easily that

$$u = \tfrac{1}{2}, \quad v = 0, \quad \tau = 0, \quad \text{and} \quad w = \sigma = 0$$

similarly.

Consider now the values of Ψ/d for the sets of values of x, y, z, t:

$$0, 1, -1, 1;\quad 0, 1, 2, 0;\quad 0, 2, -2, 3;\quad 1, 1, 2, 0;\quad 0, 1, 3, 3;\quad 0, -1, 1, 1;$$

$$1-f;\qquad 2f-6;\qquad 9-4f;\qquad 2f-7;\qquad 3f-7;\qquad 4-f;$$

where $f = 8\nu/5$. Since $|\Psi/d| \geqslant 1$, we find that

$$f = 2 \quad \text{or} \quad f \geqslant 5.$$

The first case gives

$$\frac{\Delta}{d^4} = \tfrac{25}{4}; \quad \frac{\Psi}{d} = t^2+tx-x^2+2(y^2+yz-z^2), \tag{4}$$

a form which is not null.

The second case gives

$$\frac{\Delta}{d^4} \geqslant \tfrac{205}{16} > 10. \tag{5}$$

Case I_2. $\mu_0 = \frac{3}{2}$, $\mu_1 = \frac{3}{2}$. As in Case I_1, we find that

$$u = \tfrac{1}{2}, \quad v = 0, \quad \tau = \pm\tfrac{2}{5}, \quad w = \tfrac{1}{2}, \quad \sigma = \pm\tfrac{1}{5}.$$

We may suppose that τ is positive. If $\sigma > 0$, consider the values of Ψ/d for the sets

$$0, -1, 1, 0;\quad 0, -1, 1, 1;\quad 1, -1, 1, 1;\quad 1, 2, -1, 3;$$

$$\tfrac{1}{5}-f;\qquad \tfrac{11}{5}-f;\qquad \tfrac{16}{5}-f;\qquad \tfrac{47}{5}-2f \quad (5f = 8\nu),$$

whence, since $|\Psi/d| \geqslant 1$,

$$f = \tfrac{6}{5} \quad \text{or} \quad \tfrac{21}{5} \quad \text{or} \quad \geqslant \tfrac{26}{5};$$

$$\frac{\Delta}{d^4} = \tfrac{9}{4}, \quad \frac{\Psi}{d} = t^2-x^2+y^2-z^2+xt+zt-xy+yz; \tag{6}$$

$$\frac{\Delta}{d^4} = \tfrac{117}{16}, \quad \frac{\Psi}{d} = t^2-x^2+y^2-z^2+xt+zt-xy+4yz; \tag{7}$$

or

$$\frac{\Delta}{d^4} \geqslant \tfrac{41}{4}. \tag{8}$$

Neither of these forms is null, each has minimum 1.

If $\sigma < 0$, consider the values of Ψ/d,

$$\tfrac{4}{5}-f;\quad \tfrac{9}{5}-f;\quad \tfrac{19}{5}-f;\quad \tfrac{29}{5}-f;$$

for the sets

$$1, -1, 1, 0; \quad 0, -1, 1, 1; \quad 1, -1, 1, 1; \quad 0, -1, 1, 2.$$

Hence $$f = \tfrac{14}{5}, \quad \tfrac{24}{5}, \quad \text{or} \quad \geqslant \tfrac{34}{5};$$

$$(9) \qquad \frac{\Delta}{d^4} = \tfrac{17}{4}, \quad \frac{\Psi}{d} = t^2 - x^2 + y^2 - z^2 + xt + zt - xy + xz + 3yz;$$

$$(10) \qquad \frac{\Delta}{d^4} = 9, \quad \frac{\Psi}{d} = t^2 - x^2 + y^2 - z^2 + xt + zt - xy + xz + 5yz;$$

or

$$(11) \qquad \frac{\Delta}{d^4} \geqslant \tfrac{65}{4}.$$

Neither form is null, each has minimum unity.

Case I_3. $\mu_0 = \frac{3}{2}, \mu_1 = \frac{5}{2}$. Here

$$u = \tfrac{1}{2}, \quad v = 0, \quad \tau = \pm\tfrac{2}{5}, \quad w = \sigma = 0.$$

We may suppose that $\tau > 0$. Consider the values of Ψ/d, $f-1$, $f-3$, $f-4$, for

$$0, 1, 1, 0; \quad 1, 1, 1, 0; \quad 2, -1, -1, -1;$$

whence (since $|\Psi/d| \geqslant 1$)

$$f = 2 \quad \text{or} \quad f \geqslant 5;$$

$$(12) \qquad \frac{\Delta}{d^4} = \tfrac{17}{4}, \quad \frac{\Psi}{d} = t^2 + tx - x^2 + y^2 + 2yz - 2z^2 - xy;$$

or

$$(13) \qquad \frac{\Delta}{d^4} \geqslant \tfrac{173}{16}.$$

Case I_4. $\mu_0 = \frac{5}{2}, \mu_1 = \frac{3}{2}$. Here

$$u = \tfrac{1}{2}, \quad v = \tau = 0, \quad w = \tfrac{1}{2}, \quad \sigma = \pm\tfrac{1}{5}.$$

We may suppose that $\sigma > 0$. Taking the sets

$$0, 1, -1, 0; \quad 1, 1, -1, 2; \quad -1, 1, -1, 3,$$

we obtain $$f = 2 \quad \text{or} \quad f \geqslant 5;$$

$$(14) \qquad \frac{\Delta}{d^4} = \tfrac{17}{4}, \quad \frac{\Psi}{d} = t^2 + tx - x^2 + 2y^2 - z^2 + 2yz + zt;$$

or

$$(15) \qquad \frac{\Delta}{d^4} \geqslant \tfrac{173}{16}.$$

Thus Case I *leads to*

$$\frac{\Delta}{d^4} = \tfrac{9}{4},\ \tfrac{17}{4},\ \tfrac{25}{4},\ \tfrac{117}{16},\ \text{or } 9,$$

with $\pm\Psi/d$ *equivalent to the respective forms*

$$t^2-x^2+y^2-z^2+xt+zt-xy+yz,$$

$$t^2-x^2+y^2-z^2+xt+zt-xy+xz+3yz,$$

$$t^2+tx-x^2+2y^2+2yz-2z^2,$$

$$t^2-x^2+y^2-z^2+xt+zt-xy+4yz,$$

$$t^2-x^2+y^2-z^2+xt+zt-xy+xz+5yz,$$

or else

$$\frac{\Delta}{d^4} > 10.$$

6. *Case* II. $\text{Max}(\mu_i, \mu_{i+1}) \geqslant \frac{25}{8}$ *for every* i. We have

$$K_i^2 = \left(\frac{5\Delta}{d^4\mu_{i+1}^2}\right) \leqslant \frac{50}{\mu_{i+1}^2}, \tag{1}$$

assuming that

$$\Delta \leqslant 10d^4. \tag{2}$$

According, therefore, as

$$\mu_{i+1} = \tfrac{3}{2},\ \tfrac{5}{2},\ \tfrac{25}{8},\ \text{or } > \frac{1}{\lambda}, \tag{3}$$

we have

$$K_i \leqslant \frac{10\sqrt{2}}{3},\quad 2\sqrt{2},\quad \frac{8\sqrt{2}}{5},\quad 5\lambda\sqrt{2}, \tag{4}$$

and so

$$g_i \leqslant 4\ \ (\mu_{i+1} = \tfrac{3}{2}),\quad g_i \leqslant 2\ \ (\text{otherwise}). \tag{5}$$

Denote the g_i corresponding to the possible values of μ_{i+1} in (3) by t_i, k_i, l_i, m_i. Not both of the classes l and m can be empty.

Every $m_i = 1$. We have the useful inequality

$$g_i + \frac{1}{1+g_{i-1}} + \frac{1}{1+g_{i+1}} < K_i. \tag{6}$$

If $g_i = m_i = 2$, then, since $g_j \leqslant 4$ (all j),

$$2+\tfrac{1}{5}+\tfrac{1}{5} < K_i < 5\lambda\sqrt{2},$$

which is false. Since $m_i \leqslant 2$, we must have $m_i = 1$.

Every $l_i = 1$. If $l_i = 2$, then

$$2+\tfrac{1}{5}+\tfrac{1}{5} < K_i \leqslant \frac{8\sqrt{2}}{5},$$

which is false. Since $l_i \leqslant 2$, we must have $l_i = 1$.

Every $k_i = 1$. For if a k_i exists, its neighbours being of class l or m are both unity and so

$$k_i+\tfrac{1}{2}+\tfrac{1}{2} < K_i < 2\sqrt{2}, \quad k_i < 2, \quad k_i = 1.$$

If no t occurs, then all the g_i are unity by the preceding work, and all the μ_{i+1} are equal and so each is $\frac{25}{8}$ or greater than $1/\lambda$. But here

$$K_i = \overset{\times}{(1)}+\overset{\times}{(0,\ 1)} = \sqrt{5} > 5\lambda\sqrt{2},$$

so that all the μ_{i+1} are $\frac{25}{8}$, and

$$\frac{\Psi}{d} = (ux+vy+wz+t)^2-\tfrac{5}{4}(x+\tau y+\sigma z)^2+\tfrac{5}{2}(y^2+yz-z^2), \tag{7}$$

$$\frac{\Delta}{d^4} = \tfrac{625}{64}, \quad -\tfrac{1}{2} < u,\ v,\ w,\ \tau,\ \sigma \leqslant \tfrac{1}{2}. \tag{8}$$

Put $z = 0$. We obtain an indefinite ternary quadratic form of minimum 1 and determinant $-\frac{25}{8}$. By Q, Theorem 2, its coefficients must be congruent to 0 or $\frac{1}{2}$ modulo 1. We find easily that

$$u = \tfrac{1}{2}, \ v = 0, \ \tau = 0; \quad \text{and} \quad w = 0, \ \sigma = 0 \tag{9}$$

likewise. Thus we have the form

$$\frac{\Psi}{d} = (t^2+tx-x^2)+\tfrac{5}{2}(y^2+yz-z^2), \quad \frac{\Delta}{d^4} = \tfrac{625}{64}, \tag{10}$$

which is not null and has minimum 1, for the form cannot represent $\pm\frac{1}{2}$ since

$$\frac{8\Psi}{d} \equiv 2T^2 \pmod 5,$$

and $2T^2 \equiv \pm 4 \pmod 5$ is not possible.

Apart then from the form (10), at least one t exists.

Every $t_i \leqslant 3$. For the neighbours of $g_i = t_i$ are unity and so

$$t_i+\tfrac{1}{2}+\tfrac{1}{2} < K_i \leqslant \frac{10\sqrt{2}}{3}, \quad t_i < 4, \quad t_i \leqslant 3.$$

All the t's are equal. For, if t_i, t_j occur and $t_i > t_j$ and so $t_i \geqslant t_j+1$, we have

$$K_i > t_i+\tfrac{1}{2}+\tfrac{1}{2} = t_i+1, \quad K_j < t_j+1+1 \leqslant t_i+1,$$

$$K_i < K_j;$$

this is a contradiction, *since the K's of the same type* (t, or k, or l) *are necessarily equal*, for they correspond to equal μ's.

Hence $t_i \geqslant 2$. For if $t_i = 1$, then every $g_i = 1$, a case just excluded.

There are t's arbitrarily far to the right and arbitrarily far to the left of any assigned g.

If only a finite number of t's occur to the right, let t_i be the one of greatest suffix. Then (for $n = 1, 2, \ldots$)

$$K_{i+2n} = (1_\infty)+(0, 1_{2n-1}, t_i, 1, \ldots)$$

$$> (1_\infty)+(0, 1_{2n-1}, 1_\infty) = \sqrt{5} > 5\lambda\sqrt{2}.$$

Hence, for $n \geqslant 1$, K_{i+2n} belongs to type k or type l and so can take only two values. But this is absurd, since K_{i+2n} decreases steadily as n increases.

The second part is proved in the same way.

Suppose now that g_0 and g_{n+1} are consecutive t's,

$$g_0 = g_{n+1} = t \quad (n \geqslant 1, \quad t = 2 \text{ or } 3).$$

The sequence (g_i) is

$$\ldots \; 1 g_0 \; 1_n \; g_{n+1} \; 1 \; \ldots.$$

We have $\mu_1 = \mu_{n+2}$, $K_0 = K_{n+1}$, and so

$$(0, 1_n, X)+Y = (0, 1_n, Y)+X, \tag{11}$$

where

$$X = F_{n+1} = (g_{n+1}, 1, \ldots), \quad Y = \frac{1}{S_1} = (g_0, 1, \ldots).$$

Hence

$$X = Y, \quad g_{n+r} = g_{1-r} \quad (r = 1, 2, 3, \ldots).$$

The sequence to the right of g_{n+1} is the reverse of the sequence to the left of g_0.

If $n \geqslant 3$, *then* $n = 3$ *and the sequence* (g_i) *is*

$$(\overset{\times}{3}\, 1\, 1\, \overset{\times}{1}).$$

We use the fact:

$$\frac{K_0}{K_2} = \frac{\mu_3}{\mu_1} = \tfrac{5}{3} \quad \text{or} \quad \geqslant \tfrac{2}{1}\tfrac{5}{2}, \tag{12}$$

according as $g_2 = 1$ belongs to class k or not.

Let $$Z = F_2 = (1_{n-1}, g_{n+1}, 1, \ldots) = (1_{n-1}, X)$$

Then $$K_2 = Z + (0, 1, X) = \frac{ZX+Z+X}{X+1},$$

$$K_0 = (0, 1, Z) + X = \frac{ZX+Z+X}{Z+1},$$

$$\frac{K_0}{K_2} = \frac{X+1}{Z+1}. \tag{13}$$

Now, since $n \geqslant 3$,

$$Z = (1, 1, \ldots) > 1 + \tfrac{1}{2},$$

$$X = (t, 1, \ldots) < t+1 \leqslant 4 < 2Z+1,$$

$$\frac{K_0}{K_2} < 2 \quad \text{and therefore} \quad \frac{K_0}{K_2} = \tfrac{5}{3}. \tag{14}$$

Let $p_r/q_r\ (r \geqslant 1)$ be the convergents to $\overset{\times}{(1)}$. Then $p_{r-1} = q_r$ and so

$$Z = \frac{p_{n-1}X + p_{n-2}}{q_{n-1}X + q_{n-2}} = \frac{\xi_{n-1}X + 1}{X - 1 + \xi_{n-1}},$$

where $\xi_r = p_r/q_r$. The equation $3(X+1) = 5(Z+1)$ [by (13) and (14)] now yields the quadratic for X,

$$f(X) \equiv 3X^2 - 5X - 3 - 2\xi_{n-1}(X+1) = 0, \tag{15}$$

the positive root being the value sought.

Now

$$\tfrac{5}{3} \geqslant \xi_r \geqslant \tfrac{8}{5} \text{ for } r \geqslant 4, \quad \xi_3 = \tfrac{3}{2}, \quad \xi_2 = 2.$$

Hence $f(3) < 0$, $f(4) > 0$, so that X lies between 3 and 4, and therefore $t = g_{n+1} = 3$. Again, if $n \geqslant 4$, $f(\frac{7}{2}) > 0$, so that

$$3 < X < 3 + \tfrac{1}{2};$$

but $$X = (g_{n+1}, 1, \ldots) > 3 + \tfrac{1}{2},$$

which is a contradiction. Thus $n = 3$, $\xi_2 = 2$, and so

$$X = \frac{9 + \sqrt{(165)}}{6} = \overset{\times}{(3, 1, 1, 1)}, \quad (g_i) = \overset{\times}{(3, 1, 1, 1)}. \tag{16}$$

The cases $n = 2$, $n = 1$ cannot occur. If $n = 2$, then

$$Z = (1, X), \quad \frac{K_0}{K_2} = \frac{X(X+1)}{2X+1},$$

which increases as X increases.

Now if $t = 2$, then all t's are 2,

$$X = (2, 1, \ldots) < (2, 1, 3) = \tfrac{11}{4},$$

$$\frac{K_0}{K_2} < \tfrac{165}{104} < \tfrac{5}{3},$$

which is false, since $g_2 = 1$ is not a t.

Hence $t = 3$, all t's are 3, $X < (3, 1, 4) = \frac{17}{5}$,

$$\frac{K_0}{K_2} < \tfrac{456}{215} < \frac{2}{3\lambda}.$$

Also $X > 3$, $K_0/K_2 > \frac{12}{7} > \frac{5}{3}$, so that

$$\frac{K_0}{K_2} = \tfrac{25}{12}, \quad 12X(X+1) = 25(2X+1),$$

$$12X = 19 + \sqrt{661}, \quad X = (3, 1, 2, \ldots),$$

which cannot be the case since any $g > 1$ is a t and all t's are equal. Thus $n = 2$ cannot occur.

Since (i) when $n \geqslant 3$, necessarily $n = 3$ and the sequence of g's is $(\overset{\times}{3}, 1, 1, \overset{\times}{1})$, (ii) $n = 2$ cannot occur, and (iii) t's occur arbitrarily far to the left and right, the sequence of g's must be either

$$(\overset{\times}{3}, 1, 1, \overset{\times}{1}) \quad \text{or} \quad (\overset{\times}{t}, \overset{\times}{1}).$$

From $(\overset{\times}{2}, \overset{\times}{1})$, $K_0 = 2\sqrt{3}$, $K_1 = \sqrt{3}$, $K_0/K_2 = 2$, which is false. From $(\overset{\times}{3}, \overset{\times}{1})$, with $g_0 = 3$, $\mu_1 = \frac{3}{2}$,

$$\frac{\phi(y, z)}{d^3} = \tfrac{3}{2}(3y^2 + 3yz - z^2), \quad \frac{\Delta}{d^4} = \tfrac{189}{20},$$

$$-\frac{\Phi}{d^2} = \tfrac{5}{4}(x + \tau y + \sigma z)^2 + \tfrac{6}{5}(3y^2 + 3yz - z^2),$$

where we may suppose that

$$0 \leqslant \sigma \leqslant \tfrac{1}{2}, \quad -\tfrac{1}{2} < \tau \leqslant \tfrac{1}{2}.$$

Now every positive value of $-\Phi/d^2$ is either $\frac{5}{4}$ or at least 2; every positive value of Φ/d^2 is at least $\frac{3}{4}$. With $x=z=1$, $y=0$,

$$0<\tfrac{5}{4}(1+\sigma)^2-\tfrac{6}{5}<2, \quad \tfrac{5}{4}(1+\sigma)^2-\tfrac{6}{5}=\tfrac{5}{4}, \quad \sigma=\tfrac{2}{5}.$$

If $\tau\leqslant 0$, take $x=0$, $-y=z=1$,

$$-\tfrac{1}{5}\leqslant\tfrac{5}{4}(\sigma-\tau)^2-\tfrac{6}{5}\leqslant-\tfrac{3}{20}.$$

If $\tau>0$, take $x=1$, $-y=z=1$,

$$-\tfrac{3}{20}\leqslant\tfrac{5}{4}(1+\tfrac{2}{5}-\tau)^2-\tfrac{6}{5}<\tfrac{5}{4}.$$

In each case we have a contradiction.

The case $g_1=3$, $\mu_2=\frac{3}{2}$ may be treated in the same way.

Our analysis thus shows that in Case II with the assumption $\Delta\leqslant 10d^4$ the sequence of g's is either $\overset{\times}{(1)}$, discussed already and leading to the form (10), or else $\overset{\times}{3}$, 1, 1, $\overset{\times}{1}$. And here, if 3 is g_0, $\mu_1=\frac{3}{2}$,

$$6F_0=\xi+9, \quad 6S_0=\xi-9, \quad \xi=\sqrt{(165)},$$

$$-\frac{\Phi}{d^2}=\tfrac{5}{4}(x+\tau y+\sigma z)^2+\tfrac{4}{5}(\tfrac{7}{2}y^2+\tfrac{9}{2}yz-\tfrac{3}{2}z^2), \quad \frac{\Delta}{d^4}=\tfrac{33}{4}, \tag{17}$$

$$0\leqslant\sigma\leqslant\tfrac{1}{2}, \quad -\tfrac{1}{2}<\tau\leqslant\tfrac{1}{2}.$$

As above, $\sigma=\frac{2}{5}$. Take $-y=z=1$ and choose the integer x so that $1\leqslant|x+\sigma-\tau|\leqslant\frac{3}{2}$. Then

$$-\tfrac{3}{4}\leqslant\tfrac{5}{4}(x+\sigma-\tau)^2-2\leqslant\tfrac{13}{16}<\tfrac{5}{4},$$

so that $\quad \tfrac{5}{4}(x+\sigma-\tau)^2-2=-\tfrac{3}{4}, \quad |x+\sigma-\tau|=1, \quad \tau=\tfrac{2}{5}.$

Thus

$$\frac{\Psi}{d}=(t+ux+vy+wz)^2-\tfrac{5}{4}(x+\tfrac{2}{5}y+\tfrac{2}{5}z)^2-\tfrac{2}{5}(7y^2+9yz-3z^2), \tag{18}$$

$$-\tfrac{1}{2}<u,\ v,\ w\leqslant\tfrac{1}{2}.$$

We find without trouble that

$$u=\tfrac{1}{2}, \quad w=0, \quad v=0.$$

Replace z by $z-y$. We obtain the form

$$\frac{\Psi}{d}=t^2-x^2+2y^2+z^2+xt-xz-6yz, \quad \frac{\Delta}{d^4}=\tfrac{33}{4}, \tag{19}$$

not null and with minimum 1.

If, however, 3 is g_1, $\mu_2 = \frac{3}{2}$; we obtain

$$(20) \qquad -\frac{\Phi}{d^2} = \tfrac{5}{4}(x+\tau y+\sigma z)^2+\tfrac{4}{5}(-\tfrac{7}{2}y^2+\tfrac{9}{2}yz+\tfrac{3}{2}z^2), \quad \sigma = \tfrac{1}{5},$$

$$(21) \qquad \frac{\Psi}{d} = (t+ux+vy+wz)^2-\tfrac{5}{4}(x+\tau y+\sigma z)^2+\tfrac{2}{5}(7y^2-9yz-3z^2).$$

Put $y = 0$ and argue as in Case I. Then

$$u = \tfrac{1}{2}, \quad w = \tfrac{1}{2}.$$

Put $y = z$ and argue as in Case I. Then

$$v = \tfrac{1}{2}, \quad \tau = -\sigma = -\tfrac{1}{5}.$$

Replace z by $z+y$, t by $t-y$. We obtain the form

$$(22) \qquad \frac{\Psi}{d} = t^2-x^2-z^2-2y^2+xt+zt-6yz, \quad \frac{\Delta}{d^4} = \tfrac{33}{4},$$

equivalent to the negative of the form above.

7. *The case* $n' \geqslant 2d^2$. Write

$$(1) \qquad n = \nu d^2, \quad n' = \nu_1 d^2, \quad d^4 = \xi\Delta,$$

so that

$$(2) \qquad 10\xi \geqslant 1; \quad \nu_1 = 2 \quad \text{or} \quad \geqslant 2{\cdot}21.$$

It is clear that $n' \geqslant n$ and (by Q, Theorem 2, applied to Φ) $3n^3 \leqslant 2d^2\Delta$. Hence $n' = n$ implies that $12\xi \leqslant 1$, whereas $10\xi \geqslant 1$. Thus $n' > n$, Φ does not represent $-n$, no section of Φ can represent $-n$. Note also that any definite binary form χ represented by Φ is negative definite, since Φ has signature -1, and that the minimum of $-\chi$ is necessarily at least n'.

Consider the representation $N(2, 10)$ of Φ. Let m be the absolute minimum of the indefinite binary quadratic form $\phi(y, z)$ of determinant $-n'\Delta$. Then (Q, Theorem 1) (Dickson, *Studies*, Theorems 62, 79)

$$(3) \qquad n'\Delta = \rho m^2, \quad \rho = \tfrac{5}{4}, \ 2, \ 2{\cdot}21 \quad \text{or} \quad \geqslant \tfrac{15}{6}\tfrac{17}{6} > 2{\cdot}24;$$

$$(4) \qquad \nu_1 d^6 = \rho\xi m^2.$$

Now any value of ϕ is the determinant of some indefinite ternary section of Ψ with minimum d. Hence either

$$(5) \qquad d^3 = \tfrac{1}{3}m \quad \text{or} \quad d^3 = \zeta m, \quad \zeta < \lambda = 0{\cdot}30033$$

(since ϕ represents either m or $-m$), by Q, Theorem 2, the other cases of that theorem being excluded since $n' \geqslant 2d^2$.

If $3d^3 = m$, then $n' = 2d^2$, $\nu_1 = 2$ (Q, Lemma 15), and [by (4)]

$$9\rho\xi = 2; \quad 9\rho \leqslant 20; \quad \rho = \tfrac{5}{4} \text{ or } 2 \text{ or } 2{\cdot}21; \tag{6}$$

by (2) and (3).

Case I. $4\rho = 5$, $\phi = m(y^2+yz-z^2)$,

$$\frac{\Psi}{d} = (t+ux+vy+wz)^2-2(x+\tau y+\sigma z)^2-\tfrac{3}{2}(y^2+yz-z^2).$$

We may suppose that

$$-\tfrac{1}{2} < u, v, w, \tau, \sigma \leqslant \tfrac{1}{2}.$$

As in § 5, I, we find that u, v, w are all zero, $\tau = \sigma = \frac{1}{2}$. But now $\Psi(-1, 1, 1, 1) = -\frac{1}{2}d$, whereas $|\Psi/d| \geqslant 1$.

Case II. $\rho = 2$, $\phi = m(y^2+2yz-z^2)$,

$$\frac{\Psi}{d} = (t+ux+vy+wz)^2-2(x+\tau y+\sigma z)^2-\tfrac{3}{2}(y^2+2yz-z^2),$$

u, v, w, τ, σ as in I. Replace z by $z+x-y$, x by $x-y$. We obtain the equivalent form

$$\frac{\Psi}{d} = t^2+z^2-3x^2-3y^2, \quad \Delta = 9d^4, \tag{7}$$

of minimum 1. This form is clearly not equivalent to the form with $\Delta = 9d^4$ found in § 5 (10).

Case III. $\rho = 2{\cdot}21$, $\phi = \frac{1}{5}m(5y^2+11yz-5z^2)$,

$$\frac{\Psi}{d} = (t+\ldots)^2-2(x+\ldots)^2-\tfrac{3}{10}(5y^2+11yz-5z^2),$$

$u, v, \ldots$ as in I. But now $\Psi(-1, 1, 3, 0) = \frac{1}{10}d$, whereas $|\Psi/d| \geqslant 1$.

Excluding then the form (7), we may suppose from now on that (5_2) holds and so

$$\rho\xi = \nu_1\zeta^2 < \lambda^2\nu_1, \quad \rho < 10\nu_1\lambda^2. \tag{8}$$

If ϕ represents $+m$, we may take $C = m$. The form $\chi = \Phi(x, y, 0)$ is negative definite and represents $-n'$, so that, by the remark below (2), the minimum of $-\chi$ is n'. Hence

$$3n'^2 \leqslant 4dm, \quad 3\nu_1{}^2\zeta \leqslant 4. \tag{9}$$

From (8) and (9), by eliminating ζ, we have

$$9\rho\xi\nu_1^3 \leqslant 16, \quad 9\rho\nu_1^3 \leqslant 160, \quad [(2)]. \tag{10}$$

If $\nu_1 = 2$, then (8) gives $\rho < 20\lambda^2 < 2$, $\rho = \frac{5}{4}$.

If $\nu_1 \geqslant 2{\cdot}21$, then (10) gives $\rho < 2$, $\rho = \frac{5}{4}$.

In any case then

$$-\frac{\Phi}{d^2} = \nu_1(x+\tau y+\sigma z)^2+\epsilon(y^2+yz-z^2), \quad \epsilon = \frac{1}{\nu_1\zeta}, \tag{11}$$

where we may suppose that

$$0 \leqslant \sigma \leqslant \tfrac{1}{2}, \quad -\tfrac{1}{2} < \tau \leqslant \tfrac{1}{2}. \tag{12}$$

Case I. $\nu_1 \geqslant 2{\cdot}21$. Every positive value of $-\Phi/d^2$ is at least ν_1 Hence

$$\nu_1\tau^2+\epsilon \geqslant \nu_1, \quad \nu_1(x+\tau+\sigma)^2+\epsilon \geqslant \nu_1 \tag{13}$$

for any integer x. Also

$$\frac{\epsilon^2}{\nu_1^2} = \frac{4}{5\xi\nu_1^3}, \quad \tfrac{9}{16} \leqslant \frac{\epsilon^2}{\nu_1^2} \leqslant \frac{8}{\nu_1^3} < 1, \tag{14}$$

by (8), (10), and (2).

Since $\epsilon < \nu_1$, we have $\nu_1(1+\sigma)^2-\epsilon > 0$, and so

$$\nu_1(1+\sigma)^2-\epsilon \geqslant \nu_1. \tag{15}$$

If $\tau \geqslant 0$, we use

$$2 = (1+\sigma)+\tau+(1-\tau-\sigma) \geqslant \left(1+\frac{\epsilon}{\nu_1}\right)^{\frac{1}{2}}+2\left(1-\frac{\epsilon}{\nu_1}\right)^{\frac{1}{2}},$$

from (13) and (15), whence

$$\frac{\epsilon}{\nu_1} \geqslant \frac{8\sqrt{6}+3}{25} > \left(\frac{8}{\nu_1^3}\right)^{\frac{1}{3}},$$

which contradicts (14).

Thus

$$\tau < 0, \quad -\tau \geqslant \left(1-\frac{\epsilon}{\nu_1}\right)^{\frac{1}{2}} = u, \text{ say.} \tag{16}$$

If $\sigma+\tau \geqslant u$ [by (13_2) with $x = 0$], then

$$\sigma \geqslant 2u > \tfrac{1}{2}.$$

Hence $$-\sigma-\tau \geqslant u, \quad -\tau \geqslant u+\sigma > \tfrac{1}{2},$$

since $u > \frac{1}{4}$ and $$1+\sigma > \left(1+\frac{\epsilon}{\nu_1}\right)^{\frac{1}{2}} \geqslant \frac{\sqrt{7}}{2} > \tfrac{5}{4}.$$

Thus Case I leads to a contradiction.

Case II. $\nu_1 = 2$. Every positive value of $-\Phi/d^2$ is now either 2 or else at least 2·21. Every positive value of Φ/d^2 is at least $\nu \geqslant \frac{3}{4}$. Consider the following values of $-\Phi/d^2$:

(17) $$X = 2(1+\sigma)^2-\epsilon, \quad Y = 2(3-2\sigma)^2-4\epsilon.$$

Since $X \geqslant 0$, $Y \geqslant 0$, we must have

(18) $$X = 2 \quad \text{or} \quad \geqslant 2{\cdot}21, \quad Y = 2 \quad \text{or} \quad \geqslant 2{\cdot}21.$$

From (17),

(19) $$40\sigma = 4X-Y+10, \quad \epsilon = 2(1+\sigma)^2-X,$$

so that,

(20) if $X = 2$, $$5\sigma \leqslant 2, \quad 25\epsilon \leqslant 48;$$

(21) if $X \geqslant 2{\cdot}21$, $$\sigma \leqslant 0{\cdot}421, \quad 25\epsilon < 48.$$

Consider now $$Z = 2(1+\tau-\sigma)^2-\epsilon.$$

If $Z \geqslant 0$, then $Z \geqslant 2$,

$$(1+\tau-\sigma)^2 \geqslant 1+\tfrac{1}{2}\epsilon, \quad 1+\tau-\sigma > \tfrac{5}{4}, \quad \tau > \tfrac{1}{4}+\sigma > \tfrac{1}{2},$$

by (18), since $\frac{1}{2}\epsilon \geqslant \frac{3}{4}$ by (14), and $X \geqslant 2$.

Thus $$Z < 0, \quad 2(1+\tau-\sigma)^2-\epsilon \leqslant -\tfrac{3}{4},$$

(22) $$1+\tau-\sigma \leqslant v, \quad v = (\tfrac{1}{2}\epsilon-\tfrac{3}{8})^{\frac{1}{2}}.$$

Now

(23) $$2\tau^2+\epsilon \geqslant 2, \quad |\tau| \geqslant (1-\tfrac{1}{2}\epsilon)^{\frac{1}{2}} = w,$$

say.

If $\tau \geqslant 0$, then, by (22) and (23),

$$\sigma = 1+\tau-(1+\tau-\sigma) \geqslant 1+w-v > 1+\tfrac{1}{5}-\tfrac{31}{40} > \sigma,$$

by (20) or (21), which is a contradiction.

Thus

(24) $$\tau < 0, \quad -\tau \geqslant w.$$

Consider now

$$2(\tau+\sigma)^2+\epsilon \geqslant 2, \quad |\tau+\sigma| \geqslant w. \tag{25}$$

If $-\tau-\sigma \geqslant w$, we have

$$-\tau \geqslant \sigma+w \geqslant (1+\tfrac{1}{2}\epsilon)^{\frac{1}{2}}+(1-\tfrac{1}{2}\epsilon)^{\frac{1}{2}}-1 > \tfrac{1}{2}, \tag{26}$$

since $25\epsilon \leqslant 48$, whereas $-\tau < \frac{1}{2}$.

We now discuss the case

$$\tau+\sigma \geqslant w, \quad \sigma \geqslant w-\tau \geqslant 2w, \quad \sigma^2 \geqslant 4-2\epsilon. \tag{27}$$

From these inequalities we have [(17), (18), (27)],

$$2(1+\sigma)^2 = X+\epsilon \geqslant 2+\epsilon \geqslant 4-\tfrac{1}{2}\sigma^2,$$

$$5\sigma^2+8\sigma-4 \geqslant 0, \quad \sigma \geqslant \tfrac{2}{5};$$

and

$$2(3-2\sigma)^2 = Y+4\epsilon \geqslant 2+4\epsilon \geqslant 10-2\sigma^2,$$

$$5\sigma^2-12\sigma+4 \geqslant 0, \quad \sigma \leqslant \tfrac{2}{5}.$$

Hence

$$\sigma = \tfrac{2}{5}, \quad X = Y = 2, \quad \epsilon = \tfrac{4}{25},$$

$$-\tau \geqslant \tfrac{1}{5}, \quad \tfrac{2}{5}+\tau = \sigma+\tau \geqslant \tfrac{1}{5}, \quad \tau = -\tfrac{1}{5}.$$

We obtain, therefore,

$$-\frac{\Phi}{d^2} = 2(x-\tfrac{1}{5}y+\tfrac{2}{5}z)^2+\tfrac{48}{25}(y^2+yz-z^2), \tag{28}$$

and so, by putting $y = 0$ in Ψ/d, the indefinite ternary form

$$(t+ux+wz)^2-2x^2-\tfrac{8}{5}xz+\tfrac{8}{5}z^2, \quad -\tfrac{1}{2} < u,\ w \leqslant \tfrac{1}{2}, \tag{29}$$

of minimum 1.

From $z = 0$ we obtain $u = 0$ and then, by $z = x$, we have $w = 0$, and a contradiction results with $t = 3$, $x = 2$, $z = 1$.

Thus "ϕ represents $+m$" leads to a contradiction. Hence ϕ cannot represent $+m$ and so, by Markoff's theorem (Q, Theorem 1),

$$4\rho \geqslant 9, \quad 4\zeta^2\nu_1 \geqslant 9\xi, \tag{30}$$

by (8).

We obtain now some inequalities for ν. We use the representation P_2, $a = n$. The form ϕ is now negative definite; we may suppose it reduced with $-C$ the minimum of $-\phi$. We then have

$$d^2 \leqslant \tfrac{4}{3}n, \quad 9n^2 \leqslant -4Cd, \quad 3C^2 \leqslant 4n\Delta, \tag{31}$$

the first inequality from (a, u, d) of minimum d and determinant α; the second from the indefinite form (α, τ, β) of minimum α, determinant dC, which does not represent $-n$; the third from the definite form ϕ.

Also from $\Psi(x, y, 0, t)$, indefinite of determinant C, we have

$$d^3 = -\eta C, \quad \eta = \tfrac{1}{3} \quad \text{or} \quad \eta < \lambda, \tag{32}$$

by Q, Theorem 2, the other possibilities of that theorem being excluded since $n' \geqslant 2d^2$.

If $3\eta = 1$, then (Q, Lemma 15) we may take

$$(\alpha, \tau, \beta) = d^2(x^2+2xy-y^2), \quad n' = 2d^2.$$

Take the representation N with y as leading variable in place of x,

$$\beta\Phi = (\quad)^2+d(Az^2-2Gzx+Cx^2).$$

The minimum of the new indefinite form $\phi = (A, -G, C)$ is clearly $-C$, and we have $3d^3 = -C$, in contradiction with (5_2).

Thus $3\eta = 1$ is impossible, so (32_3) holds. From (1) and (31), we have

$$9\nu^2\eta \leqslant 4, \quad 3\xi \leqslant 4\nu\eta^2, \tag{33}$$

$$\frac{3\xi}{4\eta^2} \leqslant \nu \leqslant \left(\frac{4}{9\eta}\right)^{\frac{1}{2}}, \tag{34}$$

$$(9\xi)^{\frac{2}{3}} \leqslant 4\eta < 4\lambda. \tag{35}$$

Return now to the form $\Psi(x, y, 0, t)$ employed in (32):

$$d\Psi(x, y, 0, t) = (dt+ux+vy)^2+\psi(x, y),$$

$$\psi = \alpha x^2+2\tau xy+\beta y^2, \quad \alpha\beta-\tau^2 = dC.$$

We may identify the indefinite form ψ with any member

$$\psi_{2i} = [A_{2i}, B_{2i}, -A_{2i+1}] \quad (A_i > 0)$$

of the chain of reduced forms to which it belongs.

Any value of ψ is a value of Φ, so that $(-)^i A_i$ is a value of Φ. Every positive value of Φ is at least n, every negative value of Φ is at most $-n'$. Hence

$$A_{2i} \geqslant n = \nu d^2, \quad A_{2i+1} \geqslant n' = \nu_1 d^2. \tag{36}$$

Since

$$K_i^2 = -\frac{4dC}{A_{i+1}^2} = \frac{4d^4}{\eta A_{i+1}^2},$$

we obtain, by (33_2), (35_2), and (2),

$$K_{2i-1}^2 \leqslant \frac{4}{\nu^2\eta}, \quad K_{2i-1} < \frac{80\lambda^3}{3},$$

$$K_{2i-1} < 4{\cdot}4, \quad g_{2i-1} \leqslant 4; \tag{37}$$

and, by (35_1), (30_2), (5_2),

$$K_{2i}^2 \leqslant \frac{4}{\nu_1^2\eta}, \quad K_{2i} < 16\lambda^2(9\xi)^{-\frac{1}{2}},$$

$$K_{2i} < 1+\tfrac{2}{3}, \quad g_{2i} = 1. \tag{38}$$

Now

$$K_i > g_i + \frac{1}{1+g_{i-1}} + \frac{1}{1+g_{i+1}}, \tag{39}$$

so that $$4{\cdot}4 > K_{2i-1} > g_{2i-1} + \tfrac{1}{2} + \tfrac{1}{2}, \quad g_{2i-1} \leqslant 3.$$

Every $g_{2i-1} \geqslant 2$. For, if $g_{2i-1} = 1$, we have

$$1+\tfrac{1}{4}+\tfrac{1}{2} < K_{2i} < 1+\tfrac{2}{3},$$

a contradiction.

One of g_{2i-1}, g_{2i+1} *is* 2 *and the other is* 3. For, if both are 2, we have

$$1+\tfrac{1}{3}+\tfrac{1}{3} < K_{2i} < 1+\tfrac{2}{3};$$

and, if both are 3,

$$4{\cdot}4 > K_{2i-1} = (3,\ 1,\ 3,\ \ldots)+(0,\ 1,\ g_{2i-3},\ \ldots) > (3,\ 1,\ 3)+(0,\ 1,\ 2) = \tfrac{5\,3}{1\,2};$$

we have a contradiction in each case.

But, now, if $g_{2i-1} = 3$, both g_{2i-3} and g_{2i+1} are 2, and g_{2i-5}, g_{2i+3} are 3, and so

$$K_{2i-1} = (3,\ 1,\ 2,\ 1,\ 3,\ \ldots)+(0,\ 1,\ 2,\ 1,\ 3,\ \ldots)$$
$$> (3,\ 1,\ 2,\ 1,\ 3)+(0,\ 1,\ 2,\ 1,\ 3) = \tfrac{6\,7}{1\,5},$$

whereas $$K_{2i-1} < 4{\cdot}4 < \tfrac{6\,7}{1\,5}.$$

Thus the case "ϕ does not represent $+m$" also leads to a contradiction, and so the conditions $n' \geqslant 2d^2$, $10\xi \geqslant 1$ lead only to the form (7) with $9\xi = 1$.

The proof of our theorem is thus completed.

TWO-DIMENSIONAL POTENTIAL PROBLEMS FOR THE SPACE OUTSIDE A RECTANGLE

By W. G. Bickley.

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Introduction and summary.

In a recent paper* general solutions of the common potential problems were given for the two-dimensional region outside a single closed boundary, in terms of the conformal transformation which represents the region outside the boundary on a half-plane, the boundary itself corresponding to the real axis. It was there pointed out, however, that, when the transformation formula had branch-points on the real axis, the interpretation and evaluation of other formulae given might present difficulties. Although a perfectly general solution taking account of possible branch-points has not yet been obtained, individual cases can be dealt with; and the present paper is devoted to a detailed account of one of these, namely where the boundary is a rectangle.

In § 1 the transformation formula is examined and put into a form in which the comparison region is that outside a circle (the half-plane was used in the previous paper only on account of its convenience for the purpose of obtaining general formulae), and the relation between the planes is expressed both in terms of series, which are useful in numerical calculations, as well as in terms of elliptic functions, which facilitate the analytical developments.

In § 2 the simpler potential distributions, whose hydrodynamical interpretations are circulation about, and flow past, an infinite rectangular prism, are obtained and illustrated.

In § 3 the problem corresponding hydrodynamically to the rotation of such a prism about its axis is solved and illustrated.

* W. G. Bickley, *Phil. Trans. Royal Soc.* (A), 228 (1929), 235–274.

The analysis of § 3 required the values of certain integrals containing products of Jacobian elliptic functions and, since I have not found them in standard treatises, a brief account of their derivation is given in the Appendix.

1. *Formulae of conformal transformation.*

The transformation formula which transforms the two-dimensional region outside a rectangle in the z-plane into the upper half of the ζ-plane, the boundary transforming into the real axis, is*

$$\frac{dz}{d\zeta} = C\,\frac{\sqrt{(\zeta^2-a^2)(\zeta^2-a^{-2})}}{(1+\zeta^2)^2}, \tag{1.01}$$

correspondence between various points being as indicated in Fig. 1, a being the value of ζ at A, and C a constant depending upon the scale and orientation of the rectangle.

If we write $\zeta = -\tan\frac{1}{2}\tau$, with $a = \tan\frac{1}{2}\alpha$, we find that

$$\begin{aligned} dz/d\tau &= -\tfrac{1}{2}C\operatorname{cosec}\alpha\{2(\cos 2\tau-\cos 2\alpha)\}^{\frac{1}{2}} \\ &= -\tfrac{1}{2}C\operatorname{cosec}\alpha\, e^{i\tau}(1-2e^{-2i\tau}\cos 2\alpha+e^{-4i\tau})^{\frac{1}{2}}. \end{aligned} \tag{1.02}$$

In the last expression, when the imaginary part of τ is negative, we may expand the radical into a series thus:

$$(1-2e^{-2i\tau}\cos 2\alpha+e^{-4i\tau})^{\frac{1}{2}} = \sum_{n=0}^{\infty} a_n e^{-2ni\tau}; \tag{1.03}$$

in this we write $\cos 2\alpha = \mu$ and derive from the known expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$ in terms of Legendre polynomials the following expressions for the coefficients:

$$\begin{aligned} a_0 &= 1, \quad a_1 = -\mu, \\ a_n &= P_n(\mu)-2\mu P_{n-1}(\mu)+P_{n-2}(\mu) \\ &= \{P_n(\mu)-\mu P_{n-1}(\mu)\}/n. \end{aligned} \tag{1.04}$$

Since, however, tables of P_n for large n are not readily available, it is useful to have a recurrence formula for the a_n. This formula is readily found by familiar methods to be

$$(n+1)a_{n+1}-(2n-1)\mu a_n+(n-2)a_{n-1} = 0. \tag{1.05}$$

We may also notice that, if $\mu = \pm 1$, the series reduces to $1\mp e^{-2i\tau}$, *i.e.*, for these values of μ, $a_n = 0$ if $n \geqslant 2$; for such values of n, therefore, a_n

* *Loc cit.*, 245.

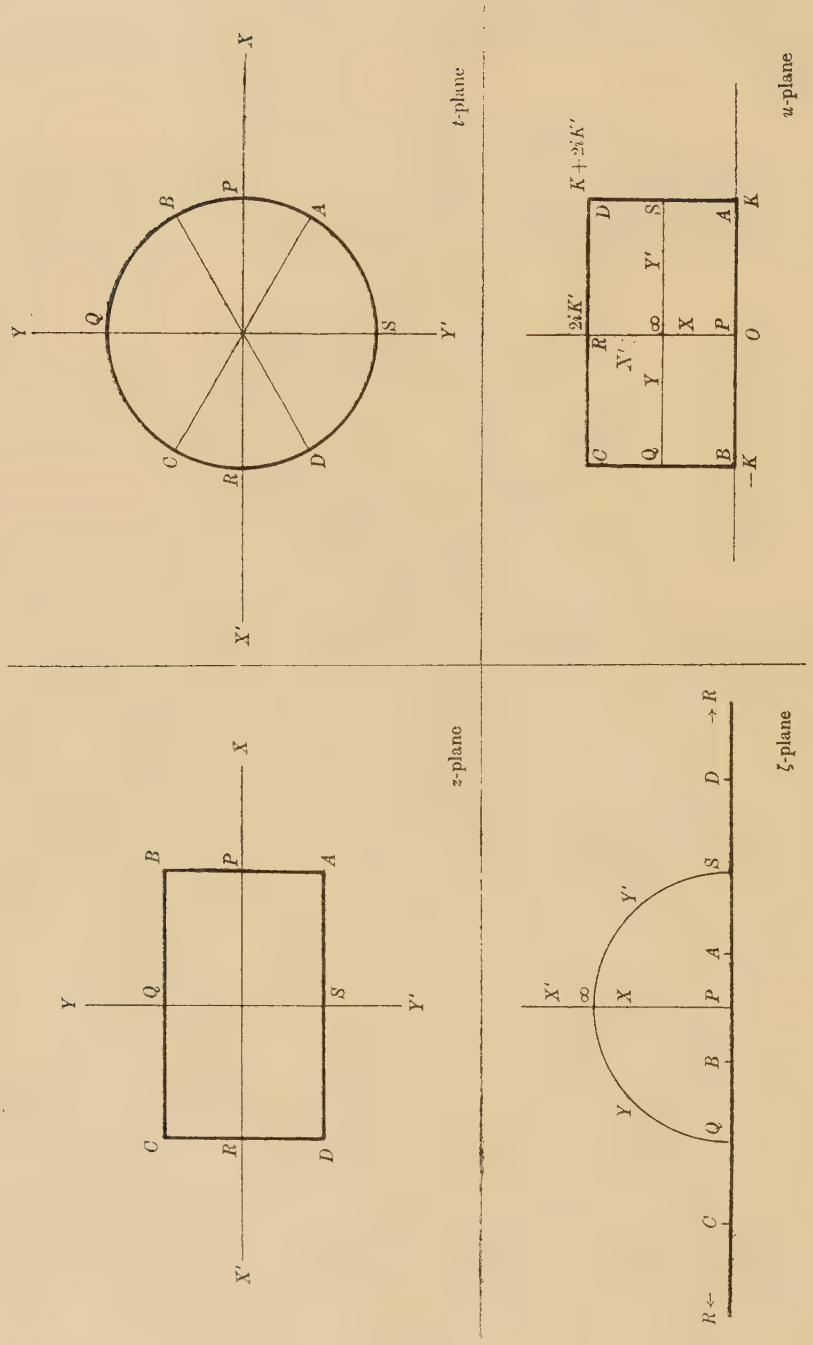

Fig. 1.

contains $1-\mu^2$ as a factor. A few early values of a_n are as follows:

$$a_2 = (1-\mu^2)/2, \quad a_3 = \mu(1-\mu^2)/2,$$
$$a_4 = -(1-\mu^2)(1-5\mu^2)/8, \quad a_5 = -\mu(1-\mu^2)(3-7\mu^2)/8,$$
$$a_6 = (1-\mu^2)(1-14\mu^2+21\mu^4)/16, \quad \ldots.$$

We may now write the transformation formula in the form

$$\frac{dz}{d\tau} = -\tfrac{1}{2}C \operatorname{cosec}\alpha \left\{e^{i\tau} + \sum_1^\infty a_n e^{-(2n-1)i\tau}\right\}, \tag{1.06}$$

so that

$$z = \tfrac{1}{2}iC \operatorname{cosec}\alpha \left\{e^{i\tau} - \sum_1^\infty a_n e^{-(2n-1)i\tau}/(2n-1)\right\} + D$$

$$= \tfrac{1}{2}iC \operatorname{cosec}\alpha \left\{e^{i\tau} + \sum_1^\infty c_n e^{-(2n-1)i\tau}\right\} + D, \tag{1.07}$$

where $c_n = -a_n/(2n-1)$, and D is a constant of integration. The recurrence formula for the c_n is

$$(n+1)(2n+1)c_{n+1} - (2n-1)^2\mu c_n + (2n-3)(n-2)c_{n-1} = 0, \tag{1.08}$$

with $$c_0 = 1, \quad c_1 = \mu.$$

If we write $i\tau = \beta + i\theta$, $\rho = e^\beta$, and $t = \rho e^{i\theta}$, the region outside the rectangle in the z-plane will be conformally represented upon the region outside the unit circle in the t-plane, the boundaries corresponding as in Fig. 1, and the points at infinity also corresponding. In order that the centre of the circle may be $t = 0$, we must have, in (1.07), $D = 0$.

The series for z just obtained is convenient for plotting the fundamental net, but it remains to determine a, which fixes the shape of the rectangle, and C, which fixes its size and orientation. We therefore integrate (1.01) in finite form. This can be done, in terms of Jacobian elliptic functions, by writing $\zeta = a\,\mathrm{sn}(u, a^2)$; but a simpler result is obtained by using (1.02), when the modulus of the elliptic functions is $k = \sin\alpha$. The change from a^2 to k is equivalent to a Landen transformation. We write

$$\sin\tau = -k\,\mathrm{sn}\,(u, k), \tag{1.09}$$

so that

$$\left.\begin{aligned} \cos\tau &= \mathrm{dn}\,(u, k), \\ \sqrt{\{2(\cos 2\tau - \cos 2\alpha)\}} &= 2k\,\mathrm{cn}\,(u, k) \end{aligned}\right\}. \tag{1.10}$$

Hence

$$dz = Ck\,\mathrm{cn}^2 u\,du = C/k\,.\,(\mathrm{dn}^2 u - k'^2)\,du, \tag{1.11}$$

and

$$z = C/k\,.\,\{\mathrm{Z}(u) + (E/K - k'^2)u\} + D, \tag{1.12}$$

D being (another) constant of integration. If $2a$ and $2b$ are the sides of the rectangle (the old significance of a is no longer required), we see that,

at P, $$u = 0, \quad z = a,$$

so that $$D = a;$$

at A, $$u = K, \quad z = a - ib,$$

so that $$a - ib = C(E - k'^2 K)/k + a,$$

i.e., $$C = -ibk/(E - k'^2 K); \tag{1.13}$$

at R, $$u = 2iK', \quad z = -a,$$

so that $$-a = C\{-i\pi + 2iK'(E - k'^2 K)\}/(kK) + a,$$

i.e., $$-2a = 2iC(K'E - k'^2 K'K - \tfrac{1}{2}\pi)/(kK)$$

$$= -2iC(E' - k^2 K')/k, \tag{1.14}$$

by Legendre's relation $KE' + K'E - KK' = \frac{1}{2}\pi$.

Thus $$C = -iak/(E' - k^2 K'). \tag{1.15}$$

It will be convenient to denote $E - k'^2 K$ and $E' - k^2 K'$ by L and L' respectively. Then

$$C = -ibk/L = -iak/L', \tag{1.16}$$

so that $$b/a = L/L'. \tag{1.17}$$

This is the equation for the modulus, in terms of the ratio of the sides of the rectangle. It is evident that, when $b \neq a$, we have a choice of two (complementary) moduli, and we shall see later that it is more convenient to work with the smaller of these. We give, in Fig. 2, a graph showing the relation between k^2 and b/a, as far as $k^2 = 0{\cdot}5$ (which, by what has just been said, is sufficient), while in Table I values of b/a at intervals of 5° in the modular angle, up to 45°, are shown. It is also evident from (1.17) that the special

TABLE I.

a	L	L'	L/L'
0°	0	1	0
5°	0·005973	0·983553	0·006073
10°	0·023773	0·945027	0·025156
15°	0·053064	0·890978	0·059557
20°	0·093280	0·825401	0·113012
25°	0·143664	0·751460	0·191153
30°	0·203149	0·671927	0·302338
35°	0·277325	0·589271	0·470624
40°	0·344626	0·505798	0·681351
45°	0·423607	0·423607	1·000000

case of the square is given by $k = k'$, or $\alpha = 45°$. There would seem to be no other case where a rational value of b/a corresponds to a simple value of k or α.

The formula for z may now be written

$$z = \frac{b}{L}[L' + i\{\mathrm{Z}(u) + Lu/K\}], \qquad (1.18)$$

which transforms the area outside a rectangle in the z-plane into the area inside a rectangle in the u-plane. Except in the special case of the square,

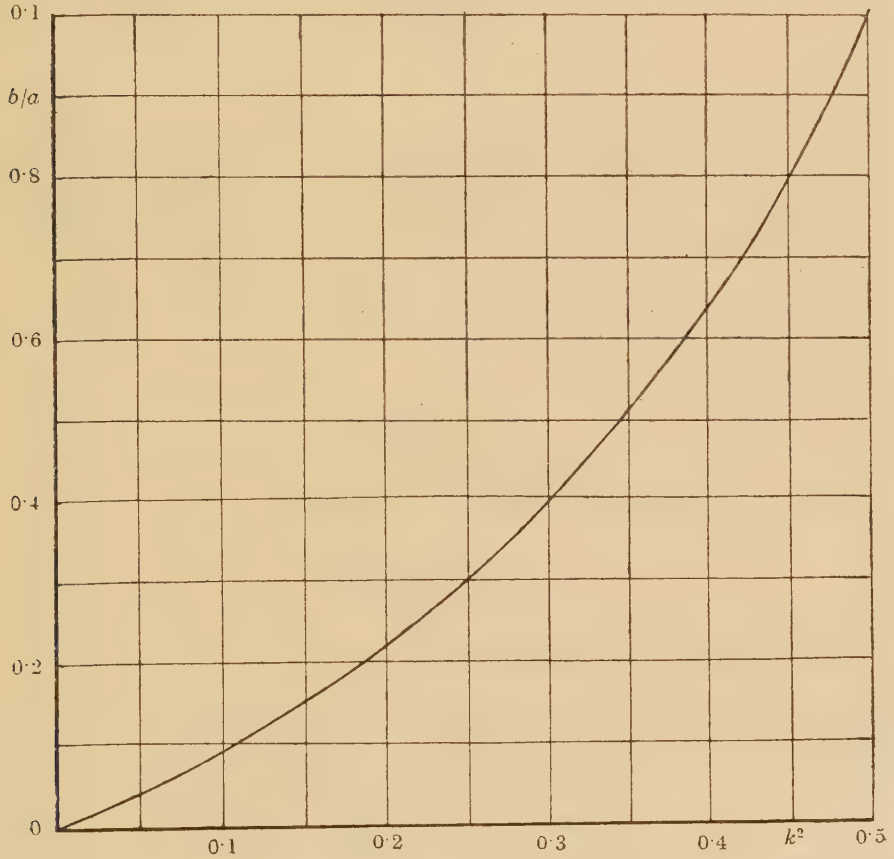

Fig. 2.

however, these rectangles are not similar, nor is the relation between the ratios of their sides simple.

It is convenient here to investigate the variation of x and y on the boundary in the z-plane in terms of the angle θ in the t-plane. Along PB, θ varies from 0 to α while u varies from 0 to $-K$. Writing $u = -u'$, we have

$$\sin\theta = k \operatorname{sn} u'.$$

Therefore $$dy = b/L \,.\, k^2 \operatorname{cn}^2 u' \, du', \tag{1.19}$$

so that $$y = b/L \,.\, \{Z(u') + Lu'/K\}, \tag{1.20}$$

while also $$d\theta = k \operatorname{cn} u' \, du'. \tag{1.21}$$

Along QB it is convenient to write $u = -K + iK' - iv$. We then have

$$\cos\theta = k' \operatorname{sn} v, \tag{1.22}$$

$$\left.\begin{aligned} dx &= a/L' \,.\, k'^2 \operatorname{cn}^2 v \, dv \\ x &= a/L' \,.\, \{Z(v) + L' v/K'\}, \end{aligned}\right\} \tag{1.23}$$

and $$d\theta = -k' \operatorname{cn} v \, dv, \tag{1.24}$$

the modulus for the elliptic functions of v being $k' = \cos\alpha$. If we put $\theta' = \frac{1}{2}\pi - \theta$, we see that the relations between u', y, and θ for PB are exactly complementary to the relations between v, x, and θ' for QB, as was to be expected. Symmetry makes it unnecessary to consider other quadrants in detail.

The special case of the square, however, deserves a little extra consideration. Here we have

$$\alpha = 45^\circ, \quad \mu = 0, \quad k = k' = 1/\sqrt{2},$$

$$K = K' = K_1, \quad L = L' = L_1.$$

Moreover, since Legendre's relation can be written

$$\frac{L}{K} + \frac{L'}{K'} = \frac{\pi}{2KK'},$$

we deduce that $$L_1 = \pi/(4K_1). \tag{1.25}$$

Again, $$\frac{dz}{d\tau} = \frac{ib}{2L_1} \sqrt{(2\cos 2\tau)}$$

$$= \frac{ib}{2L_1} e^{i\tau} \left\{1 + \tfrac{1}{2} e^{-4i\tau} - \frac{1\,.\,1}{2\,.\,4} e^{-8i\tau} \ldots\right\}, \tag{1.26}$$

i.e., all the a_n with odd suffixes vanish. Thus

$$z = \frac{b}{2L} \left\{e^{i\tau} - \frac{1}{2\,.\,3} e^{-3i\tau} + \frac{1\,.\,1}{2\,.\,4\,.\,7} e^{-7i\tau} - \ldots\right\}, \tag{1.27}$$

the general term being

$$(-)^n \frac{1\,.\,1\,.\,3\,.\,5 \ldots (2n-3)}{2\,.\,4\,.\,6\,.\,8 \ldots 2n} \frac{e^{-(4n-1)i\tau}}{4n-1}.$$

Putting $\tau = 0$ and remembering that $a = b$, we have

$$2L_1 = \left(1 - \frac{1}{2.3} + \frac{1.1}{2.4.7} - \frac{1.1.3}{2.4.6.11} + \ldots\right). \tag{1.28}$$

If we put $\tau = \frac{1}{4}\pi$, and equate real and imaginary parts, we find that

$$2L_1\sqrt{2} = \left(1 + \frac{1}{2.3} + \frac{1.1}{2.4.7} + \frac{1.1.3}{2.4.6.11} + \ldots\right). \tag{1.29}$$

Comparing these two results, we deduce that

$$+\frac{1}{2.3} + \frac{1.1}{2.4.7} + \frac{1.1.3}{2.4.6.11} + \ldots$$

$$= \sqrt{2}\left(1 - \frac{1}{2.3} + \frac{1.1}{2.4.7} - \frac{1.1.3}{2.4.6.11} + \ldots\right). \tag{1.30}$$

Another expression for L_1 can be obtained from the fact that

$$2L_1 = \int_0^{\frac{1}{4}\pi} (\cos 2\theta)^{-\frac{1}{2}}\, d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\cos\phi)^{-\frac{1}{2}}\, d\phi$$

$$= \tfrac{1}{4}\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{4}}\, dx,$$

where $x = \sin^2\phi$. The last integral is

$$\tfrac{1}{4}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}, \tag{1.31}$$

which agrees with the value given in Table I.

2. *Fundamental net and elementary potential distributions.*

Formulae in terms of elliptic functions are of little use for points off the boundary, both because they are complicated and also because tables are not easy to use, since, when making the calculations preparatory to plotting the fundamental net, it is from values of β and θ that one starts. We therefore revert to (1.07), which can now be written

$$z = \frac{b}{2L}\left(e^{i\tau} + \sum_1^\infty c_n e^{-(2n-1)i\tau}\right), \tag{2.01}$$

giving

$$x = \frac{b}{2L}\left\{e^{\beta}\cos\theta + \sum_1^\infty c_n e^{-(2n-1)\beta}\cos(2n-1)\theta\right\}, \tag{2.02}$$

$$y = \frac{b}{2L}\left\{e^{\beta}\sin\theta - \sum_1^\infty c_n e^{-(2n-1)\beta}\sin(2n-1)\theta\right\}. \tag{2.03}$$

These series converge rapidly for moderate values of β, and, even when $\beta = 0$, the terms decrease sufficiently rapidly for them to be of service in calculating x and y.

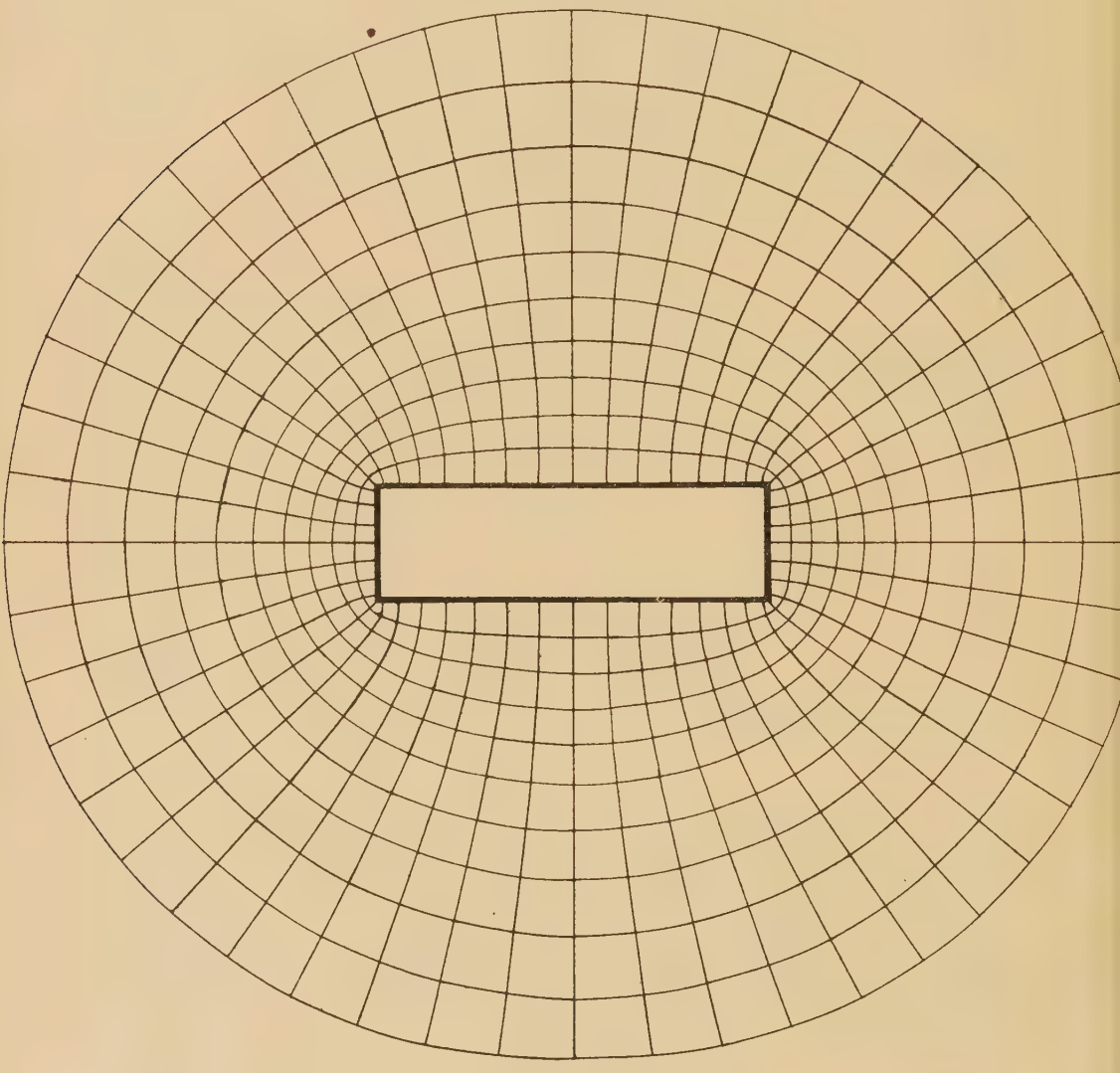

Fig. 3.

The curves $\beta = \text{const.}$ and $\theta = \text{const.}$ constitute the fundamental net. They are plotted in Fig. 3, for $\alpha = 30^\circ$ $(b/a = 0{\cdot}3023)$*. The interval is

* Figures corresponding to Figs. 3, 4, and 5 have also been drawn for the square, but have not been reproduced.

$\pi/24$ for both β and θ. They can be interpreted as

(a) Equipotentials and lines of force for a charged conducting prism in space;

(b) Stream-lines and equipotentials for two-dimensional irrotational fluid flow round a prism.

In the case of the charged prism, the surface density becomes infinite at the corners. The surface density σ is given by

$$\sigma = \frac{e}{2\pi}\left|\frac{d\tau}{dz}\right| \tag{2.04}$$

$$= \frac{eL}{2\pi b\sqrt{(\sin^2\alpha - \sin^2\theta)}}, \tag{2.05}$$

e being the charge per unit length. The average surface density is

$$\bar{\sigma} = \frac{e}{4(a+b)}, \tag{2.06}$$

so that

$$\frac{\sigma}{\bar{\sigma}} = \frac{2(L+L')}{\pi\sqrt{(\sin^2\alpha - \sin^2\theta)}}. \tag{2.07}$$

The variation of $\sigma/\bar{\sigma}$ along the sides of the prism is shown in Fig. 4, for the

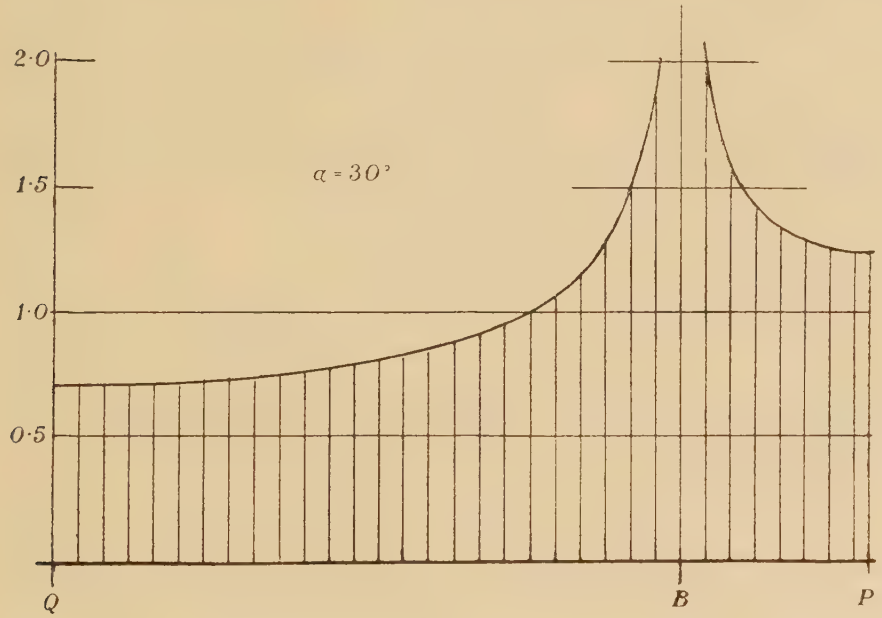

Fig. 4.

case already treated, the values shown relating to two half sides meeting in a corner which has been "flattened out" to facilitate comparison.

By means of Leathem's formula*, and the graphs given by the author†, stream-lines for flow past the prism, or equipotentials and lines of force for a conducting prism in an otherwise uniform field, are easily drawn on the fundamental net. One case is shown in Fig. 5. It may be worth while mentioning the actual process adopted to produce these figures. A tracing of the fundamental net was first made, and from it several ferro-gallic prints were obtained. On one of these, with the help of the above-mentioned graphs, the stream-lines were sketched in, and inked. The lines of the fundamental net were then bleached by a solution of oxalic acid, and the stream-lines, together with the inked boundary, remained.

If the prism is in motion in a fluid otherwise at rest, the impulse per unit length can be found from formulae given by the author‡ involving the coefficients of the expansion of (1 . 01) valid near $\zeta = i$. If U is the velocity of the prism, at an angle γ to the x-axis, we find that

$$I = \tfrac{1}{2}\pi\rho b^2 U(e^{i\gamma} - \mu e^{-i\gamma})/L^2, \tag{2 . 08}$$

ρ being the fluid density, and this including the momentum of the fluid that the prism would enclose. The inertia coefficients are thus

$$\left.\begin{aligned} m_x &= \tfrac{1}{2}\pi\rho b^2(1-\mu)/L^2 = \pi\rho ab(1-\mu)/(2LL'), \\ m_y &= \tfrac{1}{2}\pi\rho b^2(1+\mu)/L^2 = \pi\rho ab(1+\mu)/(2LL'). \end{aligned}\right\} \tag{2 . 09}$$

The inertia of the enclosed fluid would be $4\rho ab$ per unit length. The effective addition to the inertia due to the surrounding fluid thus has components

$$4\rho ab\{\pi(1\mp\mu)/(8LL') - 1\}. \tag{2 . 10}$$

In the case of a square prism, this is a fraction

$$\tfrac{1}{8}\pi/L_1^2 - 1 = 1{\cdot}1884\ldots \tag{2 . 11}$$

of the mass of the enclosed fluid, slightly greater than for a circular cylinder. In the case illustrated, $\alpha = 30°$, the fractional increases are

$$\left.\begin{aligned} &\pi/(16LL') - 1 = 0{\cdot}4384\ldots, \\ \text{and} \quad &3\pi/(16LL') - 1 = 3{\cdot}3153\ldots. \end{aligned}\right\} \tag{2 . 12}$$

* *Phil. Mag.*, 35 (1918), 119–130.
† *Loc. cit.*, 274.
‡ *Loc. cit.*, 262.

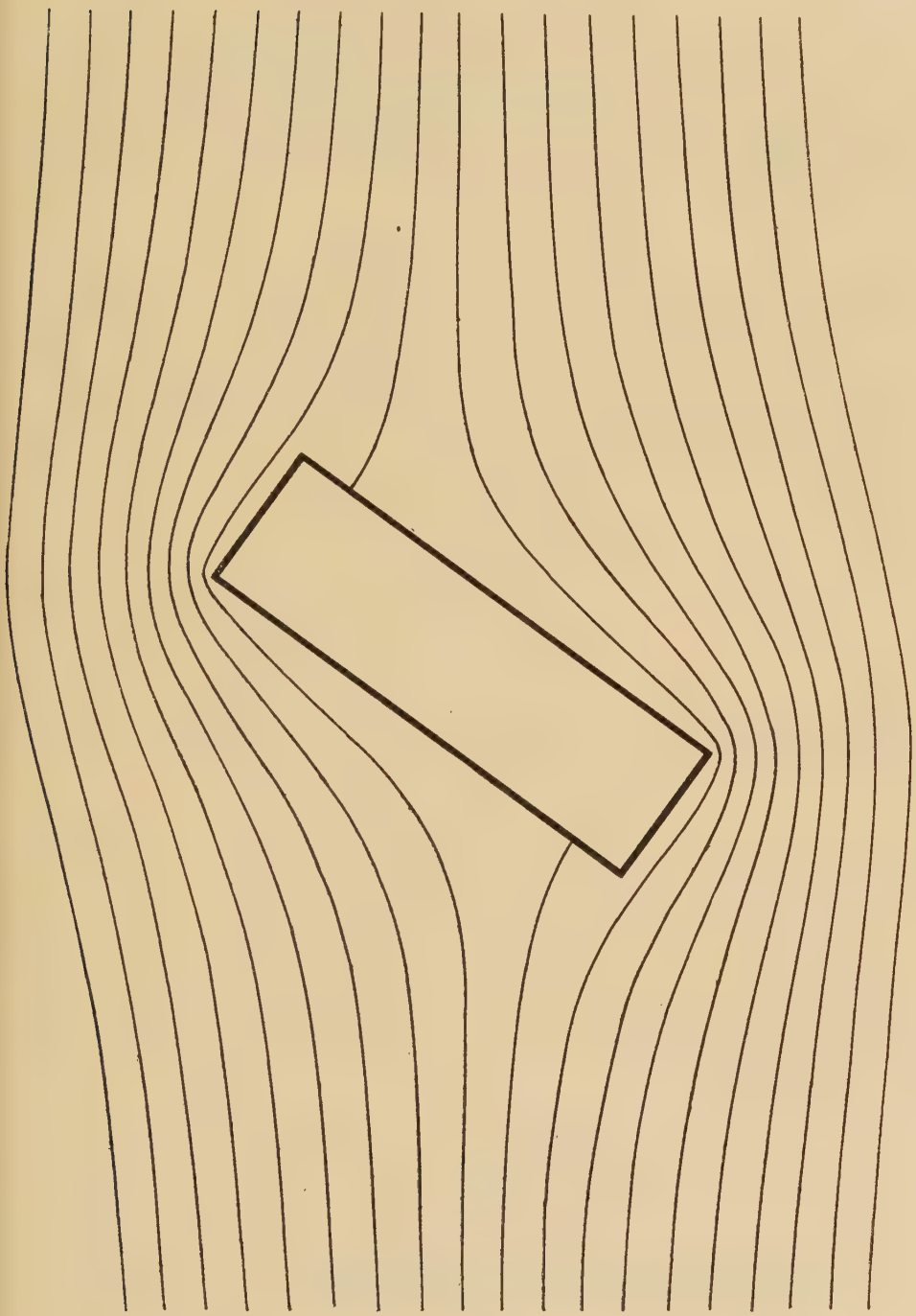

Fig. 5.

3. *Rotating prism in a fluid.*

Since any function harmonic in the z-plane transforms into a function also harmonic in the t-plane, and since the regions at infinity in the two planes correspond, any function harmonic in the z-plane, analytic at all points outside the rectangle, and finite at infinity, must be expressible in the form

$$\phi = \sum_0^\infty e^{-n\beta}(A_n \cos n\theta + B_n \sin n\theta), \tag{3.01}$$

and thus the determination of the A_n and B_n is reduced to that of finding the Fourier coefficients of the boundary values of ϕ. An important case is when the boundary value of ϕ is $|z|^2$. This gives the solution of the problem of the rotating prism surrounded by fluid, and is also of importance in biharmonic analysis*.

In the case of the rectangle, the boundary values of $|z|^2$ in the first quadrant are

$$\left.\begin{aligned} |z|^2 &= \frac{b^2}{L^2}[L'^2 + \{\mathrm{Z}(u) + Lu/K\}^2] \quad (0 \leqslant \theta \leqslant \alpha), \\ &= \frac{b^2}{L^2}[L^2 + \{\mathrm{Z}(v) + L'v/K'\}^2] \quad (\alpha \leqslant \theta \leqslant \tfrac{1}{2}\pi), \end{aligned}\right\} \tag{3.02}$$

values in other quadrants corresponding. It is also evident, taking into account considerations of symmetry, that these may be expanded in the Fourier series

$$|z|_0^2 = \frac{b^2}{L^2}\left(\gamma_0 + \sum_1^\infty \gamma_m \cos 2m\theta\right), \tag{3.03}$$

and the required harmonic function will then be

$$\phi = \frac{b^2}{L^2}\left(\gamma_0 + \sum_1^\infty \gamma_m e^{-2m\beta} \cos 2m\theta\right). \tag{3.04}$$

By direct squaring of the series for x and y given by (2.02) and (2.03), we find that

$$4\gamma_0 = 1 + \sum_1^\infty c_r^2, \quad 4\gamma_m = 2\left(c_m + \sum_{r=1}^\infty c_r c_{m+r}\right). \tag{3.05}$$

These series, however, converge rather slowly, and Love† has obtained,

* Love, *Proc. London Math. Soc.* (2), 29 (1929), 189.

† *Loc. cit.* Love's analysis is rather lengthy, and it can be considerably shortened by methods similar to those which follow.

for the *interior* of a rectangle, the γ_m in finite terms when $m \geqslant 1$, and also a rapidly convergent form for γ_0. In our case,

$$\gamma_m = \frac{4}{\pi}\Big[\int_0^{\alpha}[L'^2+\{\mathrm{Z}(u)+Lu/K\}^2]\cos 2m\theta\, d\theta + \int_{\alpha}^{\frac{1}{2}\pi}[L^2+\{\mathrm{Z}(v)+L'v/K'\}^2]\cos 2m\theta\, d\theta\Big] \quad (3.06)$$

$$= \frac{4}{\pi}\{f(\alpha)+(-)^m f(\tfrac{1}{2}\pi-\alpha)\}, \quad (3.07)$$

where $$f(\alpha) = \int_0^{\alpha}[L'^2+\{\mathrm{Z}(u)+Lu/K\}^2]\cos 2m\theta\, d\theta, \quad (3.08)$$

and the second integral in (3.06) is evaluated by writing $\theta = \frac{1}{2}\pi-\theta'$ and remembering that the modulus for the functions of v is $\sin(\frac{1}{2}\pi-\alpha)$. If we write

$$w = \mathrm{Z}(u) + Lu/K, \quad (3.09)$$

so that $$dw = k^2\,\mathrm{cn}^2 u\, du = k\,\mathrm{cn}\, u\, d\theta, \quad (3.10)$$

and integrate by parts, we have

$$\int_0^{\alpha}(L'^2+w^2)\cos 2m\theta\, d\theta = \Big[\frac{(L'^2+w^2)\sin 2m\theta}{2m}\Big]_0^{\alpha} - \int_0^{\alpha}\frac{\sin 2m\theta}{m}\, w\, dw \quad (3.11)$$

$$= \frac{(L'^2+L^2)\sin 2m\alpha}{2m} - \frac{1}{m}\int_0^{\alpha} w\sin 2m\theta\,\sqrt{(\sin^2\alpha-\sin^2\theta)}\, d\theta. \quad (3.12)$$

Now, if $$\sin 2m\theta\,\sqrt{(\sin^2\alpha-\sin^2\theta)} = F_m'(\theta), \quad (3.13)$$

we have, again integrating by parts,

$$f_m(\alpha) = \frac{(L'^2+L^2)\sin 2m\alpha}{m} - \Big[\frac{wF_m(\theta)}{m}\Big]_0^{\alpha} + \frac{1}{m}\int_0^{\alpha} F_m(\theta)\,\sqrt{(\sin^2\alpha-\sin^2\theta)}\, d\theta. \quad (3.14)$$

But $$F_m(\theta) = -2\int^{u}\frac{\sin 2m\theta}{\sin 2\theta}\, k^2\,\mathrm{cn}^2 u\, d(k\,\mathrm{cn}\, u), \quad (3.15)$$

and, if $m \neq 0$, $\sin 2m\theta/\sin 2\theta$ is expansible in powers of $\cos 2\theta$, and so in even powers of $k\,\mathrm{cn}\, u = \sqrt{\{\frac{1}{2}(\cos 2\theta-\cos 2\alpha)\}}$, with the result that $F_m(\theta)$ is a series of odd powers of $k\,\mathrm{cn}\, u$. Thus $F_m(\alpha) = 0$, provided that the constant

of integration is zero, so that $wF_m(\theta)$ then vanishes at both limits. It also follows that the final integrand is a series of even powers of $k\,\mathrm{cn}\,u$, *i.e.*, a rational trigonometric function of θ. Writing

$$G_m(\alpha) = \int_0^\alpha F_m(\theta)\sqrt{(\sin^2\alpha - \sin^2\theta)}\,d\theta, \tag{3.16}$$

we can express the final result in the form

$$\gamma_m = \frac{4}{\pi m}\{G_m(\alpha) + (-)^m G_m(\tfrac{1}{2}\pi - \alpha)\}, \tag{3.17}$$

the other terms in $f_m(\alpha)$ and $f_m(\frac{1}{2}\pi - \alpha)$ cancelling.

Carrying out the process outlined above, we find that

$$\gamma_1 = \{3\sin 4\alpha + 4(2+\cos 4\alpha)(\tfrac{1}{4}\pi - \alpha)\}/(6\pi),$$

$$\gamma_2 = -\{3\sin 2\alpha - \sin 6\alpha + 2(\cos 2\alpha - \cos 6\alpha)(\tfrac{1}{4}\pi - \alpha)\}/(15\pi),$$

$$\gamma_3 = -\{27(2\sin\alpha - \sin 8\alpha) + 4(6 + 10\cos 4\alpha - 16\cos 8\alpha)(\tfrac{1}{4}\pi - \alpha)\}/(1260\pi),$$

$$\gamma_4 = \{37\sin 10\alpha - 65\sin 6\alpha + 10\sin 2\alpha - 48(\cos 2\alpha + \cos 6\alpha - 2\cos 10\alpha)(\tfrac{1}{4}\pi - \alpha)\}/(3780\pi),$$

but the labour of calculating them grows progressively heavier.

This labour is much lightened by a recurrence formula, which is especially convenient for numerical calculations. To establish this we start from the identity

$$-4\frac{d}{d\theta}\{\cos 2m\theta(\sin^2\alpha - \sin^2\theta)^{\frac{3}{2}}\}$$
$$= \{(2m+3)\sin 2(m+1)\theta - 4m\cos 2\alpha\sin 2m\theta + (2m-3)\sin 2(m-1)\theta\}(\sin^2\alpha - \sin^2\theta)^{\frac{1}{2}}. \tag{3.18}$$

Hence, on integration,

$$-4\cos 2m\theta(\sin^2\alpha - \sin^2\theta)^{\frac{3}{2}}$$
$$= (2m+3)F_{m+1}(\theta) - 4m\cos 2\alpha F_m(\theta) + (2m-3)F_{m-1}(\theta). \tag{3.19}$$

Multiplying through by $(\sin^2\alpha - \sin^2\theta)^{\frac{1}{2}}$ and integrating from 0 to α, we find that

$$(m+1)(2m+3)f_{m+1}(\alpha) - 4m^2\cos 2\alpha f_m(\alpha) + (m-1)(2m-3)f_{m-1}(\alpha)$$
$$= -\frac{1}{4m}\left\{\frac{\sin 2(m+2)\alpha}{(m+2)(m+1)} - \frac{2\sin 2m\alpha}{m^2-1} - \frac{\sin 2(m-2)\alpha}{(m-2)(m-1)}\right\}. \tag{3.20}$$

Writing down the similar formula for $\frac{1}{2}\pi-\alpha$, we ultimately find that

$$(m+1)(2m+3)\gamma_{m+1}-4m^2\cos 2\alpha\,\gamma_m+(m-1)(2m-3)\gamma_{m-1}$$

$$=-\frac{2}{\pi m}\left\{\frac{\sin 2(m+2)\alpha}{(m+2)(m+1)}-\frac{2\sin 2m\alpha}{m^2-1}-\frac{\sin 2(m-1)\alpha}{(m-2)(m-1)}\right\}. \quad (3.21)$$

When $m=1$ or 2, some of the denominators vanish, but the formula remains true when limiting values are substituted, the explicit forms being

$$10\gamma_2-4\cos 2\alpha\,\gamma_1=-\{\sin 6\alpha+9\sin 2\alpha+24(\tfrac{1}{4}\pi-\alpha)\cos 2\alpha\}/(3\pi),$$

$$21\gamma_3-16\cos 2\alpha\,\gamma_2+\gamma_1=-\{\sin 8\alpha-8\sin 4\alpha-24(\tfrac{1}{4}\pi-\alpha)\}/(12\pi).$$

Again, the special case of the square merits a more detailed examination. It is evident from symmetry that only terms in $4m\theta$ will be present. The results of direct squaring, with the special values of the c_n, are

$$4\gamma_0=1+\sum_{n=1}^{\infty}c_n{}^2=1+\frac{1^2}{2^2\,3^2}+\frac{1^2\,1^2}{2^2\,4^2\,7^2}+\frac{1^2\,1^2\,3^2}{2^2\,4^2\,6^2\,11^2}+\cdots$$

$$={}_4F_3\left\{\begin{matrix}-\frac{1}{2}, & -\frac{1}{4}, & -\frac{1}{2}, & -\frac{1}{4}, & 1\\ \frac{3}{4}, & 1, & \frac{3}{4}, & & \end{matrix}\right\}, \quad (3.22)$$

$$4\gamma_{2m}=c_{2m}+\sum_{r=1}^{\infty}c_{2m+2r}\,c_{2r}$$

$$=c_{2m}\cdot{}_4F_3\left\{\begin{matrix}m-\frac{1}{2}, & m-\frac{1}{4}, & -\frac{1}{2}, & -\frac{1}{4}, & 1\\ & & \frac{3}{4}, & m+\frac{3}{4}, & \end{matrix}\right\}. \quad (3.23)$$

These series are of the type termed by Whipple "well-poised", but they do not seem to fall into any of the classes studied by him. It is, however, possible to express all of them except γ_0 in finite terms in the following manner.

Putting $\alpha=\frac{1}{4}\pi$ in the recurrence formula, and remembering that the γ's with odd suffixes vanish, taking m odd, and writing $m=2p+1$, we get

$$(p+1)(4p+5)\gamma_{2p+2}+p(4p-1)\gamma_{2p}=\frac{(-)^p\,4}{\pi(2p-1)(2p+1)(2p+3)}. \quad (3.24)$$

Now write $$\gamma_{2p}=(-)^p\frac{4}{\pi p}\frac{(4p-5)(4p-9)\ldots 3}{(4p+1)(4p-3)\ldots 1}D_p, \quad (3.25)$$

and the above relation becomes

$$D_{p+1}=D_p-\frac{(4p+1)(4p-3)\ldots 1}{(4p-5)(4p-9)\ldots 3}\frac{1}{(2p-1)(2p+1)(2p+3)}, \quad (3.26)$$

so that

$$\gamma_{2p} = (-)^p \frac{4}{\pi p} \frac{(4p-5)(4p-9)\dots 3}{(4p+1)(4p-3)\dots 1} \times \left(D_1 - \sum_{q=2}^{p-1} \frac{(4q+1)(4q-3)\dots 1}{(4q-5)(4q-9)\dots 3} \frac{1}{(2q-1)(2q+1)(2q+3)} \right). \quad (3.27)$$

To find the value of D_1 we use the value of γ_2 already given, and obtain $D_1 = \frac{1}{3}$.

The evaluation of γ_0 is much less simple, and it has not been found possible to express it in finite terms. The recurrence formula fails, for terms which would contain γ_0 always have zero coefficients. We have

$$\gamma_0 = \frac{2}{\pi} \left\{ \int_0^\alpha (L'^2+w^2)\, d\theta + \int_\alpha^{\frac{1}{2}\pi} (L^2+w'^2)\, d\theta \right\}. \quad (3.28)$$

Now

$$\int_0^\alpha (L'^2+w^2)\, d\theta = \int_0^K \left\{ L'^2 + \left(\mathrm{Z}(u) + \frac{Lu}{K} \right)^2 \right\} k \,\mathrm{cn}\, u\, du \quad (3.29)$$

$$= (L'^2+L^2) \int_0^K k \,\mathrm{cn}\, u\, du + \int_0^K \mathrm{Z}^2(u)\, k \,\mathrm{cn}\, u\, du + \frac{2L}{K} \int_0^K u\, \mathrm{Z}(u)\, k \,\mathrm{cn}\, u\, du - \frac{L^2}{K^2} \int_0^K (K^2-u^2)\, k \,\mathrm{cn}\, u\, du. \quad (3.30)$$

The values of the definite integrals are (see Appendix)

$$\int_0^K k \,\mathrm{cn}\, u\, du = \sin^{-1} k = \alpha, \quad (3.31)$$

$$H = \int_0^K \mathrm{Z}^2(u)\, k \,\mathrm{cn}\, u\, du = \left\{ \frac{3-2k^2}{2} - \frac{2E}{K} - \frac{\pi^2}{4K^2} \right\} \alpha + \frac{3kk'}{2} + \frac{\pi^2}{K^2} \sum_0^\infty \frac{(-)^n}{(2n+1)\, \cosh^3 (n+\frac{1}{2})\, \pi K'/K}, \quad (3.32)$$

$$I = \int_0^K u\, \mathrm{Z}(u)\, k \,\mathrm{cn}\, u\, du = \alpha - 2 \sum_0^\infty \frac{(-)^n \sinh (n+\frac{1}{2})\, \pi K'/K}{(2n+1)^2 \cosh^2 (n+\frac{1}{2})\, \pi K'/K}, \quad (3.33)$$

$$J = \int_0^\alpha (K^2-u^2)\, k \,\mathrm{cn}\, u\, du = \frac{16K^2}{\pi^2} \sum_0^\infty \frac{(-)^n}{(2n+1)^3 \cosh (n+\frac{1}{2})\, \pi K'/K}, \quad (3.34)$$

with complementary formulae for the complementary integrals. The series involving hyperbolic functions of $\frac{1}{2}\pi K'/K$ converge rapidly when $K' \geqslant K$, but then the complementary series converge much less rapidly. There is, however, a relation between the complementary integrals which enables us to use always the more convergent form. This relation can be

obtained by integrating $(L'^2+w^2)\,k\,\mathrm{cn}\,u\,du$ round the rectangle $BADC$ in the u-plane (Fig. 1). If we write

$$\left.\begin{aligned} Q &= \int_0^K (L'^2+w^2)\,k\,\mathrm{cn}\,u\,du, \\ Q' &= \int_0^{K'} (L^2+w'^2)\,k'\,\mathrm{cn}\,v\,dv, \end{aligned}\right\} \tag{3.35}$$

the integral along BA contributes $2Q$; since $\mathrm{Z}(u+2iK') = \mathrm{Z}(u)-i\pi/K$, and $\mathrm{cn}(u+2iK') = -\mathrm{cn}\,u$, the contribution from DC becomes

$$\int_{-K}^{K} \{L'^2+(w-2iL')^2\}\,k\,\mathrm{cn}\,u\,du = 2Q-8L'^2\alpha\,;$$

along AD we put $u = K+iK'+iv$, so that

$$\mathrm{Z}(u) = -\{i\mathrm{Z}(v)+\tfrac{1}{2}i\pi/K+i\pi v/(2KK')\},$$

and $k\,\mathrm{cn}\,u\,du = k'\,\mathrm{cn}\,v\,dv$, and the contribution becomes

$$\int_{-K'}^{K'} \{2L^2-(L^2+w'^2)-2w'(L'+iL)-2iLL')\}\,k'\,\mathrm{cn}\,v\,dv$$
$$= 4(L^2-iLL')(\tfrac{1}{2}\pi-\alpha)-2Q'\,;$$

similarly the contribution from CB is found to be

$$4(L^2+iLL')(\tfrac{1}{2}\pi-\alpha)-2Q'.$$

The residue of the integrand at the only pole in the rectangle, iK', is found to be $\frac{1}{2}i(1-2k^2)$. Collecting results, we have

$$4Q-8L'^2\alpha-4Q'+8L^2(\tfrac{1}{2}\pi-\alpha) = -\pi(1-2k^2),$$

or
$$Q+2L^2(\tfrac{1}{2}\pi-\alpha)-\tfrac{1}{4}\pi k^2 = Q'+2L'^2\alpha-\tfrac{1}{4}\pi k'^2,$$

so that
$$Q+Q' = 2Q+2L^2(\tfrac{1}{2}\pi-\alpha)-2L'^2\alpha+\tfrac{1}{4}\pi(1-2k^2). \tag{3.36}$$

Thus γ_0, the value of which is $2(Q+Q')/\pi$, is made to depend upon Q only. The result is somewhat simplified by writing

$$Q = (L^2+L'^2)\alpha+Q_1, \tag{3.37}$$

so that
$$Q_1 = H+2LI/K-L^2J/K^2, \tag{3.38}$$

and
$$\gamma_0 = 2L^2+4Q_1/\pi+\tfrac{1}{2}(1-2k^2). \tag{3.39}$$

In virtue of the relation $a/L' = b/L$, we may rewrite (3.03) in the form

$$|z|^2 = \frac{a^2}{L'^2}\,(\gamma_0+\Sigma\gamma_m\cos 2m\theta)$$
$$= a^2(\delta_0+\Sigma\delta_m\cos 2m\theta), \tag{3.40}$$

where $$\delta_m = \gamma_m / L'^2.$$

Values of the δ_m have been calculated for $\alpha = 45°$ and $\alpha = 30°$, the former by means of (3.27), and the latter from the recurrence formula. In the former case the calculations were carried to the stage where δ_{2m} was less than 0·00001. Putting $\theta = 0$, we should have

$$\sum_0^\infty \delta_{2m} = 1.$$

Actually $$\sum_{m=0}^{57} \delta_{2m} = 1{\cdot}000006.$$

Again, putting $\theta = \frac{1}{4}\pi$, we have

$$\sum_0^\infty (-)^m \delta_{2m} = 2,$$

while, as computed,

$$\sum_{m=0}^{57} (-)^m \delta_{2m} = 1{\cdot}999631.$$

Considering the slowness of the convergence of the last series, all of the terms of which are positive, the result is about what was to be expected. Methods of estimating the sum to infinity of a slowly convergent series point to an error of not more than 2 or 3 in the sixth place. Similar agreement, although not pushed so far, was obtained in the other case. Some early values of the δ_m are given in Table II.

TABLE II.

m	δ_{2m} $(\alpha = 45°)$	δ_m $(\alpha = 30°)$
0	1·43241	0·70163
1	−0·47304	0·48986
2	0·05256	−0·15902
3	−0·01713	−0·04452
4	0·00798	−0·00085
5	−0·00446	0·00998
6	0·00279	0·00637
7	−0·00188	0·00015
8	0·00134	−0·00288
9	−0·00099	−0·00222
10	0·00076	−0·00006
11	−0·00060	0·00132
12	0·00048	0·00109

By means of these values of the δ_m, and the series

$$\phi = \Sigma \delta_m e^{-2m\beta} \cos 2m\theta,$$

the values of ϕ have been calculated at the lattice points of the fundamental

nets, and, by interpolation, Figures 6*a* and 6*b*, showing the equi-ϕ's, have been drawn. These curves may be interpreted as the stream-lines in the two-dimensional fluid motion due to the rotation of the prism about its longitudinal axis.

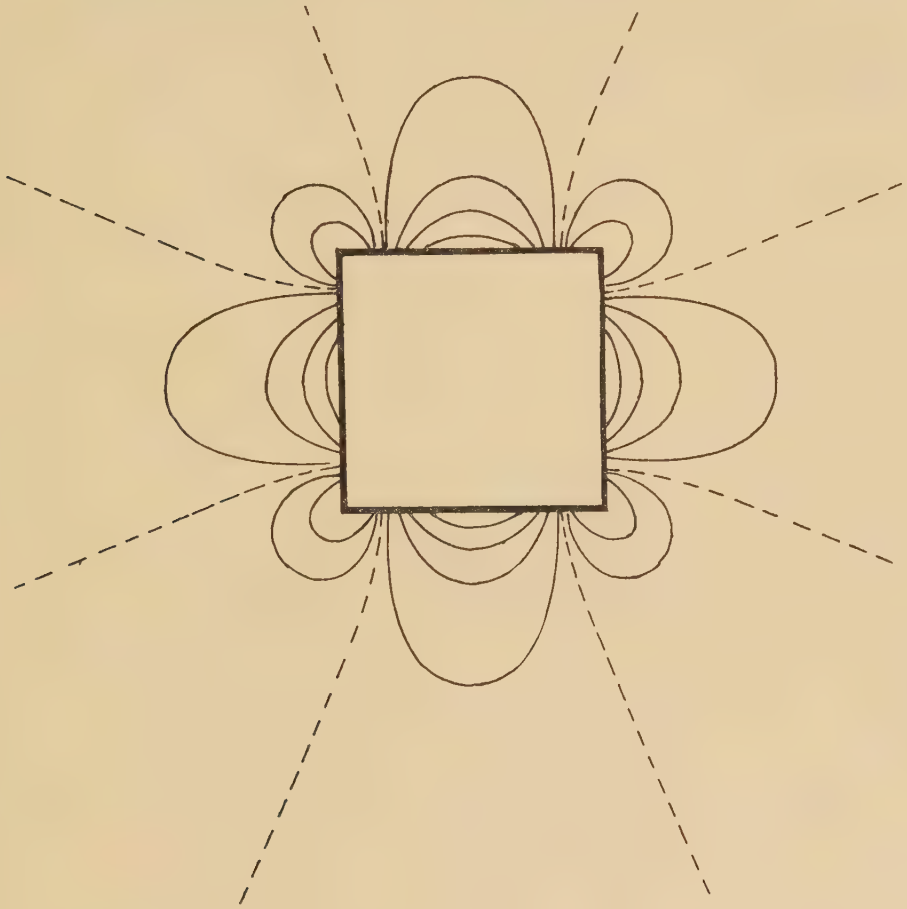

Fig. 6*a*.

The solution of this hydrodynamical problem is given by the velocity-potential-stream-function $\phi+i\psi$, provided that ψ has the value $\frac{1}{2}\omega|z|^2$ on the boundary, ω being the angular velocity. The function ψ here required is thus $\frac{1}{2}\omega$ times the function which we have just determined, so that

$$\phi+i\psi = \tfrac{1}{2}\omega a^2 i \Sigma \delta_m e^{-2m(\beta+i\theta)}, \tag{3.41}$$

i.e.,

$$\left.\begin{aligned}\phi &= \tfrac{1}{2}\omega a^2 \Sigma \delta_m e^{-2m\beta} \sin 2m\theta,\\ \psi &= \tfrac{1}{2}\omega a^2 \Sigma \delta_m e^{-2m\beta} \cos 2m\theta.\end{aligned}\right\} \tag{3.42}$$

The kinetic energy of the fluid motion is given by

$$2T = \left[\rho \int \phi d\psi\right], \tag{3.43}$$

the square brackets implying that the difference between the values of the integral taken round the circle at infinity, and the boundary, in the z-plane, is to be used. Here the value of the integral round the circle at infinity vanishes, and, on the boundary,

$$\left.\begin{aligned} \phi &= \tfrac{1}{2}\omega a^2 \Sigma \delta_m \sin 2m\theta, \\ d\psi &= \tfrac{1}{2}\omega a^2 \Sigma 2m\delta_m \sin 2m\theta \, d\theta, \end{aligned}\right\} \tag{3.44}$$

so that

$$2T = \tfrac{1}{2}\pi\rho\omega^2 a^4 \Sigma m\delta_m{}^2. \tag{3.45}$$

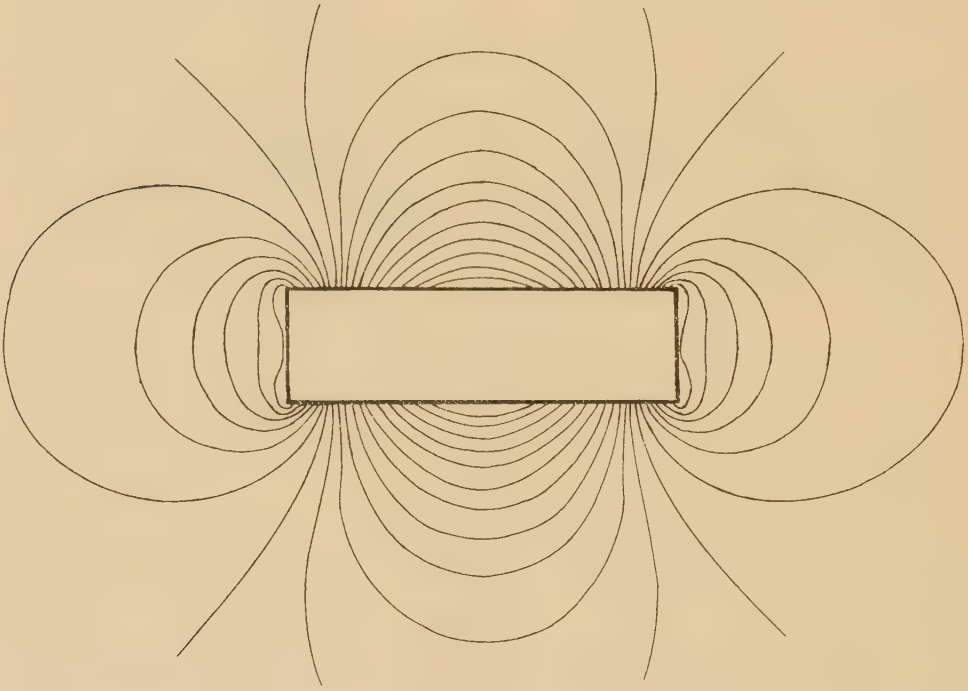

Fig. 6*b*.

It follows that the increase in the apparent moment of inertia due to the presence of the fluid is

$$\tfrac{1}{2}\pi\rho a^4 \Sigma m\delta_m{}^2$$

per unit length. The moment of inertia of the fluid that the prism would contain, per unit length, about its axis, is $4\rho ab(a^2+b^2)/3$, so that the

apparent increase is a fraction

$$\frac{3\pi}{8}\frac{a^4}{ab(a^2+b^2)}\Sigma m\delta_m{}^2=\frac{3\pi}{8}\frac{L'^3}{L(L^2+L'^2)}\Sigma m\delta_m{}^2$$

of this. When $\alpha=45°$, the fraction is 0·23313..., and when $\alpha=30°$, it is 1·0615....

In conclusion, the author would like to thank those who have assisted him in preparing this paper, and in particular Miss V. F. White, M.Sc., who computed the δ_m for the square; Miss V. Hatcher, who made the tracings of the fundamental nets; and Mr. T. McVey, who suggested the use of ferro-gallic prints for the original diagrams, and prepared them.

APPENDIX.

The evaluation of certain definite integrals involving Jacobian elliptic functions.

If we write $\lambda=\frac{1}{2}\pi K'/K$, and $\theta=\frac{1}{2}\pi u/K$, the Fourier series for $\operatorname{cn} u$ and $Z(u)$ become

$$k\operatorname{cn}u=\frac{\pi}{K}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{\cosh(2n+1)\lambda}, \tag{A 1}$$

$$Z(u)=\frac{\pi}{K}\sum_{n=0}^{\infty}\frac{\sin 2n\theta}{\sinh 2n\lambda}. \tag{A 2}$$

From the first of these, using the known result

$$\int_0^K k\operatorname{cn}u\,du=\sin^{-1}k=\alpha,$$

we have

$$\alpha=2\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}\pi}\frac{\cos(2n+1)\theta}{\cosh(2n+1)\lambda}\,d\theta,$$

$$=2\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)\cosh(2n+1)\lambda}, \tag{A 3}$$

a result which we shall use later.

Again, from (A 1) and the relation between u and θ, we have

$$J=\int_0^K(K^2-u^2)k\operatorname{cn}u\,du=\frac{2K^2}{\pi^2}\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}\pi}\frac{(\pi^2-4\theta^2)\cos(2n+1)\theta}{\cosh(2n+1)\lambda}\,d\theta$$

$$=\frac{16K^2}{\pi^2}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)^3\cosh(2n+1)\lambda}. \tag{A 4}$$

By direct multiplication, or more simply by integrating

$$Z(u)\,k\operatorname{cn}u\,e^{(2n+1)(\theta-i\lambda)}$$

round the rectangle $ABCD$ in the u-plane (Fig. 1), we derive the Fourier series

$$\mathrm{Z}(u)\, k\, \mathrm{cn}\, u = \frac{\pi^2}{2K^2} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta}{\cosh(2n+1)\lambda} \{2n+1-\tanh(2n+1)\lambda\}$$

$$= \frac{\pi^2}{2K^2} \sum_{n=0}^{\infty} C_n \sin(2n+1)\theta.$$

Thus

$$I = \int_0^K u\, \mathrm{Z}(u)\, k\, \mathrm{cn}\, u\, du = 2 \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}\pi} \theta\, C_n \sin(2n+1)\theta\, d\theta$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-)^n C_n}{(2n+1)^2}$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)\cosh(2n+1)\lambda} - 2 \sum_{n=0}^{\infty} \frac{(-)^n \sinh(2n+1)\lambda}{(2n+1)^2 \cosh^2(2n+1)\lambda},$$

so that

$$I = \alpha - 2 \sum_{n=0}^{\infty} \frac{(-)^n \sinh(2n+1)\lambda}{(2n+1)^2 \cosh^2(2n+1)\lambda}. \qquad \text{(A 5)}$$

Similarly we find that

$$\mathrm{Z}^2(u)\, k\, \mathrm{cn}\, u = \frac{\pi}{K} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{\cosh(2n+1)\lambda}$$

$$\times \left\{ \frac{3-2k^2}{2} - \frac{2E}{K} + \frac{\pi^2}{4K^2} - \frac{\pi^2}{2K^2} \{n+\tfrac{1}{2}-\tanh(2n+1)\lambda\}^2 \right\}$$

$$= \frac{\pi}{K} \sum_{n=0}^{\infty} D_n \cos(2n+1)\theta.$$

Thus

$$H = \int_0^K \mathrm{Z}^2(u)\, k\, \mathrm{cn}\, u\, du = 2 \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}\pi} D_n \cos(2n+1)\theta\, d\theta$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-)^n D_n}{(2n+1)}$$

$$= 2 \left\{ \frac{3-2k^2}{2} - \frac{2E}{K} - \frac{\pi^2}{4K^2} \right\} \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)\cosh(2n+1)\lambda} - \frac{\pi^2}{4K^2} \sum_{n=0}^{\infty} \frac{(-)^n (2n+1)}{\cosh(2n+1)\lambda}$$

$$+ \frac{\pi^2}{K^2} \sum_{n=0}^{\infty} \frac{(-)^n \sinh(2n+1)\lambda}{\cosh^2(2n+1)\lambda} + \frac{\pi^2}{K^2} \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)\cosh^3(2n+1)\lambda}. \qquad \text{(A 6)}$$

Differentiating both sides of (A 1) with respect to θ, we deduce that

$$\sum_{n=0}^{\infty} \frac{(2n+1)\sin(2n+1)\theta}{\cosh(2n+1)\lambda} = \frac{2kK^2}{\pi^2}\, \mathrm{sn}\, u\, \mathrm{dn}\, u,$$

and, putting $\theta = \frac{1}{2}\pi$, we get

$$\sum_{n=0}^{\infty} \frac{(-)^n (2n+1)}{\cosh(2n+1)\lambda} = \frac{2kK^2}{\pi^2} \operatorname{sn} K \operatorname{dn} K = \frac{2kk'K^2}{\pi^2}. \tag{A 7}$$

Also, from the Fourier series for $\mathrm{Z}(u)\, k \operatorname{cn} u$ and $\operatorname{sn} u \operatorname{dn} u$, we derive

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta \sinh(2n+1)\lambda}{\cosh^2(2n+1)\lambda} = \frac{2kK^2}{\pi^2} \{\operatorname{sn} u \operatorname{dn} u - \operatorname{cn} u\, \mathrm{Z}(u)\}.$$

And, putting $\theta = \frac{1}{2}\pi$ in this, we find that

$$\sum_{n=0}^{\infty} \frac{(-)^n \sinh(2n+1)\lambda}{\cosh^2(2n+1)\lambda} = \frac{2kk'K^2}{\pi^2}. \tag{A 8}$$

Collecting results, we have

$$H = \int_0^K \mathrm{Z}^2(u)\, k \operatorname{cn} u\, du$$

$$= \left(\frac{3-2k^2}{2} - \frac{2E}{K} - \frac{\pi^2}{4K^2}\right) \sin^{-1} k + \frac{3kk'}{2} + \frac{\pi^2}{K^2} \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)\cosh^3(2n+1)\lambda}. \tag{A 9}$$

It has not been found possible to express the series remaining in the above results in finite terms, but, if $K' > K$, and even if $K' = K$, they converge quite rapidly. Evidently complementary results are obtained for the integrals with the complementary modulus, and then the series may converge much more slowly. By the method already used to obtain the relation between Q and Q', relations between complementary integrals can be deduced. The actual relations are found to be

$$\tfrac{1}{2}\pi K^2 - J = \tfrac{1}{2}\pi K'^2 - J', \tag{A 10}$$

$$I + I' = \tfrac{1}{2}\pi - \frac{\pi^2(K^2+K'^2)}{8KK'} + \frac{\pi(J+J')}{4KK'}, \tag{A 11}$$

$$H + \frac{\pi I}{2KK'} - \frac{\pi^2 J}{8K^2K'^2} - \frac{\pi^3}{16K^2} - \tfrac{1}{4}\pi k^2 + \frac{\pi L}{2K}$$

$$= H' + \frac{\pi I'}{2KK'} - \frac{\pi^2 J'}{8K^2K'^2} - \frac{\pi^3}{16K'^2} - \tfrac{1}{4}\pi k'^2 + \frac{\pi L'}{2K'}, \tag{A 12}$$

and from these identities the relation between Q and Q' could have been deduced.

POWER VECTORS

By H. W. Turnbull.

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Introduction.

The following investigation arose out of a study of the simplest non-singular matrix which is a solution of a commutantal matrix equation $f(U)X = XU$, U being the auxiliary unit matrix (§ 2), and $f(U)$ being a power series in the single matrix argument U, starting with a term $a_0 U$ and ascending in integral powers with scalar coefficients a_i. The reason why this came to be studied is given in § 12, where the relation between the matrices which reduce a given matrix A and a functional matrix $f(A)$ each to canonical form is discussed. Such reducing matrices are identical only in the simplest cases when A possesses linear elementary divisors.

Originally I was led to the results concerning the solution P, namely the matrix discussed in §§ 13 and 14, whose elements bear a strong analogy to the gradients of binary invariant theory—each row of elements being derived by the well-known Sylvester operator Ω from the previous row. The nature of such problems became clearer through the work of Mr. D. E. Rutherford (cited in § 8), who found a neat form for the reciprocal matrix P^{-1}. In searching next for a more fundamental explanation of the type of element arising, I was considerably helped by a happy conjecture of Dr. E. T. Copson as to the character of the generating function. For this hint and also for pointing out at a later stage the connection between part of this work and the method of Arbogast my best thanks are due.

Eventually a theory of *induced matrices* (§ 4) arose which completely accounts for matrix problems of this kind, and which seems worth while elaborating in some detail. § 1 is introductory and shows the further scope of the same ideas. The solution, when p and q may differ, of the equation $YU^p = U^q Y$ in § 9, is perhaps foreign to the theory, but is necessary to complete the solution of the equations discussed in the other sections.

1. *Matrix factors of a power series.*

Considerable light is often thrown upon ordinary scalar algebra and analysis by the use of non-commutative methods. For example, it is always possible to resolve a scalar quantity into matrix factors. Thus a quadratic form may be expressed as follows:

$$\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_ix_j = [x_0, x_1, x_2, \ldots]\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}, \tag{1}$$

where $a_{ij} = a_{ji}$, and each row and each column of these three matrix factors contains $n+1$ elements (n being finite or infinite). If these factors are denoted, in order, by $[x]$, A, $\{x\}$, then the product $[x]\,A$ is by definition a single row matrix (or row vector), while the continued product of this and the final column vector $\{x\}$ is a single element, a matrix consisting of one row and one column. All this follows directly from the definition of matrix multiplication. I shall use the notation $\{x\}$ always to denote a column vector, and shall often denote it row-wise, in order to save space, by

$$\{x\} = \{x_0, x_1, x_2, \ldots\}. \tag{2}$$

The corresponding $[x]$ enclosed in square brackets will invariably denote the row vector.

Suppose now that the x_i take the values x^i given by actual powers of a variable x. Then the quadratic form becomes a power series

$$z = f(x) = a_0 + a_1x + a_2x^2 + \ldots, \tag{3}$$

where $a_r = \Sigma a_{ij}$, summed for the $r+1$ values of the suffixes for which $i+j = r$. Thus

$$a_{00} = a_0, \quad a_{01}+a_{10} = a_1, \quad a_{02}+a_{11}+a_{20} = a_2, \quad \ldots. \tag{4}$$

Again, let $[a] = [a_0, a_1, a_2, \ldots]$ denote the row vector whose elements are these coefficients in the power series. Then, evidently, the series can at once be broken up into two matrix factors

$$z = [a]\{x\} = [a_0, a_1, a_2, \ldots]\{1, x, x^2, x^3, \ldots\}, \tag{5}$$

since the product of a row matrix followed by a column matrix is scalar. (Equally well it can be transposed and written $z = [x]\{a\}$).

Or again, by choosing a set of numbers a_{ij} ($= a_{ji}$) to satisfy the above relation (4), which is possible in many ways, *the power series can always*

be represented as a quadratic form whose variables are in geometrical progression. The simplest such representation is

$$f(x) = [1,\ x,\ x^2,\ \ldots] \begin{bmatrix} a_0 & \frac{1}{2}a_1 & . & . & \ldots \\ \frac{1}{2}a_1 & a_2 & \frac{1}{2}a_3 & . & \ldots \\ . & \frac{1}{2}a_3 & a_4 & \frac{1}{2}a_5 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix}, \qquad (6)$$

where the central matrix factor is palpably symmetrical (so that $a_{ij} = a_{ji}$), containing three consecutive non-zero elements on each row and in each column, except the first and last which have only two. Any other symmetric matrix will equally avail, provided that the sum of its elements in the r-th *secondary diagonal* is a_r.

As an illustration of the use to which this may be put, let us consider the following problem:

To find sufficient conditions for the sum of a given series, finite or infinite,

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

never to be negative for real values of the coefficients and the variable.

To find these it is only necessary to throw the series into the quadratic form (3), which must therefore never be negative. In other words, it must be positive definite; and this property is secured* for all real values of the variables, and therefore for the particular real values $\{1, x, x^2, \ldots\}$, if the following principal minor determinants are never negative:

$$a_0 \geqslant 0, \quad \begin{vmatrix} a_0 & \frac{1}{2}a_1 \\ \frac{1}{2}a_1 & a_2 \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} a_0 & \frac{1}{2}a_1 & . \\ \frac{1}{2}a_1 & a_2 & \frac{1}{2}a_3 \\ . & \frac{1}{2}a_3 & a_4 \end{vmatrix} \geqslant 0, \quad \ldots . \qquad (7)$$

2. *Semi-persymmetric matrices.*

The *auxiliary unit matrix* of n rows and columns is denoted by

$$U = \mathrm{diag}_1(1,\ 1,\ \ldots,\ 1) \equiv \mathrm{diag}_1(1),$$

and always consists of $n-1$ units arranged on the first over-diagonal, with n^2-n+1 zero elements in the residual positions. By the law of multiplication of matrices it follows that, when the product of U and

* *Cf.* Turnbull and Aitken, *Canonical matrices* (1932), 90.

another matrix A is formed either way, UA is a matrix in which the rows of A have risen one place, while AU is a matrix in which the columns of A have moved to the right one place. In particular, it is useful to denote powers of U by

$$U^2 = \mathrm{diag}_2(1), \quad U^3 = \mathrm{diag}_3(1), \quad \ldots, \quad U^n = 0, \tag{1}$$

where the suffix denotes the position of the diagonal as counted from the principal diagonal towards the right. For instance, if $n = 4$,

$$U = \begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ . & . & . & . \end{bmatrix}, \quad U^2 = \begin{bmatrix} . & . & 1 & . \\ . & . & . & 1 \\ . & . & . & . \\ . & . & . & . \end{bmatrix}, \quad U^3 = \begin{bmatrix} . & . & . & 1 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}, \quad U^4 = 0. \tag{2}$$

Since U has a zero determinant, it is a singular matrix, to which it is impossible to attach zero or negative exponents. But by multiplying each matrix of positive exponent U^i by a scalar a_{i-1} and then adding the results, a matrix polynomial $f(U)$ is obtained, such as

$$V \equiv f(U) = a_0 I + a_0 U + a_1 U^2 + \ldots + a_{n-2} U^{n-1}, \tag{3}$$

where I denotes the *unit matrix*

$$\mathrm{diag}_0(1, 1, \ldots, 1) \equiv [\delta_{ij}]. \tag{4}$$

This matrix V is of *semi-persymmetric* type

$$\begin{bmatrix} a_0 & a_0 & a_1 & a_2 & \ldots \\ . & a_0 & a_0 & a_1 & \ldots \\ . & . & a_0 & a_0 & \ldots \\ . & . & . & a_0 & \ldots \\ \ldots & & \ldots & & \ldots \end{bmatrix}, \tag{5}$$

with equal elements throughout each diagonal, but with zeros everywhere below the principal diagonal. Conversely, every semi-persymmetric matrix is expressible as a polynomial in U of order $n-1$.

A fully persymmetric matrix, such as

$$\begin{bmatrix} a_0 & a_0 & a_1 \\ a_0' & a_0 & a_0 \\ a_1' & a_0' & a_0 \end{bmatrix},$$

contains $2n-1$ parameters a_i, a_0, a_i', and is essentially a polynomial matrix function of *two* arguments

$$f(U', U) \equiv \sum_{i=0}^{n-2} a_i' U'^{i+1} + a_0 I + \sum_{i=0}^{n-2} a_i U^{i+1},$$

U' being the transposed auxiliary unit matrix

$$\mathrm{diag}_{-1}(1, 1, \ldots, 1)$$

with $n-1$ units placed upon the first under-diagonal.

In the case of infinite matrices ($n = \infty$) the above conditions may be partially relaxed. For a finite n the product UU' consists of $n-1$ units followed by a single zero element upon the principal diagonal, so that

$$UU' = \mathrm{diag}_0(1, 1, \ldots, 1, 0);$$

but this is indistinguishable from I the unit matrix when $n \to \infty$. The infinite matrix U can then be said to possess a reciprocal U^{-1} such that $UU^{-1} = I$; but it is never the case that UU^{-1} and $U^{-1}U$ are equal. In both finite and infinite cases we have the relation

$$U'U = \mathrm{diag}_0(0, 1, 1, \ldots, 1) \neq UU'.$$

3. *Power vectors.*

I shall call a vector whose elements are in geometrical progression a *power vector*.

The simplest power vector starts with unity for first element, and depends upon a scalar argument x:

$$\{x\} = \{1, x, x^2, \ldots, x^{n-1}\}. \tag{1}$$

Such a notation implies that $\{x\}$ is a column vector of ascending powers. The corresponding *negative power vector* will be denoted by $\{\bar{x}\}$, and the corresponding row vector will be

$$[\bar{x}] = [1, x^{-1}, x^{-2}, \ldots, x^{1-n}]. \tag{2}$$

By scalar multiplication we have

$$\lambda\{x\} = \{\lambda, \lambda x, \lambda x^2, \ldots, \lambda x^{n-1}\} = \{x\}\lambda,$$

and, in particular,

$$x\{x\} = \{x, x^2, x^3, \ldots, x^n\} = \{x\}x.$$

In accordance with a well-known concept of analysis let $O(x^r)$ denote an expression, whether matrix or scalar, in which every term of every element involved contains x to the order r. The cases to be contemplated are terms that arise from polynomials in x, and from rational functions of x which are capable of expansion in ascending, and convergent, power series in x. Then the above result may be written

$$\begin{aligned} x\{x\} &= \{x, x^2, \ldots, x^{n-1}, 0\} + \{0, \ldots, 0, x^n\} \\ &= \{x, x^2, \ldots, x^{n-1}, 0\} + O(x^n). \end{aligned} \tag{3}$$

The first expression on the right consists of a *column* of elements, obtained from the original column $\{x\}$ by moving them all one place upwards, vacating the last place, and deleting unity, the first element. But this is precisely the property of premultiplying the vector by the auxiliary unit matrix U. Hence

$$x\{x\} = U\{x\} + O(x^n). \tag{4}$$

For example, when $n = 3$,

$$U = \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ . & . & . \end{bmatrix}, \quad U\{x\} = \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ . & . & . \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = \begin{bmatrix} x \\ x^2 \\ . \end{bmatrix};$$

that is $$U\{x\} = \{x, x^2, .\}.$$

Now suppose that $f(x)$ is a polynomial, or else a convergent power series, in x:

$$f(x) = a_0 + a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_{n-2} x^{n-1} + \ldots, \tag{5}$$

then, by successive applications of this principle,

$$f(U)\{x\} = f(x)\{x\} + O(x^n). \tag{6}$$

Owing to the property that the n-th power of U vanishes, it follows that the expression $f(U)$ is a polynomial in U of order not exceeding n, into which no coefficient a_r can enter, whose suffix exceeds $n-2$. For example, when $n = 3$,

$$U^2\{x\} = U^2\{1, x, x^2\} = \{x^2, 0, 0\} = x^2\{x\} + O(x^3);$$

and again,

$$\begin{aligned} f(x)\{1, x, x^2\} &= (a_0 + a_0 U + a_1 U^2)\{1, x, x^2\} + O(x^3) \\ &= \{a_0 + a_1 x + a_2 x^2, a_0 x + a_1 x^2, a_0 x^2\} + O(x^3). \end{aligned}$$

The expression (6) is of some importance, because it renders flexible the matrix product $f(X)\{x\}$ whose two factors cannot of course be commuted. On the other hand, the scalar factor $f(x)$ may stand fore or aft; so that

$$\{x\}f(x) = f(x)\{x\} = f(U)\{x\} + O(x^n). \tag{7}$$

Again, when rows and columns are completely interchanged in the usual manner by *transposition*, then the column $\{x\}$ becomes the row $[x]$ of *ascending* powers, and U becomes U'. Thus

$$\{x\}' = [x] = [1, x, x^2, \ldots, x^{n-1}], \tag{8}$$

and

$$f(x)[x] = [x]f(x) = [x]f(U') + O(x^n), \tag{9}$$

where it will be seen that the vector precedes the function of U' in the expression on the right of the identity.

Furthermore, if the negative power vector $[\bar{x}]$ is multiplied by the scalar x, the elements are translated one place to the right, except the last, which disappears. Such a process is equivalent to postmultiplication with the auxiliary U, except for the initial element in the result. The actual relation is

$$\begin{aligned} x[\bar{x}] &= [0, 1, x^{-1}, \ldots, x^{2-n}] + [x, 0, \ldots, 0] \\ &= [\bar{x}]U + O(x); \end{aligned} \tag{10}$$

from which it follows that

$$f(x)[\bar{x}] = [\bar{x}]f(x) = [\bar{x}]f(U) + O(x) \tag{11}$$

in the manner of (7) above. This last identity may, if necessary, be transposed, analogously to (9). Also, if a rational integral form of the result is preferred, the relation may be multiplied throughout by x^{n-1}, which clears the denominators. This replaces the descending power vector $[\bar{x}]$ by the *reverse vector*

$$[x]'' = [x^{n-1}, x^{n-2}, \ldots, x, 1] = x^{n-1}[\bar{x}], \tag{12}$$

and the aggregate of terms $O(x)$ by $O(x^n)$. Thus

$$x[x]'' = [x]''U + O(x^n), \tag{13}$$

and

$$f(x)[x]'' = [x]''f(U) + O(x^n). \tag{14}$$

If $[x]'''$ denotes the transposed of $[x]''$, that is

$$[x]''' = \{x^{n-1}, x^{n-2}, \ldots, x, 1\}, \tag{15}$$

which is a column vector, then the last relation can also be transposed, as follows:

$$f(x)\,[x]''' = f(U')\,[x]''' + O(x^n). \tag{16}$$

Another mode of statement is to make further use of the distributive law by writing relation (6), for example, as

$$\big(f(U) - f(x)\big)\{x\} = O(x^n). \tag{17}$$

Strictly it is usual to insert the factor I, denoting the unit matrix, and to write $f(U) - f(x)\,I$ as the matrix prefactor of the vector $\{x\}$; but it would be pedantic always to insist upon this, in cases where no misunderstanding can arise.

Once more, it is worth while to put these results in still another way by the use of the *secondary unit matrix* J, which consists of n units placed upon the *secondary diagonal* (and $n^2 - n$ zeros elsewhere in the square array), so that the positions $(n, 1)$ to $(1, n)$ inclusive are occupied by units. Such a matrix satisfies the fundamental relations

$$J^2 = I, \quad JUJ = U', \quad JU'J = U, \tag{18}$$

while JX, XJ, and $JXJ = X''$ are respectively the matrices obtained by reflecting the array X in a horizontal central axis, a vertical central axis, and in the centre. For example, when $[x_{ij}] = X$, and $n = 3$, then

$$J = \begin{bmatrix} . & . & 1 \\ . & 1 & . \\ 1 & . & . \end{bmatrix}, \qquad JX = \begin{bmatrix} x_{31} & x_{32} & x_{33} \\ x_{21} & x_{22} & x_{23} \\ x_{11} & x_{12} & x_{13} \end{bmatrix}, \qquad XJ = \begin{bmatrix} x_{13} & x_{12} & x_{11} \\ x_{23} & x_{22} & x_{21} \\ x_{33} & x_{32} & x_{31} \end{bmatrix},$$

$$JXJ = \begin{bmatrix} x_{33} & x_{32} & x_{31} \\ x_{23} & x_{22} & x_{21} \\ x_{13} & x_{12} & x_{11} \end{bmatrix} = X''. \tag{19}$$

This last matrix is called the reverse of X; and evidently the relation between X and its reverse is symmetrical. Like the process of transposition (indicated by the single accent), that of *reversion* (indicated by the double accent) is fundamental; it can always be applied to a matrix, a matrix function, or a matrix equation. Thus

$$(X \pm Y)'' = J(X \pm Y)\,J = JXJ \pm JYJ = X'' \pm Y'',$$

and (since $J^2 = I$)

$$(XY)'' = J(XY)\,J = JXJ^2\,YJ = X''\,Y'', \tag{20}$$

a result which may be contrasted with the transposition identity $(XY)' = Y'X'$ which reverses the order of the factors. It follows also by iteration that

$$JU^2J = JUJJUJ = U'^2, \quad JU^rJ = U'^r,$$

and, in general, that $U = U''$ and

$$Jf(U)J = f(U'). \tag{21}$$

In other words, *the transposed is identical with the reverse of* U, *or of* $f(U)$. For the general matrix the processes of transposition and reversion commute, so that, let us say, they give rise unambiguously to a *converse* matrix (which is indicated by three accents)

$$(X')'' = (X'')' = X'''. \tag{22}$$

As regards vectors and power vectors the corresponding relations are evidently

$$\{x\}'' = J\{x\}, \quad [x] = [x]''J, \quad \{x\}''' = [x]J. \tag{23}$$

For the purpose of applying the foregoing concepts to the solution of a certain commutantal equation

$$Xf(U) - g(U)X = 0,$$

I shall now consider four lemmas.

LEMMA 1. *If* $A = [a_{ij}]$ *is a matrix independent of* x, *the relation*

$$A\{x\} = O(x^n)$$

is an identity in x, *when and only when* A *is the null matrix.*

Proof. The left-hand member of this equation is a column vector, whose i-th element is the series $\Sigma a_{ij}x^{j-1}$, summed for j from 1 to n inclusive. Since each power of x here is less than n, and since the right-hand member contains x^n in each term, this series must vanish independently of terms on the right. This can only happen for all values of x if each coefficient a_{ij} vanishes. And since this argument holds for each of the n values of i, it follows that $[a_{ij}] = 0$; that is, A is the null matrix.

LEMMA 2. *If* $X = A$ *is a solution of the commutantal equation* $Xf(U) = g(U)X$, *then* $X = JA'J = A'''$ *is a solution of the equation* $Xg(U) = f(U)X$, *where* J *is the secondary unit matrix.*

Proof. If $Af(U) = g(U)A$, then, by transposition, $f(U')A' = A'g(U')$; that is,

$$Jf(U')JJA'J = JA'JJg(U')J.$$

But $Jf(U')J = f(U)$ and $Jg(U')J = g(U)$; whence the result follows immediately.

LEMMA 3. *The following four relations are equivalent; any three may be derived from the fourth by transposition or reversion:*

$$X'f(U) = g(U)X', \qquad Xg(U') = f(U')X,$$

$$X'''f(U') = g(U')X''', \quad X''g(U) = f(U)X''.$$

Proof. This follows at once from the foregoing results.

LEMMA 4. *If $X = A$ is a solution of the equation $Xf(U) = g(U)X$, then $X = JA$ is a solution of the equation $Xf(U) = g(U')X$.*

Proof. This again follows at once since $Jg(U')J = g(U)$.

Hence again all such equations involving $f(\theta)$ and $g(\theta)$, where θ is either U or U', in either function f or g, are derivable, in the manner explained, from one another.

4. *Induced matrix of a function.*

Let y be an explicit function of x defined by the polynomial or absolutely convergent power series

$$y = f(x) = a_0 + a_0 x + a_1 x^2 + \ldots a_{n-2} x^{n-1} + \ldots. \tag{1}$$

Let the powers of y be arranged in a power vector

$$\{y\} = \{1, y, y^2, \ldots, y^{n-1}\}. \tag{2}$$

If each element is now expressed in terms of x, and if each term of order x^n or more is segregated, and if next the coefficient of the $(j-1)$-th power of x occurring in the i-th element is called f_{ij}, then the totality of these f_{ij} define a matrix

$$F = [f_{ij}],$$

consisting of n rows and columns, such that

$$\{y\} = F\{x\} + O(x^n). \tag{3}$$

This matrix F, which evidently consists of elements polynomial in the first n coefficients of the power series $f(x)$, will be called the *induced matrix* of the function $f(x)$.

For example, if $y = a_0 + a_0 x + a_1 x^2$, then

$$\{1, y, y^2\} = \{1, a_0 + a_0 x + a_1 x^2, a_0^2 + 2a_0 a_0 x + (a_0^2 + 2a_0 a_1) x^2\} + O(x^3)$$

$$= \begin{bmatrix} 1 & . & . \\ a_0 & a_0 & a_1 \\ a_0^2 & 2a_0 a_0 & a_0^2 + 2a_0 a_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} + O(x^3), \tag{4}$$

where the induced matrix F is explicitly written out in full.

It will later be seen that the term a_0 plays a relatively unimportant part. For instance, here and also in the general case the determinant of F is independent of a_0, being in fact an integral power of a_0.

In a similar fashion it is possible to form the *negatively induced matrix* $\bar{F}$, from the same function, by first expressing each element of the descending power vector $[\bar{y}]$ in terms of x, then expanding each element in ascending powers of x, and finally segregating the terms $O(x)$ involving all positive powers of x. Thus

$$[\bar{y}] = [\bar{x}]\,\bar{F} + O(x). \tag{5}$$

Once more, attention may be called to the fact that, in the product term here, the row vector precedes the factor $\bar{F}$ which is a square array. If $a_0 \neq 0$, then $\bar{F}$ is a relatively trivial matrix consisting of the vector $[\bar{a}_0]$ for its top row, and of zeros throughout the remaining rows. But if $a_0 = 0$, then $\bar{F}$ is much more interesting, and plays a useful part in the discussion.

A *converse induced matrix*, $F''' \equiv H$, may also be obtained in the same manner, either by forming the converse of the relation (3) or else by calculating the descending power vector of y directly in terms of x and then segregating powers of order n or higher. The new relation is

$$[y]'' = [x]''\,H + O(x^n), \quad H = F'''. \tag{6}$$

5. *Analogy between the matrix and the functional notation.*

It is interesting to note that the analogy which already exists between the relations

$$y = f(x), \quad \{y\} = F\{x\} + O(x^n) \tag{1}$$

extends to the inverse and transitive operations. If y is a function of u,

and again u is a function of x, then y is a function of x. Thus, if $y = f(u)$ and $u = g(x)$, then we may write

$$y = f\big(g(x)\big), \quad \{y\} = FG\{x\} + O(x^n). \tag{2}$$

In more detail we have $\{y\} = F\{u\} + O(u^n)$, $\{u\} = G\{x\} + O(x^n)$, and the result follows by the associative law, provided that the order of u is at least that of x.

Or again, if $y = f(x)$ is a power series starting with the non-zero term $a_0 x$, so that $f'(0) \neq 0$, then $x = f^{-1}(y)$ exists as a similar power series starting with the term $a_0^{-1} y$. But by premultiplication with F^{-1} (which obviously exists if $a_0 \neq 0$), we have, from (1),

$$x = f^{-1}(y), \quad \{x\} = F^{-1}\{y\} + O(y^n). \tag{3}$$

The coefficients in the series $f^{-1}(y)$ of powers of y are at once given by the elements in the second row of the reciprocal matrix F^{-1}, whose subsequent rows give the coefficients in the power series for x^2, x^3, ... respectively.

The ordinary functional notation, in so far as it pertains to power series with integral exponents, is virtually an illustration of matrix theory. Reversion of series is part of the theory of reciprocal matrices.

6. *Theorems on induced matrices.*

The leading properties of these induced matrices are summarized in the following theorem.

THEOREM I. *The induced matrix F of a scalar function*

$$y = f(x) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots$$

satisfies the matrix equation

$$Ff(U) = UF. \tag{1}$$

The negatively induced matrix and also the converse induced matrix satisfy the further equations

$$f(U)\,\bar{F} = \bar{F}U, \quad f(U)\,H = HU. \tag{2}$$

Proof. Whether or not a_0 is zero it is evident that all terms in y, of order n, are included among all terms in x of this order. We may therefore write

$$O(y^n) \subset O(x^n). \tag{3}$$

Also, from the identity

$$y\{y\} = U\{y\} + O(y^n),$$

it follows by substitution that

$$f(x)\{y\} = U\{y\} + O(x^n),$$

and again, after using § 4 (3), that

$$f(x)\, F\{x\} = UF\{x\} + O(x^n).$$

Since $f(x)$ is scalar, the left product may be written as $Ff(x)\{x\}$. Furthermore, by § 3 (6),

$$f(x)\{x\} = f(U)\{x\} + O(x^n).$$

Hence, by substitution, we have the identity

$$\left(Ff(U) - UF\right)\{x\} = O(x^n). \tag{4}$$

But this matrix prefactor of $\{x\}$ is a constant, and therefore Lemma I applies, so that

$$Ff(U) = UF,$$

which proves the first part of the theorem.

The second part of the theorem is proved in the same way by substituting for y in terms of x in the identity

$$y[\bar{y}] = [\bar{y}]\, U + O(y),$$

where obviously the terms $O(y)$ are included in the terms $O(x)$. Thus

$$f(x)[\bar{x}]\,\bar{F} = [\bar{x}]\,\bar{F}U + O(x).$$

After using the identity (11) of § 2 upon the product $f(x)[\bar{x}]$, we obtain

$$[\bar{x}]\left(f(U)\,\bar{F} - \bar{F}U\right) = O(x).$$

Again, by Lemma I, it follows that the first equation (2) is satisfied.

As for the third matrix H, it satisfies the same equation since this is the converse of the equation (1) satisfied by F (Lemma 2). Alternatively, a direct proof may be supplied, as in the second part of the theorem, by considering the identity

$$y[y]'' = [y]''\, U + O(y^n).$$

This completes the proof of the theorem.

COROLLARY 1. *If $y=f(x)$ is the power series in which the constant term a_0 may appear, then the corresponding induced matrices of the function $f(x)-a_0$ satisfy the equations*

$$Ff(U)=(U+a_0 I)F,$$

$$f(U)H=H(U+a_0 I), \tag{5}$$

and similarly for the negatively induced matrix.

The corollary is trivial in that it contains the same term in a_0 on each side of the equation. The actual equation, when this term is removed, is that of (1); in expanded form it is

$$F(a_0 U+a_1 U^2+\ldots+a_{n-2}U^{n-1})=UF, \tag{6}$$

with an analogous equation for H and for $\bar{F}$.

COROLLARY 2. *If $a_0=0$, $a_0\neq 0$, then the inverse power series $f^{-1}(y)$ exists; and the matrix F^{-1}, induced by this inverse function, satisfies the matrix equations*

$$f(U)F^{-1}=F^{-1}U, \quad F^{-1}f^{-1}(U)=UF^{-1}.$$

Proof. The first equation follows at once from the theorem, since the determinant of F is seen to be non-zero when $a_0=0$, $a_0\neq 0$. The second follows by interchanging the rôles of x and y throughout the theorem after starting with the identity

$$\{x\}=F^{-1}\{y\}+O(y^n).$$

If $y=f(x)=\alpha+a_0x+a_1x^2+\ldots$ is a power series with a non-zero constant term α, and if the induced matrix is generated by the formula

$$\{y\}=F_\alpha\{x\}+O(x^n),$$

then it is no longer true that F_α satisfies the above commutantal equation. Instead, the actual state of things is as follows:

$$\big(F_\alpha f(U)-UF_\alpha\big)\{x\}=\{0,\ 0,\ \ldots,\ 0,\ y^n-O(x^n)\}. \tag{7}$$

That is, the vector implied by evaluating the product on the left is equal to a vector of zero components with the exception of the last and n-th, which consists of the first n terms in the power series for y^n. (Evidently no such terms exist if $\alpha=0$; and the statement reverts to that of Theorem I.)

In proof of this, one may proceed exactly as in this theorem; only the initial identity must now be written

$$y\{y\} = U\{y\} + \{0,\ 0,\ \ldots,\ 0,\ y^n\};$$

for relation (2) is no longer valid when $a \neq 0$.

For example, if $n = 3$ and if $y = a + bx$, then

$$F_a = \begin{bmatrix} 1 & . & . \\ a & b & . \\ a^2 & 2ab & b^2 \end{bmatrix}, \quad f(U) = aI + bU,$$

and

$$F_a f(U) = \begin{bmatrix} a & b & . \\ a^2 & 2ab & b^2 \\ a^3 & 3a^2 b & 3ab^2 \end{bmatrix},$$

so that

$$\left(F_a f(U) - U F_a\right)\{x\} = \{0,\ 0,\ a^3 + 3a^2 bx + 3ab^2 x^2\},$$

where the final element is identical with y^3 except for the term involving x^3.

THEOREM 2. *In the case of infinite matrices (when $n \to \infty$) the equation*

$$Xf(U) - UX = 0$$

is satisfied by the matrix F_a induced by the power series $y = f(x)$, provided that this series is absolutely convergent.

Proof. In this case the square of y can be formed term by term, as an absolutely convergent power series in x, and so can each successive power of y. Hence the vectors $\{x\}$ and $\{y\}$ exist (and each contains an infinity of elements columnwise).

Also the fundamental identity now assumes the simpler form

$$y\{y\} = U\{y\}, \tag{8}$$

without an additional modifying term; and, as before, the result follows from Lemma I by substituting throughout for y in terms of x. This proves the theorem.

As for the related matrices $\bar{F}$ and F''', the former is trivial unless $a = 0$, and the latter has no place for infinite matrices of the type which has just been utilized (although a theory of infinite matrices with an end term but with no initial term may be contemplated).

THEOREM 3. *The determinants of the induced matrix F_α and of its converse F'''_α are each equal to $a_0{}^m$, where $m = \frac{1}{2}n(n-1)$.*

Proof. Let there be n different values assigned to x, namely $x, x', x'', \ldots$, and let $y, y', y'', \ldots$ be the corresponding values of y. Further, let these values of x be small and all of the same order. Then the equation giving the vector $\{y\}$ may be written

$$\{y\} = \big(F + O(x)\big)\{x\},$$

where $O(x)$ denotes terms which contain x to a positive integral power. A similar equation holds for each of the n different values of x.

Now let two *square* matrices (of the type known as *alternants*) be formed:

$$X = [\{x\}, \{x'\}, \ldots], \quad Y = [\{y\}, \{y'\}, \ldots],$$

whose columns in order are indicated by the column power vectors $\{x\}$, etc. Then, since $O(x) = O(x') = \ldots$, we may write the matrix Y in the following form:

$$\begin{aligned} Y &= \Big[\big(F + O(x)\big)\{x\},\ \big(F + O(x)\big)\{x'\},\ \big(F + O(x)\big)\{x''\},\ \ldots\Big] \\ &= \big(F + O(x)\big)X. \end{aligned}$$

The determinant $|F|$ is evidently given by finding the limit of the determinantal quotient $|Y|/|X|$ when $x \to 0$. Also, by hypothesis, the denominator $|X|$ is an alternant which must satisfy the relation

$$|X| \equiv |x^0 x'^1 x''^2 \ldots| = \prod_{i \neq k} (x^{(i)} - x^{(k)}) \neq 0.$$

Once more, since $y^{(i)} - y^{(k)} = a_0(x^{(i)} - x^{(k)}) + O(x^2)$, it follows that, in the determinant $|Y| \equiv \Pi(y^{(i)} - y^{(k)})$, the terms of lowest order in x are expressible as $\mu|X|$, where

$$\mu = a_0{}^0 a_0{}^1 a_0{}^2 \ldots a_0{}^{n-1} \equiv a_0{}^m, \quad m = \tfrac{1}{2}n(n-1).$$

Accordingly the determinant $|F|$, which is a constant free from x, is equal to μ; and so too is $|H|$, by a similar proof.

An example of such a determinant is that of the three-rowed matrix in § 4 (4), where its value is $a_0{}^3$.

COROLLARY. *If, and only if, a_0 is non-zero, then both F and H are non-singular and possess reciprocals. If, further, $\alpha = 0$, then the negatively*

induced matrix $\bar{F}$ *is non-singular and possesses a determinant equal in value to* a_0^{-m}, *that of these reciprocal determinants.*

The first part of this corollary is evident; the second part follows by computing the product of the diagonal elements of $\bar{F}$, since all lower elements are zero if a vanishes.

7. *The commutantal equation.*

$$Xf(U) = g(U)X. \tag{1}$$

I now propose to apply the foregoing theory to the solution of this commutantal equation; namely *to find the most general square matrix* X *of order* n *which satisfies this equation*, where $f(x)$ and $g(x)$ are two ascending power series in a single argument x:

$$f(x) = a + a_0 x + a_1 x^2 + \ldots, \quad g(x) = \beta + b_0 x + b_1 x^2 + \ldots,$$

so that

$$\left.\begin{aligned} f(U) &= aI + a_0 U + a_1 U^2 + \ldots + a_{n-2} U^{n-1}, \\ \text{and} \qquad g(U) &= \beta I + b_0 U + b_1 U^2 + \ldots + b_{n-2} U^{n-1}. \end{aligned}\right\} \tag{2}$$

In the first place, the initial terms involving a and β may be rejected at once. For if $a = \beta$ these terms evidently cancel out of the equation; and, if $a \neq \beta$, then $X = 0$ is the only possible solution. This last follows from the general theory*, which states that the equation $XA = BX$ has, besides a zero, a non-zero solution only when the matrices A and B have one or more latent roots in common; and here each latent root of $f(U)$ is equal to a, and each of $g(U)$ to β, since the characteristic determinants of such semipersymmetric matrices are evidently

$$|f(U) - \lambda I| \equiv (a - \lambda)^n, \quad |g(U) - \lambda I| \equiv (\beta - \lambda)^n. \tag{3}$$

Another way of reaching the same conclusion is by equating corresponding elements on each side of relation (1) and then solving the resulting recurrence relations.

When the constant terms a and β are rejected, importance attaches to the first coefficients in the series for $f(x)$ and $g(x)$ which do not vanish identically.

Let us suppose that they are the coefficients of x^p and x^q respectively, and that the functions are now rewritten in the form of two convergent

* *Canonical matrices*, 148.

power series,

$$\left.\begin{aligned} f_p(x) &= a_0 x^p + a_1 x^{p+1} + a_2 x^{p+2} + \ldots, \\ g_q(x) &= b_0 x^q + b_1 x^{q+1} + b_2 x^{q+2} + \ldots, \end{aligned}\right\} \tag{4}$$

where p and q are positive integers, and where $a_0 \neq 0$, $b_0 \neq 0$. It will be shown that, by a functional transformation which induces a linear transformation upon the matrix X, it is possible to reduce the commutantal equation

$$X f_p(U) - g_q(U) X = 0$$

to the canonical form

$$Y U^p - U^q Y = 0; \tag{6}$$

and, further, that this can easily be solved. Let them be called commutantal equations of the orders (p, q).

8. *Reduction to canonical form.*

I shall give three methods, each of which possesses some interest, for reducing the commutantal equation of orders (p, q),

$$X f_p(U) - g_q(U) X = 0, \tag{1}$$

to the canonical form

$$Y U^p - U^q Y = 0.$$

The first is a rational, while the second and third are irrational methods.

(i) *The rational method.* Consider the *open range* power vector

$$\{\eta\} = \{1, 0, \ldots, y^p, 0, \ldots, y^{2p}, 0, \ldots, \ldots\}, \tag{2}$$

that is, a vector which consists of consecutive powers of y^p, starting with unity, and each separated by $p-1$ zero elements. Then, clearly, we have the relation

$$y^p \{\eta\} = U^p \{\eta\} + O(y^n). \tag{3}$$

Now let the vector $\{\eta\}$ be expressed in terms of x by means of the transformation

$$y^p = f_p(x) = a_0 x^p + a_1 x^{p+1} + \ldots \quad (a_0 \neq 0), \tag{4}$$

where, it will be noted, the variables x and y are of the same order (since $a_0 \neq 0$), so that $O(x^n) = O(y^n)$. Suppose that the result is written

$$\{\eta\} = F_0 \{x\} + O(x^n). \tag{5}$$

Then it follows at once that this matrix F_0, which is induced by the given open range vector $\{\eta\}$, satisfies the equation

$$F_0 f_p(U) - U^p F_0 = 0. \tag{6}$$

In fact,

$$\left(F_0 f_p(U) - U^p F_0\right)\{x\} = y^p\{\eta\} - U^p\{\eta\} + O(x^n);$$

so that the result follows by (3) and by Lemma I. Comparing the relations (2) and (5), we infer that each row of F_0 is zero, except the first, the $(p+1)$-th, the $(2p+1)$-th, and so on; F_0 is, in fact, singular unless p is merely unity.

There are, however, other possible solutions of (6); and, indeed, each of the following p matrices is a solution:

$$F_0, \quad U'F_0U, \quad U'^2F_0U^2, \quad \ldots, \quad U'^{p-1}F_0U^{p-1}. \tag{7}$$

Also each of these matrices occupies a different set of non-zero rows, in such wise that their sum involves all the rows. For example, $U'F_0U$ is generated from F_0 by translating each element of F_0 one place rightwards and one place downwards—that is, one place diagonally at a gradient of 1:1—and deleting the final row and column. Similarly for each successive matrix. It can then be seen intuitively that each such matrix satisfies (6); but an analytical proof can be composed as follows.

Let F_0 be denoted by $(1, 2, 3, \ldots, n)$, these indices referring to its rows. Then $U'F_0$ may be written as $(\cdot, 1, 2, \ldots, n-1)$ and UF_0 as $(2, 3, \ldots, n, \cdot)$. Then $U'F_0U$ will satisfy equation (6) if

$$U'F_0Uf_p(U) = U^pU'F_0U,$$

or, what is the same thing, if

$$(U'U^p - U^pU')F_0U = 0,$$

since U commutes with $f_p(U)$, and since $F_0f_p(U) = U^pF_0$. This means that the expression

$$\{(\cdot, p+1, p+2, \ldots, n, \cdot, \ldots) - (p, p+1, \ldots, n-1, \cdot, \ldots)\}U$$

must vanish, which is clearly the case, provided that row p and row n vanish when the matrix F_0 is post-multiplied by U. In any case, row p of F_0 is zero and row n has at most *one* non-zero element, in the final position; and this is deleted when postmultiplied by U. This proves that $U'F_0U$ is also a solution of (6); and a similar proof holds for the rest of the sequence (7).

Now if the sum of these is denoted by

$$F = F_0 + \sum_{r=1}^{p-1} U'^r F_0 U^r, \tag{8}$$

then F itself must be a solution of (6). Moreover, it is a non-singular solution, since every element upon its principal diagonal is either unity or a power of a_0, while every lower element is zero. This is, in fact, the solution which has been given by D. E. Rutherford*, who obtained it by solving the n^2 equations, implied in the commutantal equation (1), explicitly.

For example, if $F(a_0 U^2 + a_1 U^3 + a_2 U^4) = U^2 F$, then

$$F_0 = \begin{bmatrix} 1 & . & . & . & . & \dots \\ & . & . & . & . & \dots \\ & & a_0 & a_1 & a_2 & \dots \\ & & & . & . & \dots \\ & & & & a_0^2 & \dots \\ \dots & & \dots & \dots & \dots & \dots \end{bmatrix}, \qquad U' F_0 U = \begin{bmatrix} . & . & . & . & . & . & \dots \\ & 1 & . & . & . & . & \dots \\ & & . & . & . & . & \dots \\ & & & a_0 & a_1 & a_2 & \dots \\ & & & & . & . & \dots \\ & & & & & a_0^2 & \dots \\ \dots & & \dots & & \dots & & \dots \end{bmatrix},$$

and $F = F_0 + U' F_0 U$, where $|F| = a_0^{\rho}$ and $\rho = 1+1+2+2+\dots$ to $n-2$ terms.

Since F is a non-singular solution of (6), we may now write

$$f_p(U) = F^{-1} U^p F, \tag{9}$$

which puts in evidence, what is already known from the general theory of canonical matrices, that $f_p(U)$ is capable of such a rational reduction to its equivalent form U^p.

To apply this to the given equation (1), let the corresponding rational matrix G be constructed for which

$$g_q(U) = G^{-1} U^q G \quad (|G| \neq 0). \tag{10}$$

Then the equation $X f_p(U) - g_q(U) X = 0$ becomes

$$YU^p - U^q Y = 0, \tag{11}$$

where $Y = GXF^{-1}$. In this way the equation has been reduced to canonical form. The method is suitable also for infinite matrices.

* "On the canonical form of a rational integral function of a matrix", *Proc. Edin. Math. Soc.* (2), 3 (1932), 135-143 (139).

(ii) *The corresponding irrational method.* By taking the p-th root of both sides in equation (4), we obtain a relation

$$y = a_i(x+\beta_1 x^2+\beta_2 x^3+\ldots), \tag{12}$$

where a_i is a p-th root of $a_0 (\neq 0)$, and where the coefficients β_k are rational functions of the original coefficients a_k together with a_i. Let the corresponding induced matrix be given by

$$\{y\} = F_i\{x\}+O(x^n), \tag{13}$$

where the *close range* power vector $\{1, y, y^2, \ldots\}$ which has no zero gaps is employed, so that F_i is a non-singular matrix. Similarly, let G_j be the non-singular matrix induced by any of the q-th roots of the function $g_q(x)$. Then, exactly as before, the matrix

$$Y = G_j X F_j^{-1} \tag{14}$$

will satisfy the canonical commutantal equation

$$YU^p - U^q Y = 0 \tag{15}$$

for each of the p possible values of i, and of the q possible values of j. But usually this will be an irrational procedure. Like the preceding, this method applies to infinite matrices.

If, instead of taking the single F_i, we sum these p matrices, then the result is

$$pF_0 = F_1+F_2+\ldots+F_p, \tag{16}$$

where F_0 is the rational open range matrix of the previous method. Similarly, $qG_0 = \Sigma G_j$. Another further step, introducing a geometrical progression of solutions, such as (8), is then necessary, in order to construct a non-singular alternative to F_i and G_j.

(iii) *The bilinear method.* As before, let the non-singular matrices F_1 and G_1 be induced by the functions $\sqrt[p]{f_p(x)}$ and $\sqrt[q]{g_q(x)}$ respectively according to the transformations

$$y^p = f_p(x), \quad z^q = g_q(x); \tag{17}$$

so that the power vectors of y and z satisfy the relations

$$\{y\} = F_1\{x\}+O(x^n), \quad \{z\} = G_1\{x\}+O(x^n), \tag{18}$$

where $|F_1| \neq 0$, $|G_1| \neq 0$. Since, by these hypotheses, the variables x, y, z are all of the same order, and the matrices F_1 and G_1 are non-singular, it follows that

$$F_1^{-1}\{y\}+O(y^n) = \{x\} = G_1^{-1}\{z\}+O(z^n). \tag{19}$$

Now, if X is a solution of the commutantal equation, then

$$0 = \left(Xf_p(U) - g_q(U)\, X \right) \{x\}$$
$$= \left(Xf_p(x) - g_q(U)\, X \right) \{x\} + O(x^n).$$

On placing the scalar factor $f_p(x)$ first in the first term and then substituting for x in terms of y, we have

$$0 = \left(y^p\, XF_1^{-1} - g_q(U)\, XF_1^{-1} \right) \{y\} + O(y^n).$$

When the scalar factor y^p of the first term is placed last and then replaced by U^p [by § 3 (5)], this becomes

$$0 = \left(ZU^p - g_q(U)\, Z \right) \{y\} + O(y^n) \quad (Z = XF_1^{-1}).$$

Since the prefactor of $\{y\}$ is constant, Lemma I applies. Hence the original equation has been brought to what may be called semi-canonical form,

$$0 = ZU^p - g_q(U)\, Z, \quad Z = XF_1^{-1}. \tag{20}$$

Now let this be premultiplied by $[\bar{x}]$, the negative power vector; and let $[\bar{z}]$ be the corresponding vector of z, so that, by (18),

$$[\bar{z}] = [\bar{x}]\, \bar{G}_1 + O(x). \tag{21}$$

Since x and z are of the same order, $\bar{G}_1$ is non-singular and the correlative relation exists,

$$[\bar{x}] = [\bar{z}]\, \bar{G}_1^{-1} + O(z). \tag{22}$$

Let (20) be premultiplied by $[\bar{x}]$ and then transformed into terms of z instead of x. Then

$$0 = [\bar{x}] \left(ZU^p - g_q(U)\, Z \right) = [\bar{x}](ZU^p - z^q\, Z) + O(x)$$
$$= [\bar{z}](KZU^p - U^q\, KZ) + O(z),$$

where $K = \bar{G}_1^{-1}$. Again, by Lemma I, $KZU^p - U^q\, KZ$ must vanish. That is,

$$YU^p - U^q\, Y = 0, \tag{23}$$

where

$$Y = KZ = \bar{G}_1^{-1}\, XF_1^{-1}. \tag{24}$$

This applies to finite or infinite matrices. Alternatively, for the finite case only, the semi-canonical form (20) may be reduced to the form (23) by using the reversed vector $[x]''$ of descending positive integral powers, instead of $[\bar{x}]$, and the consequent relation

$$[z]'' = [x']\, G_1''' + O(x^n) \tag{25}$$

instead of (21). The same procedure now yields equation (23), where

$$Y = HZ = HXF_1^{-1}, \quad H^{-1} = G_1''. \tag{26}$$

Since both H and F are non-singular, it follows that the matrices X and Y are equivalent in the field $\mathfrak{F}$ to which the elements of H and F belong; that is, in the field containing all the coefficients a_i and b_i, together with the p-th root of a_0 and the q-th root of b_0. The general solution for X is now given by solving this simpler equation for Y.

9. *Solution of the canonical equation.*

To solve the canonical equation

$$YU^p - U^q Y = 0, \tag{1}$$

let $Y = [c_{ij}]$, so that c_{ij} is the element of the i-th row and the j-th column. Then the equation is coextensive with n^2 equations

$$c_{i,\,j-p} - c_{i+q,\,j} = 0, \tag{2}$$

where each of i and j range from 1 to n inclusive, it being understood that, when a value renders either suffix less than 1 or greater than n, then the element is automatically zero.

If $p \geqslant n$, $q \geqslant n$, then (1) is a trivial identity, since neither coefficient U^p nor U^q exists. Apart from this there are three cases possible, the first two being again comparatively trivial:

(i) $p \geqslant n > q$, (ii) $p < n \leqslant q$, (iii) $p < n$, $q < n$.

Evidently, in case (i), the general solution Y contains nq arbitrary elements, filling the first q rows and followed by $n-q$ zero rows.

Evidently, in case (ii), the general Y contains np arbitrary elements, filling the last p columns and preceded by $n-p$ zero columns.

In the remaining case (iii) [which will break up into three subcases (iii) α, β, γ] it follows from (2) that Y is a *persymmetric matrix of gradient* q/p; that is, every element c is equal to each element which lies on a straight line drawn through c at a falling gradient measured by q rows downwards and p columns to the right. This simple rule must, however, be modified, owing to the fact that, for certain admissible values of the suffixes i and j, one or both terms in equation (2) are zero. The matrix Y, in fact, possesses certain elements which are essentially zero.

It will now be shown that, in every case, these essentially zero elements lie in a simply connected region separated from what will now be called the persymmetric region by a zigzag boundary; and that the precise

shape of this boundary depends upon the relative values of the integers n, p, q. But, before giving the actual conditions, it is simplest to take a few typical examples. In each case, the leading submatrix of Y, common to the first q rows and the first p columns, will be called C_1, while the reverse submatrix, common to the last q rows and the last q columns, will be called C_1''.

(α) If $n = 5, p = 2, q = 1$, then

$$Y = \begin{bmatrix} a & b & c & d & e \\ . & . & a & b & c \\ . & . & . & . & a \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}, \quad \begin{aligned} C_1 &= [c_{11}, c_{12}] = [a, b], \\ C_1'' &= [c_{54}, c_{55}] = 0. \end{aligned} \tag{3}$$

There are nq arbitrary elements.

(β) If $n = 5, p = 1, q = 2$, then

$$Y = \begin{bmatrix} . & . & a & c & e \\ . & . & . & b & d \\ . & . & . & a & c \\ . & . & . & . & b \\ . & . & . & . & a \end{bmatrix}, \quad C_1 = \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} = 0, \quad C_1'' = \begin{bmatrix} c_{45} \\ c_{55} \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}. \tag{4}$$

There are np arbitrary elements.

(γ) If $n = 5, p = 4, q = 3$, then

$$Y = \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & k & l & m & n \\ . & 0 & 0 & 0 & a \\ . & 0 & 0 & 0 & f \end{bmatrix}, \quad C_1 = \begin{bmatrix} a & b & c & d \\ f & g & h & i \\ . & k & l & m \end{bmatrix}, \quad C_1'' = \begin{bmatrix} k & l & m & n \\ . & . & . & a \\ . & . & . & f \end{bmatrix}, \tag{5}$$

and there are fourteen arbitrary elements (the general formula will follow later).

In each case a submatrix C_1 occupies the head of the first p columns; and, in order to satisfy relation (2), the remaining and lower elements of

these p columns must be zero. Similarly, C_1'' occupies the foot of the last p columns; and to the left of this submatrix all the remaining elements of the last q rows must be zero. Evidently, by the property of persymmetry, these zero blocks, of columns and of rows, are repeated in staircase fashion, each being bounded by a staircase barrier at a gradient of q in p. There are now three possibilities, as exemplified above. In example (α) the descending staircase boundary lies entirely above that of C_1'', which is an ascending boundary. Consequently the latter is irrelevant and is rejected. In example (β) the ascending staircase boundary of C_1'' lies entirely to the right of that belonging to C_1, which in turn is rejected. In example (γ) the corresponding boundaries overlap, causing certain elements (marked 0) belonging to either of C_1 and C_1'' to vanish. The resultant boundary, in case of doubt, is that which always lies uppermost and most to the right.

The analytical criteria for these various cases can now be given. Let p' be the quotient and r the remainder when n is divided by p; and let q' and s similarly refer to q, so that

$$pp'+r=n=qq'+s, \quad r<p, \quad s<q. \tag{6}$$

Then either $p'<q'$ or $p'>q'$ or $p'=q'$; and these three alternatives correspond to the above examples.

(iii α) If $p'<q'$, as in example (α), then the descending staircase prevails; and the matrix can be written in the form

$$Y=\begin{bmatrix} C_1 & C_2 & C_3 & \dots \\ . & C_1 & C_2 & \dots \\ . & . & C_1 & \dots \\ . & . & . & \dots \end{bmatrix}, \tag{7}$$

containing nq distinct arbitrary constants in the first q rows, which are repeated persymmetrically below. Each C_i has q rows and p columns, with the exception of $C_{p'+1}$ (if it exists) which has r columns.

(iii β) If $p'>q'$, as in example (β), then the ascending staircase prevails; and

$$Y=\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & . & C_1'' & C_2'' & C_3'' \\ \dots & . & . & C_1'' & C_2'' \\ \dots & . & . & . & C_1'' \end{bmatrix}, \tag{8}$$

containing np distinct arbitrary constants in the last p columns, per-symmetrically repeated leftwards. Each C_i'' has q rows and p columns, with the exception of $C_{q'+1}''$ (if it exists) which has s rows.

(iii γ) If $p' = q'$, the overlapping case, as in example (γ), then either of the above formulae for Y, as partitioned into submatrices C_i or C_i'', will apply, provided that the leading submatrix C_1 or C_1'' is suitably modified.

As for C_1, it must lose a rectangular array of $r(q-s)$ elements situated in the first r columns and last $q-s$ rows; and, correlatively, C_1'' loses $s(p-r)$ elements from its first $p-r$ columns and last s rows. The number of arbitrary elements in Y is now ν, where

$$np-s(p-r) = \nu = nq-r(q-s). \tag{9}$$

The equality of these two forms of ν follows at once by eliminating p' and q' from the above relations (6), when $p' = q'$. The result may be written

$$Y = \begin{bmatrix} A & C_2 & C_3 & \dots \\ . & A & C_2 & \dots \\ . & . & A & \dots \\ . & . & . & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & . & B & C_2'' & C_3'' \\ \dots & . & . & B & C_2'' \\ \dots & . & . & . & B \end{bmatrix}, \tag{10}$$

where A and B are these modified initial submatrices, while the remaining submatrices C_i and C_i'' are as before.

This case includes those in which one or both remainders, r and s, are zero. When both are zero there is complete overlap, and each matrix A coincides with a B and contains its full complement of non-zero elements.

A similar analysis applies to the solution of the equation

$$YU^p - V^q Y = 0,$$

where V is an auxiliary unit square matrix of order m, so that Y must have m rows and n columns. In this case,

$$n = pp' + r, \quad m = qq' + s. \tag{11}$$

In the case of *infinite matrices* $[x_{ij}]$, when the suffixes i and j take all positive integral values, the solution of the equation

$$YU^p - U^q Y = 0$$

is simply

$$Y = \begin{bmatrix} C_1 & C_2 & C_3 & \dots \\ . & C_1 & C_2 & \dots \\ . & . & C_1 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \tag{12}$$

where, as before, each C_i is an arbitrary submatrix of q rows and p columns. The other cases, which are necessary in the finite case, do not arise here.

It is convenient to introduce the notation

$$Y = \text{arb}\,\phi\,(p, q) \tag{13}$$

to denote the general solution of the above equation in all its cases. The number of arbitrary constants will be denoted by μ, where

$$\mu = nq,\ np,\ \nu, \tag{14}$$

according to the above cases. Since these constants λ_i enter linearly into Y, we may write

$$Y = \lambda_1 Y_1 + \lambda_2 Y_2 + \ldots + \lambda_\mu Y_\mu. \tag{15}$$

For instance, in example (α) above, $\mu = 5$; and the five constants are denoted by a, b, c, d, e. The matrix Y_1 is that obtained from Y by putting $a = 1$ and the remaining four constants zero; Y_1 contains twenty-two zero and three unit elements. Each Y_1, here and always, contains a single row of non-zero (and unit) elements at a gradient of q/p.

The number of non-zero elements possessed by Y_i depends upon the actual position of its corresponding constant λ_i in the array Y. It is useful to call those among the Y_i which contain most non-zero elements *key matrices*. In the cited example, Y_1 alone is a key matrix; but in example (γ) the coefficient of a *or* f is a key matrix. It is most convenient to choose, as key matrix, that belonging to the non-zero element of Y which stands nearest to the top left-hand corner. Let this be Y_1 in all cases. Then, by the fundamental properties of persymmetry, each Y_i is derivable from Y_1 by a relation of the form

$$Y_i = U'^{q_i} Y_1 U^{p_i}, \tag{16}$$

where p_i, q_i are the numbers of steps, rightwards and downwards, to the uppermost non-zero element of Y_i relative to Y_1. This enables us to write Y explicitly in terms of the key Y_1 and of the auxiliary unit matrices U and U', as follows:

$$Y = \sum_{i=1}^{\mu} \lambda_i U'^{q_i} Y_1 U^{p_i}, \tag{17}$$

where the λ_i are the complete set, μ in number, of arbitrary constants which it is possible for Y to possess.

10. *Solution of the commutantal equation.*

The general solution of the equation

$$Xf_p(U)-g_q(U)X=0 \tag{1}$$

is now obtained at once by first reducing the equation to the canonical form

$$YU^p-U^qY=0, \tag{2}$$

and then solving for Y. According to the three methods of reduction employed (or four, in the case of finite matrices), the value of X is

$$GY_{(1)}F^{-1}, \quad G_jY_{(2)}F^{-1}, \quad \bar{G}_jY_{(3)}F_i, \quad G_j'''Y_{(4)}F_i, \tag{3}$$

where the four $Y_{(k)}$ are general solutions of (2), each containing μ arbitrary constants. Of these the first solution alone is always rational; and the fourth applies only to finite matrices. Furthermore, each of these general solutions for Y can be expressed as the sum of μ terms, which lead, from (17) of the last section, to a corresponding general solution for X,

$$X=G\left(\sum_{r=1}^{\mu}\lambda_rU'^{q_r}Y_1U^{p_r}\right)F^{-1} \tag{4}$$

[or, alternatively, the fore and after factors may be any pair among the four cases (3) above]. By putting all but one of λ_i equal to zero, and considering the value of $G^{-1}XF$, it follows that this particular λ_i must be present in X; otherwise X could vanish for a case when Y does not, which is impossible. *Hence X contains exactly μ arbitrary constants* and may be written

$$X=\lambda_1X_1+\lambda_2X_2+\ldots+\lambda_\mu X_\mu, \tag{5}$$

where the precise character of each X_r is evident from (4).

Example. As an example of the fourth alternative method of solution, consider the equation

$$X(U^2-\alpha U^3)-(U^3+\beta U^4)X=0.$$

Here $f(x)=x^2-\alpha x^3=z^2$, and $g(x)=x^3+\beta x^4=y^3$. Hence, if the simplest value of z is chosen, and if $|x|<1$,

$$z=x(1-\alpha x)^{\frac{1}{2}}=x-\tfrac{1}{2}\alpha x^2-\tfrac{1}{8}\alpha^2x^3-\ldots.$$

Similarly,
$$y=x+\tfrac{1}{3}\beta x^2-\tfrac{1}{9}\beta^2x^3-\ldots.$$

On calculating the various powers of y and z in terms of x, we obtain

the induced matrices F and G'''. Thus

$$F = \begin{bmatrix} 1 & . & . & . & \dots \\ . & 1 & -\frac{1}{2}\alpha & -\frac{1}{8}\alpha^2 & \dots \\ . & . & 1 & -\alpha & \dots \\ . & . & . & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}, \quad G''' = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & 1 & \frac{2}{3}\beta & -\frac{1}{9}\beta^2 & . \\ \dots & . & 1 & \frac{1}{3}\beta & . \\ \dots & . & . & 1 & . \\ \dots & . & . & . & 1 \end{bmatrix}.$$

Also the general matrix Y which satisfies $YU^2 = U^3 Y$ is

$$Y = \begin{bmatrix} \dots & \dots & \dots \\ \dots & a & b & g & h \\ \dots & . & . & e & f \\ \dots & . & . & c & d \\ \dots & . & . & a & b \end{bmatrix}, \quad C_1'' = \begin{bmatrix} c & f \\ c & d \\ a & b \end{bmatrix},$$

provided that $n > 3$, that is, that we are dealing with four or more rows and columns. The general solution is therefore

$$X = G''' Y F,$$

where $a, b, \dots$ are arbitrary constants.

11. *Characteristic relations between power matrices.*

It is interesting to compare the alternative values of X, and particularly the last two. If they are equated and if the final non-singular factor is deleted, the result may be expressed as

$$\bar{G}_j \operatorname{arb} \psi(p, q) = G_j''' \operatorname{arb} \chi(p, q), \tag{1}$$

where either solution ψ or χ may be taken arbitrarily, it being then understood that the other is determined thereby. This is due to more fundamental relations which subsist between the three typical induced matrices $H, \bar{H}, H'''$ belonging to any power series $h(x)$. The relations are

$$H\bar{H} = \omega_{(1)}(U), \quad HH''' = \omega_{(2)}(U), \tag{2}$$

where $\omega_{(1)}$ and $\omega_{(2)}$ are certain specific polynomials in U depending upon the coefficients of the power series $h(x)$. Briefly these relations characterize

the *close range* matrices F_i, G_j as opposed to the open range matrices F_0 (when $p > 1$) and G_0 (when $q > 1$).

To prove that G_j satisfies these relations (2), it must be shown that both $G_j \overline{G}_j$ and $G_j G_j'''$ are expressible as polynomials in U. The proofs are similar. Consider the expression

$$K \equiv G_j \sqrt[q]{\{g_q(U)\}} - U G_j.$$

When this is postmultiplied by the vector $\{x\}$, and then expressed in terms of z by § 8 (17), it vanishes. Hence Lemma I applies, and K itself is identically zero; and so also is its converse K'''. Thus

$$\sqrt[q]{\{g_q(U)\}}\, G_j''' - G_j''' U = 0.$$

Hence, after premultiplication by G_j,

$$U G_j G_j''' - G_j G_j''' U = 0; \tag{3}$$

that is, $G_j G_j'''$ must be a solution of the equation $YU^p = U^q Y$, where $p = q = 1$. Such a solution is persymmetric and of gradient $1:1$, so that it is a polynomial in U. Hence G_j, and any such induced matrix represented by H, satisfies the second relation (2). Similarly for the first.

Conversely it readily follows that, if G_j is a non-singular matrix satisfying the properties (2), then it satisfies relations such as (1).

In illustration of the second relation $HH''' = \omega_{(2)}(U)$, the three-rowed case can be exhibited:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} h_{33} & h_{23} & h_{13} \\ h_{32} & h_{22} & h_{12} \\ h_{31} & h_{21} & h_{11} \end{bmatrix} = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 \\ . & \omega_0 & \omega_1 \\ . & . & \omega_0 \end{bmatrix}. \tag{4}$$

It would be interesting to find out the nature of the most general matrix $[h_{ij}]$ which possesses this property. For the present I merely remark that, when this is an induced matrix, all the elements h_{ij} $(i > j)$ situated below the principal diagonal are zero, and the elements of the i-th row are the coefficients of the ascending powers of x in the expression for y^{i-1}, where

$$y = \sum_{r=1}^{n-1} h_{2,1+r} x^r = h(x). \tag{5}$$

If $y = xz$, then z is a polynomial in x with h_{22} for constant term. The relation (4), for n rowed matrices, then states that each element upon the k-th diagonal of the left-hand product is equal to ω_k; and this is

equivalent to stating that the coefficient of x^k in the expansion of each of the products $z^{n-1}z^0$, $z^{n-2}z$, $z^{n-3}z^2$, ... is the same, which is manifestly true.

Incidentally this shows that the polynomial $\omega_{(2)}(U)$ is determined by the relation

$$z^{n-1} = \left(\frac{y}{x}\right)^{n-1} = \left(\frac{h(x)}{x}\right)^{n-1} = \omega_{(2)}(x) + O(x^n), \tag{6}$$

obtained by considering the first of these alternative equivalent products $z^{n-r}z^r$. The formula is true for all orders p of $h(x)$, that is, for the case when the lowest power of x occurring in y is the p-th. The first non-vanishing element is ω_{p-1}; also the final element ω_{n-1} always vanishes, as is seen by multiplying the top row by the last column in (4). The polynomial $\omega_{(2)}(U)$ is therefore of degree $n-2$ or less.

Another explicit mode for relation (4) is obtained by considering the elements in the final column of the product matrix HH''' (which must be the ω_k in reverse order). A little consideration shows that (4) can be written

$$HH''' = \left((x^{n-1})\right) y \frac{y^{n-1}I - U^{n-1}}{yI - U},$$

where $y = h(x)$, and $\{y\} = H\{x\} + O(x^n)$, and where $\left((x^{n-1})\right)$ means the coefficient of x^{n-1} in the expression following it.

12. *Application to matrices in general.*

The place of the foregoing investigation is seen in problems such as the following:

If A is a general matrix, of n rows and columns, which has been reduced to the classical canonical form HCH^{-1}, and if $f(A)$ is a polynomial in A which has been reduced to the classical canonical form KDK^{-1}, then it is required to find the value of K in terms of H and $f(C)$.

For example, let the classical canonical form of a matrix A be

$$F(\alpha, \beta) \equiv \operatorname{diag}\left(\begin{bmatrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{bmatrix}, \begin{bmatrix} \alpha & 1 \\ & \alpha \end{bmatrix}, \begin{bmatrix} \beta & 1 \\ & \beta \end{bmatrix}, \beta \right), \tag{1}$$

where $n = 8$, and where there are two distinct non-zero latent roots α, β

occurring in elementary divisors of exponents 3, 2 and 2, 1 respectively; and let

$$f(A) = A^2$$

be the function under view. Then $A = HCH^{-1}$, $A^2 = HC^2H^{-1}$, so that D must be the classical canonical form of C^2, that is of

$$\operatorname{diag}\left(\begin{bmatrix} \alpha^2 & 2\alpha & 1 \\ & \alpha^2 & 2\alpha \\ & & \alpha^2 \end{bmatrix}, \begin{bmatrix} \alpha^2 & 2\alpha \\ & \alpha^2 \end{bmatrix}, \begin{bmatrix} \beta^2 & 2\beta \\ & \beta \end{bmatrix}, \beta^2\right). \tag{2}$$

The actual value of D is the same in form as (1), but with α^2 replacing α and β^2 replacing β. (Complications would have arisen, had the case when $\alpha\beta = 0$ been considered.) What is therefore required is the nature of the non-singular matrix P_1 which will satisfy the relation

$$P_1 \begin{bmatrix} \alpha^2 & 2\alpha & 1 \\ & \alpha^2 & 2\alpha \\ & & \alpha^2 \end{bmatrix} P_1^{-1} = \begin{bmatrix} \alpha^2 & 1 & \\ & \alpha^2 & 1 \\ & & \alpha^2 \end{bmatrix}. \tag{3}$$

Four such matrices P_1, P_2, P_3, P_4 (of respective orders 3, 2, 2, 1) are needed; and, when they are found, the composite matrix

$$P = \operatorname{diag}(P_1, P_2, P_3, P_4)$$

at once transforms the semi-reduced expression (2) into its canonical form $F(\alpha^2, \beta^2)$. The general problem, in the case of $f(A)$, is that of finding these submatrices P_i, where, for example,

$$P_1\left(f(\alpha)\,I + f'(\alpha)\,U + \frac{1}{2!}f''(\alpha)\,U^2 + \ldots\right) = \left(f(\alpha)\,I + U\right)P_1,$$

with the proviso that $f'(\alpha) \neq 0$; and this is the type of commutantal equation already discussed, where $p = q = 1$.

The simplest non-singular P_1 is, according to Theorem I, the matrix induced by the function

$$y = f(\alpha + x) - f(\alpha) \quad [f'(\alpha) \neq 0],$$

so that $\{y\} = P_1\{x\} + O(x^e)$, where e is the exponent of the elementary divisor (here $e = 3$) which concerns P_1.

Summing these results up, we may write

$$A = HCH^{-1}, \quad f(A) = KDK^{-1}, \quad f(A) = Hf(C)\,H^{-1},$$

by the fundamental property of the collineatory group in which A is equivalent to C and $f(A)$ to $f(C)$. Each diagonal submatrix of $f(C)$, analogous to (2), is reduced by a non-singular P_i to canonical form, as in (3). Hence

$$D = Pf(C)\,P^{-1}.$$

The value of K which reduces $f(A)$ directly to canonical form is therefore

$$K = HP^{-1}.$$

The above example, which has illustrated the foregoing methods, is confined to the case when $f'(\lambda)$ is non-zero for each latent root λ of A; that is, for the case* when A and $f(A)$ possess exactly the same set of characteristic exponents e_{ij}. The complexities of the case when $f'(\lambda)$ may vanish must, at this stage, be left aside.

13. *Generating functions.*

There are some interesting properties involved in the induced matrices of a first order commutantal equation (when $p = q = 1$),

$$Xf(U) - UX = 0, \tag{1}$$

where the function $f(x)$ is an absolutely convergent power series starting with an initial term $a_0 x$. Let y and z be functions of x defined by the relations

$$y = xz, \quad z = a_0 + a_1 x + a_2 x^2 + \ldots, \quad a_0 \neq 0, \tag{2}$$

and let F be the non-singular matrix induced by y, so that $Ff(U) = UF$; that is, $f(U) = F^{-1}UF$. Hence

$$YU = UY, \tag{3}$$

where $Y = XF^{-1}$. The general solution for Y is $\phi(U)$, so that

$$X = \phi(U)\,F = (p_0 I + p_1 U + p_2 U^2 + \ldots + p_{n-1} U^{n-1})\,F \tag{4}$$

is the general solution for X, where the p_i are the n arbitrary constants. (In this case $\mu = n$.) Alternatively, the solution can be written in the form

$$X = F\psi(U) = F(q_0 I + q_1 U + q_2 U^2 + \ldots + q_{n-1} U^{n-1}), \tag{5}$$

where the q_i are arbitrary constants; and the reason is as follows.

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* *Canonical matrices* (1932), 75.

Let $V=f(U)$. Then, since $FV=UF$, it follows that $FV^r=U^rF$. Whence

$$\phi(U)F=F\phi(V), \tag{6}$$

so that

$$\psi(U)=\phi\big(f(U)\big). \tag{7}$$

Hence the coefficients q_i are determined as linear forms in the p_i, with coefficients which depend upon the original a_i. We have, in fact,

$$q_0+q_1\lambda+q_2\lambda^2+\ldots=p_0+p_1f(\lambda)+p_2\big(f(\lambda)\big)^2+\ldots \tag{8}$$

identically for all values of λ. Conversely, the p_i may be given linearly in terms of the q_i; and the whole matter is clearly related to the type of expansion associated with the name of Arbogast*. If

$$\phi(a_0+a_1x+a_2x^2+\ldots)\equiv\phi(z)$$

is expanded in the form

$$c_0+c_1x+c_2x^2+\ldots,$$

then the coefficients c_r satisfy the formula

$$(r+1)c_{r+1}=\left(a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+\ldots\right)c_r\equiv\Delta c_r. \tag{9}$$

This result is a consequence of the identity

$$\Delta\phi(z)=\frac{d}{dx}\phi(z), \tag{10}$$

which is easily verifiable.

Let us now consider the elements a_{rs} of the matrix F induced by the power series y. For this purpose it is well to adopt a provisional convention for the suffixes r and s, by assuming that r denotes the row, and s the diagonal, both being numbered from zero onwards. In fact, let

$$F=(a_{rs})=\begin{bmatrix}1 & . & . & . & \ldots\\ . & a_{10} & a_{11} & a_{12} & \ldots\\ . & . & a_{20} & a_{21} & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\end{bmatrix}, \qquad a_{00}=1, \qquad a_{0s}=0 \quad (s>1). \tag{11}$$

Since the elements of the r-th row are, by hypothesis, generated by the

* *Calcul des dérivations* (Strasburg, 1800). *Cf.* Edwards, *Differential calculus* (1900), 449–450, where the operational method due to Jolliffe is quoted.

series for y^r, we can write

$$y^r = \Big(f(x)\Big)^r = (a_0 x + a_1 x^2 + \ldots)^r = a_{r0} x^r + a_{r1} x^{r+1} + \ldots, \tag{12}$$

where r is zero or else a positive integer. Again, this leads clearly to an identity

$$\Big(\sum_{s=0}^{\infty} a_{rs} x^s\Big)\Big(\sum_{s=0}^{\infty} a_{1s} x^s\Big) = \sum_{s=1}^{\infty} a_{r+1,s} x^s, \tag{13}$$

in which the coefficients of terms containing x^s may be equated; so that

$$a_{r,0}\, a_{1,s} + a_{r,1}\, a_{1,s-1} + \ldots + a_{r,s}\, a_{1,0} = a_{r+1,s}, \tag{14}$$

which is precisely the condition that the element in the r-th row and the s-th diagonal should be the same in the product matrix $Ff(U)$ as in the product matrix UF. This gives another proof that F is a solution of equation (1).

In terms of (11) the general solution (5) of this equation (1) is

$$X = \begin{bmatrix} q_0, & q_1, & q_2, & \ldots, & q_{n-1} \\ \cdot, & a_{10} q_0, & a_{11} q_0 + a_{10} q_1, & \ldots, & \ldots \\ \cdot, & \cdot, & a_{20} q_0, & \ldots, & \ldots \\ \ldots & \ldots & \ldots \quad \ldots & \ldots & \ldots \end{bmatrix}, \tag{15}$$

where all the elements of the top row are arbitrary constants, while those of subsequent rows are completely determined thereby.

Again, the reciprocal equation

$$f(U)\, X - XU = 0 \tag{16}$$

is reduced in the same way to $YU = UY$ by taking $Y = FX$. Hence the general solution of (16) is

$$X = F^{-1} \phi_1(U) = F^{-1}(p_0' I + p_1' U + \ldots + p_{n-1}' U^{n-1}), \tag{17}$$

where the p_i' are arbitrary constants. Evidently this function $\phi_1(U)$ is the reciprocal of the original $\phi(U)$ in (4). A similar solution of type $\psi_1(U)\, F^{-1}$ can also be given.

This matrix F^{-1}, which will be denoted by P, can be evaluated from F as given in (11), where $a_{r0} = a_0^r \neq 0$, being written in the form

$$P = F^{-1} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \ldots \\ \cdot & u_{10} & u_{11} & u_{12} & \ldots \\ \cdot & \cdot & u_{20} & u_{21} & \ldots \\ \cdot & \cdot & \cdot & u_{30} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}. \tag{18}$$

Since P is a solution of the equation (16), the (r, s)-th element of $f(U)\,P$ must be equal to that of PU. Hence

$$a_0\,u_{r+1,\,s-1}+a_1\,u_{r+2,\,s-2}+\ldots+a_{s-1}\,u_{r+s,\,0}=u_{r,\,s-1}, \tag{19}$$

where $r \geqslant 0$, $s > 0$. This provides a set of recurrence relations analogous to (14), but of greater interest. The generating function of these elements u_{rs} is given by

$$\frac{y'}{y^{r+1}}=\frac{f'(x)}{\big(f(x)\big)^{r+1}}=\frac{u_{r,0}}{x^{r+1}}+\frac{u_{r-1,1}}{x^r}+\frac{u_{r-2,2}}{x^{r-1}}+\ldots, \tag{20}$$

where $r \geqslant 0$, and where terms involving x to a zero or positive power are immaterial as far as the matrix P is concerned. In proof of this, it is necessary to multiply $y'y^{-r-s-1}$ throughout by $f(x)$, then to form an identity analogous to (13), and finally to extract the terms in x^{-r-1}, which at once gives the recurrence formula (19). This shows that the u_{rs} as defined by (20) form a triangular matrix which is a solution of (16), and which therefore is included in (17), but is not yet proved to be identical with (18).

To complete the proof it must be shown that in (20) the coefficients $u_{0s}\,(s > 0)$ are, in fact, zero, and that $u_{00} = 1$. This would make the top row elements (and therefore all lower row elements) agree with those of P. Now these coefficients u_{0s} are zero if, for each value of r, the coefficient of x^{-1} in the series (20) is zero. But this must be so, otherwise, upon integrating the relation, as it stands, with regard to x, we should obtain an expansion for $-1/(ry^r)$ in terms of ascending powers of x, and including a term in $\log x$; this is impossible.

A direct verification also is afforded by showing that the r-th derivative of $x^{r+1}y'/y^{r+1}$ vanishes with x. But $y = xz$, and

$$\frac{d^r}{dx^r}\left(\frac{y'}{z^{r+1}}\right)=\frac{d^r}{dx^r}\left(\frac{1}{z^r}+\frac{xz'}{z^{r+1}}\right)=x\,\frac{d^r}{dx^r}\left(\frac{z'}{z^{r+1}}\right)$$

by Leibniz' theorem. Since z does not vanish with x, the verification is complete.

If the numerator $f'(x)$ of (20) is replaced by an arbitrary power series $\sum_{m=0}^{\infty} c_m x^m$, then the analogous right-hand series generates the elements in the r-th row of the *general* solution of the equation (16); for the coefficients continue to satisfy the same recurrence relations (19). Only in this case the u_{0s} are not necessarily zero; they are linear functions

of the first n coefficients c_m, which, in fact, may be taken as an alternative set to the arbitrary constants p_i' of (17).

The integrated form of relation (20) can be written

$$\frac{1}{y^r} = \frac{1}{\big(f(x)\big)^r} = \frac{u_{r,0}}{x^r} + \frac{ru_{r-1,1}}{(r-1)\,x^{r-1}} + \frac{ru_{r-2,2}}{(r-2)\,x^{r-2}} + \ldots, \tag{21}$$

where the coefficients of the negative and zero powers of x are the elements in the $(r+1)$-th column of the negative induced matrix $\bar{F}$, as given by the relation $[\bar{y}] = [\bar{x}]\,\bar{F} + O(x)$. For this is just what (21) implies.

14. *The relation between the reciprocal and the negative induced matrices.*

Among the solutions of the same equation $f(U)\,X = XU$ is this negative induced matrix, let us say,

$$Q = \bar{F} = \begin{bmatrix} 1 & v_{01} & v_{02} & v_{03} & \cdots \\ . & v_{10} & v_{11} & v_{12} & \cdots \\ . & . & v_{20} & v_{21} & \cdots \\ . & . & . & v_{30} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{1}$$

where the elements v_{rs} also satisfy the recurrence relation (19) of the last section. Let the generating series (20) be multiplied throughout by x. Then, since $u_{0s} = 0$, there is now no constant term in the series, and all the terms except those involving negative powers of x may be denoted by $O(x)$. If we tabulate the series for all values of r, they at once yield the matrix formula

$$\left[1,\ \frac{xy'}{y},\ \frac{xy'}{y^2},\ \ldots,\ \frac{xy'}{y^{n-1}}\right] = [\bar{x}]\,P + O(x), \tag{2}$$

where $y' = dy/dx$ and $\bar{x} = [1, x^{-1}, x^{-2}, \ldots, x^{1-n}]$. Now

$$\frac{xy'}{y^{r+1}} = \frac{z'}{x^{r-1}z^{r+1}} + \frac{1}{y^r}, \tag{3}$$

where, as before, $z = y/x$, so that the power series for z'/z^{-r-1} in ascending powers of x has no negative indices, when $a_0 \neq 0$. Furthermore, by hypothesis,

$$[\bar{y}] = [\bar{x}]\,Q + O(x). \tag{4}$$

Hence the left-hand side of relation (2) may be rewritten as

$$\left[0,\ \frac{z'}{z^2},\ \frac{z'}{xz^3},\ \ldots\right]+[\bar{y}],$$

that is

$$\frac{z'}{z^2}\,[\bar{y}]\,U+[\bar{y}];$$

and the whole relation becomes

$$\frac{z'}{z^2}\,[\bar{x}]\,QU+[\bar{x}]\,Q=[\bar{x}]\,P+O(x), \tag{5}$$

since z'/z^2 can contribute no negative powers of x. Now let

$$\frac{1}{z}=\frac{h_0-h_1x+h_2x^2-\ldots}{a_0},$$

so that

$$\frac{z'}{z^2}=\frac{h_1}{a_0}-\frac{2h_2x}{a_0}+\frac{3h_3x^2}{a_0}-\ldots. \tag{6}$$

Then

$$\frac{z'}{z^2}\,[\bar{x}]=\left[\frac{h_1}{a_0},\ \frac{h_1}{a_0x}-\frac{2h_2}{a_0},\ \frac{h_1}{a_0x^2}-\frac{2h_2}{a_0x}+\frac{3h_3}{a_0},\ \ldots\right]+O(x),$$

where negative powers and constant terms are alone explicitly stated. This can be written

$$\frac{z'}{z^2}\,[\bar{x}]=[\bar{x}]\left(\frac{h_1}{a_0}\,I-\frac{2h_2}{a_0}\,U+\frac{3h_3}{a_0}\,U^2-\ldots\right)+O(x)$$

$$=[\bar{x}]\,\frac{a_1I+2a_2U+3a_3U^2+\ldots}{(a_0I+a_1U+a_2U^2+\ldots)^2}+O(x)$$

by (6), where the fractional form in U is justified since $a_0\neq 0$. On substituting in (5) and applying Lemma I, we have

$$\frac{a_1I+2a_2U+3a_3U^2+\ldots}{(a_0I+a_1U+a_2U^2+\ldots)^2}\,QU+Q=P. \tag{7}$$

But $QU=f(U)\,Q$. After substituting and simplifying by ordinary algebra, we obtain

$$P=\frac{a_0I+2a_1U+3a_2U^2+\ldots}{a_0I+a_1U+a_2U^2+\ldots}\,Q\quad(a_0\neq 0), \tag{8}$$

which gives the explicit relation between these two non-singular solutions, P and Q, of the equation

$$f(U)\,X-XU=0,$$

where $$f(U) = U(a_0 I + a_1 U + a_2 U^2 + \ldots).$$

Alternatively the denominator of (8) may be cleared and the result written as

$$PU = (a_0 U + 2a_1 U^2 + 3a_2 U^3 + \ldots) Q. \tag{9}$$

The actual values of P and Q in terms of the original coefficients a_i are

$$P = \begin{bmatrix} 1 & . & . & . & . & \ldots \\ . & \frac{1}{a_0}, & -\frac{a_1}{a_0^3}, & \frac{2a_1^2 - a_0 a_2}{a_0^5}, & \frac{5a_0 a_1 a_2 - a_0^2 a_3 + 5a_1^3}{a_0^7}, & \ldots \\ . & . & \frac{1}{a_0^2}, & -\frac{2a_1}{a_0^4}, & \frac{5a_1^2 - 2a_0 a_2}{a_0^6}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \tag{10}$$

$$Q = \begin{bmatrix} 1, & -\frac{a_1}{a_0^2}, & \frac{3a_1^2 - 2a_0 a_2}{a_0^4}, & \frac{-10a_1^3 + 12a_0 a_1 a_2 - 3a_0^2 a_3}{a_0^6}, & \ldots \\ . & \frac{1}{a_0}, & -\frac{2a_1}{a_0^3}, & \frac{6a_1^2 - 3a_0 a_2}{a_0^5}, & \ldots \\ . & . & \frac{1}{a_0^2}, & -\frac{3a_1}{a_0^4}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}. \tag{11}$$

It will be seen on inspection that all these numerators are gradients in the coefficients, those which stand in the s-th diagonal possessing a weight s; also that the denominators ascend in powers of a_0^2 along each row. According to a well-known theorem*, there are certain gradients $\sigma_2, \sigma_3, \sigma_4, \ldots$ which are annihilated by the operator

$$\Omega = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \ldots; \tag{12}$$

that is, $\Omega \sigma_i = 0$, where σ_i has weight i. They are

$$\sigma_2 = a_0 a_2 - a_1^2, \quad \sigma_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3, \quad \ldots,$$

and they are called seminvariants of a binary form

$$(a_0, a_1, a_2, \ldots)(\xi, 1)^n.$$

* Elliott, *Algebra of quantics* (1895), 112.

If the relation (8) is written in the form

$$(a_0 I+a_1 U+a_2 U^2+\ldots) P=(a_0 I+2a_1 U+3a_2 U^2+\ldots) Q=R, \quad (13)$$

then it is a remarkable fact that *the elements of the top row of R from the third onwards have for their numerators these seminvariants σ_2, σ_3, ...* Also, *successive rows of P are connected by the relation*

$$\Omega \text{ row}_r=-r \text{ row}_{r+1} \quad (r \geqslant 0). \quad (14)$$

In proof of this, if y has the usual value, then

$$\Omega y^{-r}=-ry^{-r-1}(a_0 x^2+2a_1 x^3+\ldots)$$

$$=-rx^2\left(\frac{u_{r,0}}{x^{r+1}}+\frac{u_{r-1,1}}{x^r}+\ldots\right) \quad (15)$$

by § 13 (20). If this is differentiated with respect to x, then

$$\Omega \frac{d(y^{-r})}{dx}=\frac{r(r-1)u_{r,0}}{x^r}+\frac{r(r-2)u_{r-1,1}}{x^{r-1}}+\ldots.$$

Hence, by using (20) again and comparing coefficients, we have

$$\Omega u_{r-s,s}=-(r-s)u_{r-s+1,s-1}, \quad (16)$$

which is equivalent to the relation (14) above. Now from the value of R in terms of P in (13), the elements of its top row are a_0, a_1/a_0, and then w_s at the s-th position, where

$$a_1 u_{1,s-1}+a_2 u_{2,s-2}+\ldots+a_s u_{s,0}=w_s. \quad (17)$$

Hence $\Omega w_s=a_0 u_{1,s-1}+a_1 u_{2,s-2}+\ldots+a_{s-1} u_{s,0}$, by the relation just found, and by (19) of the last section this is equal to $u_{0,s-1}$ which is known to be zero. So w_s can be equated to σ_s, multiplied by a constant factor. But each has unity for coefficient of the term containing a_s, so that they are equal; this proves the above statement.

In much the same way it can be proved for Q that

$$\Omega \text{ row}_r=-(r+1) \text{ row}_{r+1} \quad (r \geqslant 0); \quad (18)$$

that is, *each element below the topmost row of Q is derivable by operating with Ω upon the element immediately above it, changing its sign, and then dividing by the number of its row.*

In proof of this it is sufficient to compare (21) of the last section with the above (1). The elements $v_{r-s,s}$ of the $(r+1)$-th column refer to the expansion of y^{-r}. Hence

$$(r-s)v_{r-s,s}=ru_{r-s,s} \quad (r>s), \quad (19)$$

while the element $v_{0,r}$ is given by the constant term in (21), for each positive integral value of r. The result now follows on writing

$$\frac{1}{y^r} = \frac{v_{r,0}}{x^r} + \frac{v_{r-1,1}}{x^{r-1}} + \ldots + v_{0,r} + O(x) \tag{20}$$

and using (15).

The actual elements, as written out in (10) and (11), illustrate the relation (19), seeing that all the elements on each row below the top agree, except for a numerical factor. The top row elements of Q are due to the term independent of x in the ascending expansion of each power y^{-r}; and they can easily be evaluated by the multinomial theorem.

The above has been written in general terms. For examples of the application of semipersymmetric determinants to the cases where $f(x)$ is of special type, compare the work of J. W. L. Glaisher, "Expressions for Laplace's coefficients, Bernoullian and Eulerian Numbers, etc., as determinants", *Messenger of Math.*, 6 (1877), 49–63.

ON SUMMABILITY BY TYPICAL MEANS

By A. C. OFFORD.

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1. *Introduction.* Let us consider functions defined by the relation

$$f(s) = \int_0^\infty e^{-su}\, dA(u), \tag{1}$$

where

$$s = \sigma + it,$$

σ and t being real, and $A(u)$ is a function of bounded variation such that $A(+0) = 0$. These functions are closely related to those given by the Dirichlet's series

$$\sum_1^\infty a_n e^{-\lambda_n s}, \tag{2}$$

where $\{\lambda_n\}$ is a monotone increasing sequence of real positive numbers such that $\lambda_n \to \infty$. We can express the series (2) as an integral of the form (1) by writing

$$A(\omega) = \sum_{\lambda_n \leqslant \omega} a_n.$$

It is largely a matter of taste which of these forms we adopt, and in this paper it makes very little difference. Since, however, (1) is slightly more general, I shall use this form but state the results so that they apply to either form.

The integral (1) is said to be summable (R, γ), *i.e.* by Riesz's typical means of the first kind of order γ, $\gamma \geqslant 0$, at the point s if

$$\omega^{-\gamma} A^\gamma(s, \omega) = \omega^{-\gamma} \int_0^\omega (\omega - u)^\gamma e^{-su}\, dA(u) \tag{3}$$

tends to a limit as $\omega \to \infty$ (**9**, 19). In the case of Dirichlet's series, we write

$$A^\gamma(s, \omega) = \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^\gamma a_n e^{-\lambda_n s},$$

and the summability of the series is defined in the same way (**6**, 21). It is convenient to write

$$A^\gamma(0, \omega) = \int_0^\omega (\omega - u)^\gamma\, dA(u) = A^\gamma(\omega),$$

Then it is known that

$$A^{\gamma+\delta}(\omega) = \frac{\Gamma(\gamma+\delta+1)}{\Gamma(\gamma+1)\,\Gamma(\delta)} \int_0^\omega (\omega-u)^{\delta-1} A^\gamma(u)\,du, \tag{4}$$

where $\delta > 0$ and $\gamma \geqslant 0$ (**6**, 27).

M. Riesz (**9**, 19) has given the following theorem. It is an extension of a result he had previously given on the convergence of Dirichlet's series (**8**, 350).

THEOREM A. *If*

$$A^\gamma(\omega) = o(\omega^\gamma e^{\omega c}), \tag{5}$$

where $c > 0$, and $\gamma \geqslant 0$, then the integral (1) *is uniformly summable (R, γ) in every finite domain of the half plane $\sigma \geqslant c+\delta > c$ and defines there a regular function $f(s)$. Further,* (1) *is summable (R, γ) at every point of $\sigma = c$ at which $f(s)$ is regular. Condition* (5) *is a necessary condition for summability at a point on $\sigma = c$.*

Riesz (**9**, 20) has also given extensions of this theorem. In this paper I give some sufficient conditions for summability of a somewhat different type from those of Riesz, and analogous to those given by Bosanquet (**1**) for power series. In proving his results, Bosanquet made use of a formula, given in (**7**, 470), expressing the behaviour of the Cesàro partial sum of a power series, at a point on the circle of convergence, in terms of the behaviour of the function in the immediate neighbourhood of this point. In § 2, I obtain the corresponding formula for $A^\alpha(c, \omega)$. The only hypothesis I make is (5) and this is known to be a necessary condition for summability (R, α). With this formula, it is easy to construct new proofs of Theorem A and its extensions.

In § 3, I give two theorems; the first is the theorem for the integral (1) corresponding to Bosanquet's Theorem 1 (**1**, 760) on power series, and the second is a slight extension of this. Theorem 4 of § 4 is a converse theorem, analogous to Bosanquet's Theorem 2 (**1**, 766).

I have to thank Dr. L. S. Bosanquet for his valuable suggestions and criticisms.

2. *The fundamental reduction.*

THEOREM 1. *If $c > 0$ and if* (5) *is satisfied, then, for all $\alpha \geqslant \gamma$ and $\delta > 0$, we have*

$$A^\alpha(c, \omega) = \frac{\Gamma(\alpha+1)}{2\pi} \int_{-\delta}^{\delta} \frac{f(s)}{s^{\alpha+1}} e^{\omega(s-c)} Q(s)\,dt + o(\omega^\alpha),$$

where $s = c+1/\omega+it$ *and*

$$Q(s) = \left(\frac{s}{s-c}\right)^{\alpha+1} + c_0 + \frac{c_1}{s}, \tag{6}$$

c_0, c_1 *being constants which depend on* δ.

The proof of this theorem is similar to that of the corresponding theorem for power series (**7**, 470). Two new difficulties, however, are introduced. The first is due to the fact that we are now dealing with an infinite integral and occurs in the terms which are denoted by J_1 and J_1' in the proof of Theorem 1. The second arises because a continuous function $A^\alpha(x)$ replaces the sequence $\{a_n\}$ and that part of the range of integration along $(\omega-1, \omega+1)$ requires special treatment. The following lemmas are required.

LEMMA 1. *If* (5) *holds, then, for all* s *for which* $\sigma > c$, *the integral* (1) *is summable* (R, γ) *to a regular function* $f(s)$ *which is given by*

$$f(s) = \frac{s^{\gamma+1}}{\Gamma(\gamma+1)} \int_0^\infty A^\gamma(u)\, e^{-su}\, du,$$

this integral being convergent.

This result is due to Riesz (**9**, 26) and it can be proved by the argument used by Hardy and Riesz (**6**, 40–43).

LEMMA 2. *If* (5) *is satisfied and* $s = \sigma+it$, $\sigma > c$, $\sigma > \sigma_0 \geqq 0$, *then*

$$A^\gamma(s_0, \omega) = \frac{\Gamma(\gamma+1)}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s)}{(s-s_0)^{\gamma+1}}\, e^{\omega(s-s_0)}\, ds.$$

When $\gamma = 0$ the integral may be improper, but its principal value $\lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT}$ certainly exists and this will therefore be the sense in which we interpret it. The result, in the form in which we quote it, is due to M. Riesz, and is an extension of theorems due to Hadamard and Perron. The proof for Dirichlet's series is given by Hardy and Riesz (**6**, 12 and 50), and the proof for the integral is very much the same. Another proof has been given by Hardy (**4**).

LEMMA 3. *Write*

$$F_1(s) = \int_0^t \frac{f(s_1)}{s_1^{\alpha+1}}\, e^{\omega(s_1-c)}\, du + C,$$

where t *is positive or negative,* $s_1 = c+1/\omega+iu$, *and*

$$C = \frac{e}{\Gamma(\alpha+1)} \left\{\int_0^{\omega-1} + \int_{\omega-1}^{\infty}\right\} \frac{A^\alpha(x)\, e^{-(c+1/\omega)x}}{i(x-\omega)}\, dx.$$

Then, under the hypotheses of Theorem 1,

$$|F_1(s)| < \epsilon\omega^{a+1} + \epsilon|t|\omega^a,$$

for all t *and* $\omega \geqslant \omega_0(\epsilon)$.

Proof. From Lemma 1 we have

$$\begin{aligned} F_1(s) &= \frac{1}{\Gamma(a+1)} \int_0^t e^{1+i\omega u}\, du \int_0^\infty A^a(x)\, e^{-s_1 x}\, dx + C \\ &= \frac{1}{\Gamma(a+1)} \int_0^t e^{1+i\omega u}\, du \int_{\omega-1}^{\omega+1} A^a(x)\, e^{-s_1 x}\, dx \\ &\qquad + \frac{ie^{-\omega c}}{\Gamma(a+1)} \left\{ \int_0^{\omega-1} + \int_{\omega+1}^\infty \right\} \frac{A^a(x)}{x-\omega} \left\{ e^{-s(x-\omega)} - e^{-(c+1/\omega)(x-\omega)} \right\} dx + C \\ &= \frac{1}{\Gamma(a+1)} \int_0^t e^{1+i\omega u}\, du \int_{\omega-1}^{\omega+1} A^a(x)\, e^{-s_1 x}\, dx \\ &\qquad + \frac{ie^{-\omega c}}{\Gamma(a+1)} \left\{ \int_0^{\omega-1} + \int_{\omega+1}^\infty \right\} \frac{A^a(x)}{x-\omega}\, e^{-s(x-\omega)}\, dx, \qquad (7) \end{aligned}$$

on inverting the order of integration. This step is clearly permissible since, in virtue of (5), the integral of $A^a(x)e^{-sx}$ converges uniformly in t for any given value of ω. Hence, using (5), we get, for $\omega \geqslant \omega_0(\epsilon)$,

$$\begin{aligned} |F_1(s)| &< \epsilon \left| \int_0^t du \int_{\omega-1}^{\omega+1} x^a e^{-x/\omega}\, dx \right| + \epsilon \left\{ \int_0^{\omega-1} + \int_{\omega+1}^\infty \right\} \frac{x^a}{x-\omega}\, e^{-x/\omega}\, dx \\ &< \epsilon|t|\omega^a + \epsilon\omega^{a+1}, \end{aligned}$$

which is the required result.

LEMMA 4. *Write*

$$F_2(s) = \int_0^t F_1(s_1)\, du,$$

where t *is positive or negative and* $s_1 = c + 1/\omega + iu$. *Then*

$$F_2(s) = \int_0^t (t-u)\, B(\omega, u)\, du + o(\omega^a), \qquad (8)$$

where the o *is uniform in* t *and*

$$B(\omega, u) = \frac{e}{\Gamma(a+1)} \int_{\omega-1}^{\omega+1} A^a(x)\, e^{-s_1 x + i\omega u}\, dx, \qquad (9)$$

and, for $|t| \geqslant 1$, $\epsilon > 0$, $\omega \geqslant \omega_0(\epsilon)$,

$$|F_2(s)| < \epsilon t^2 \omega^a. \qquad (10)$$

Proof. From (7) we have

$$F_2(s) = \int_0^t dv \int_0^v B(\omega, u)\, du$$

$$- \frac{e^{-\omega c}}{\Gamma(\alpha+1)} \left\{ \int_0^{\omega-1} + \int_{\omega+1}^{\infty} \right\} \frac{A^\alpha(x)}{(x-\omega)^2} \{ e^{-s(x-\omega)} - e^{-(c+1/\omega)(x-\omega)} \}\, dx$$

$$= J_1 + J_2 + J_3.$$

Here we have again inverted the order of integration, but the argument given before clearly justifies this. Inverting the order of integration in J_1, we get

$$J_1 = \int_0^t (t-u)\, B(\omega, u)\, du.$$

Again, using (5),

$$J_3 = o(1) \int_{\omega+1}^{\infty} \frac{x^\alpha}{(x-\omega)^2} e^{-x/\omega}\, dx$$

$$= o(\omega^{\alpha-1}) \left\{ \int_{1+1/\omega}^{2} + \int_2^{\infty} \right\} \frac{\xi^\alpha e^{-\xi}}{(\xi-1)^2}\, d\xi$$

$$= o(\omega^{\alpha-1}) \left\{ \int_{1/\omega}^{1} \frac{d\xi}{\xi^2} + \int_0^{\infty} \xi^\alpha e^{-\xi}\, d\xi \right\}$$

$$= o(\omega^\alpha)$$

uniformly in t. Similarly $J_2 = o(\omega^\alpha)$, uniformly in t. Also it is obvious from (5) that

$$B(\omega, u) = o(\omega^\alpha) \tag{11}$$

uniformly in u. On substituting (11) in J_1 we get (10), and the proof is complete.

Proof of Theorem 1. From (4) and (5) we have

$$A^{\alpha+1}(\omega) = o(1) \int_0^{\omega} u^\alpha e^{cu}\, du = o(\omega^\alpha e^{\omega c}).$$

Hence, applying Lemma 2 in the case $s_0 = 0$, we see that, for $\sigma > c$ and $p = \alpha,\ \alpha+1$, where $\alpha \geqslant \gamma$,

$$\frac{\Gamma(p+1)}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s)}{s^{p+1}} e^{\omega(s-c)}\, ds = A^p(\omega)\, e^{-\omega c} = o(\omega^\alpha). \tag{12}$$

Let δ be independent of ω. Then choose c_0, c_1 of (6) so that

$$Q(c+i\delta) = Q(c-i\delta) = 0.$$

Since $Q(s)$ is regular everywhere, except at $s = 0$, c, we have

$$\frac{Q(s)}{s-c-i\delta} \to Q'(c+i\delta) \quad \text{as} \quad s \to c+i\delta,$$

and therefore $$|Q(c+1/\omega+i\delta)| < M/\omega, \tag{13}$$

where we use M to denote a constant independent of ω. Similarly

$$|Q(c+1/\omega-i\delta)| < M/\omega.$$

Since c_0, c_1 are independent of ω, on combining the result of Lemma 2 and (12), we get

$$A^\alpha(c, \omega) = \frac{\Gamma(\alpha+1)}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{f(s)}{s^{\alpha+1}} Q(s)\, e^{\omega(s-c)}\, dt + o(\omega^\alpha).$$

Write

$$\begin{aligned} J &= \int_{\delta}^{T} \frac{f(s)}{s^{\alpha+1}} Q(s)\, e^{\omega(s-c)}\, dt \\ &= {}_{\delta}^{T}\Big[F_1(s)\, Q(s)\Big] - {}_{\delta}^{T}\Big[F_2(s)\, Q'(s)\Big] + \int_{\delta}^{T} F_2(s)\, Q''(s)\, dt \\ &= J_1 + J_2 + J_3, \end{aligned}$$

and similarly

$$J' = \int_{-T}^{-\delta} = J_1' + J_2' + J_3'.$$

Then

$$A^\alpha(c, \omega) = \frac{\Gamma(\alpha+1)}{2\pi} \int_{-\delta}^{\delta} \frac{f(s)}{s^{\alpha+1}} Q(s)\, e^{\omega(s-c)}\, dt + \frac{\Gamma(\alpha+1)}{2\pi} \lim_{T\to\infty} (J+J') + o(\omega^\alpha).$$

The proof is now reduced to showing that $\lim (J+J')$ is $o(\omega^\alpha)$, and for this it is clearly sufficient to consider the terms J_1+J_1', J_2, and J_3.

From (13) and Lemma 3 we have

$$|F_1(c+1/\omega+i\delta)\, Q(c+1/\omega+i\delta)| < \epsilon\delta\omega^{\alpha+1} M/\omega = o(\omega^\alpha).$$

Writing $w = c+1/\omega+iT$, we therefore get

$$\begin{aligned} J_1 + J_1' &= F_1(w)\, Q(w) - F_1(\bar{w})\, Q(\bar{w}) + o(\omega^\alpha) \\ &= Q(w)\{F_1(w) - F_1(\bar{w})\} + F_1(\bar{w})\{Q(w) - Q(\bar{w})\} + o(\omega^\alpha) \\ &= Q(w) \int_{-T}^{T} \frac{f(s)}{s^{\alpha+1}}\, e^{\omega(s-c)}\, dt + F_1(\bar{w})\{Q(w) - Q(\bar{w})\} + o(\omega^\alpha) \\ &= J_{1.1} + J_{1.2} + o(\omega^\alpha). \end{aligned}$$

Now $Q(w) \to 1+c_0$, as $T \to \infty$, so, from Lemma 2,

$$\lim_{T\to\infty} J_{1.1} = (1+c_0)\, e^{-\omega c} A^{\alpha}(\omega) = o(\omega^{\alpha}).$$

Again, by an elementary calculation,

$$|Q(w)-Q(\bar{w})| < M/T,$$

so that, by Lemma 3,

$$|J_{1.2}| < \epsilon\omega^{\alpha+1}/T + \epsilon\omega^{\alpha},$$

i.e.

$$\lim_{T\to\infty} J_{1.2} = o(\omega^{\alpha}),$$

which is of the desired form.

Again, it is easy to see that

$$Q'(s) \sim \frac{(\alpha+1)c+c_1}{t^2} \quad \text{as} \quad t\to\infty.$$

Hence

$$|J_2| \leqslant |Q'(w)\, F_2(w)| + |Q'(c+1/\omega+i\delta)\, F_2(c+1/\omega+i\delta)| = o(\omega^{\alpha}),$$

by Lemma 4.

By an elementary calculation we see that, if $s = c+1/\omega+it$, then

$$Q''(s) = \frac{P}{t^3} + \frac{R(t)}{t^4},$$

where P is a bounded function of $\omega \geqslant \omega_0$ and $R(t)$ is a bounded function of $t > 0$ and $\omega \geqslant \omega_0$. Hence, using Lemma 4 and (10), we get

$$\begin{aligned} J_3 &= P\int_{\delta}^{T} \frac{F_2(s)}{t^3}\, dt + \int_{\delta}^{T} \frac{R(t)\, F_2(s)}{t^4}\, dt \\ &= P\int_{\delta}^{T} \frac{dt}{t^3} \int_0^t (t-u)\, B(\omega, u)\, du + o(\omega^{\alpha}), \end{aligned}$$

the o being uniform in T. Therefore

$$\begin{aligned} J_3 &= P\int_{\delta}^{T} B(\omega, u)\, du \int_u^T \frac{t-u}{t^3}\, dt + P\int_0^{\delta} B(\omega, u)\, du \int_{\delta}^{T} \frac{t-u}{t^3}\, dt + o(\omega^{\alpha}) \\ &= P\int_{\delta}^{T} \left(\frac{1}{2u} - \frac{1}{T} + \frac{u}{2T^2}\right) B(\omega, u)\, du + o(\omega^{\alpha}), \quad \text{by (11)}, \\ &= \tfrac{1}{2}P\int_{\delta}^{T} \frac{B(\omega, u)}{u}\, du + o(\omega^{\alpha}). \end{aligned}$$

Hence, inverting the order of integration, we get

$$J_3 = \frac{eP}{2\Gamma(a+1)} \int_{\omega-1}^{\omega+1} A^a(x)\, e^{-(c+1/\omega)x}\, dx \int_\delta^T \frac{e^{-i(x-\omega)u}}{u}\, du + o(\omega^a),$$

i.e.

$$|J_3| = o(\omega^a) \int_{\omega-1}^{\omega+1} dx \left| \int_\delta^T \frac{e^{-i(x-\omega)u}}{u}\, du \right| + o(\omega^a).$$

Now

$$J_{3.1} = \int_\omega^{\omega+1} dx \left| \int_\delta^T \frac{e^{-i(x-\omega)u}}{u}\, du \right| = \int_0^1 dx \left| \int_{\delta x}^{Tx} \frac{e^{-iu}}{u}\, du \right|$$

$$\leqslant \int_0^1 dx \int_{\delta x}^1 \frac{du}{u} + \int_0^1 dx \left| \int_1^{Tx} \frac{e^{-iu}}{u}\, du \right| = 1 - \log\delta + \int_0^1 O(1)\, dx = O(1).$$

Similarly for the integral along $(\omega-1, \omega)$; it follows that

$$\lim_{T\to\infty} J_3 = o(\omega^a),$$

which completes the proof.

3. *A sufficient condition for summability.* In this section I make use of the fundamental reduction to prove two theorems. Theorem 2 is the analogue, for integrals of the form (1), of Bosanquet's Theorem 1 (**1**, 760), and Theorem 3 is a slight extension of this.

The following functions and definitions are required. A fuller account of these has been given by Bosanquet and Miss Cartwright (**2**). Write

$$f_\beta(s) = \frac{1}{\Gamma(\beta)} \int_c^s (s-u)^{\beta-1} f(u)\, du,$$

the integral being taken along the straight line (c, s) and interpreted as $\lim_{\xi\to c} \int_\xi^s$, where $\xi \to c$ along the line (c, s). The function $f_\beta(s)$ is called the fractional integral of $f(s)$ of order β taken at $s = c$. Writing $f_0(s) = f(s)$, we know (**1**, 760) that, for all $\kappa \geqslant 1$,

$$f_\kappa(s) = \int_c^s f_{\kappa-1}(u)\, du.$$

If

$$(s-c)^{-\beta}\, \Gamma(\beta+1)\, f_\beta(s) \to A$$

as $s \to c$ in $\sigma > c$, where $\beta \geqslant 0$, we say that $f(s) \to A\,(C, \beta)$ as $s \to c$ in $\sigma > c$. Further, for all $\kappa > 0$, we have (**2**, 182)

$$f_{\beta+\kappa}(s) = \frac{1}{\Gamma(\kappa)} \int_c^s (s-u)^{\kappa-1} f_\beta(u)\, du. \qquad (14)$$

Whence it follows that, if $f(s) \to A(C, \beta)$ as $s \to c$ in $\sigma > c$, then $f(s) \to A(C, \beta+\kappa)$ provided that $\kappa > 0$.

THEOREM 2. *If the integral* (1) *is such that condition* (5) *is satisfied and* $f(s) \to A(C, \beta)$, $\beta \geqslant 0$, *as* $s \to c$ *in any manner in* $\sigma > c > 0$, *then* (1) *is summable* (R, α) *to sum* A *at* $s = c$ *provided that* $\alpha > \beta$ *and* $\alpha \geqslant \gamma$.

THEOREM 3. *If the integral* (1) *is such that condition* (5) *is satisfied and* $f(s)$ *is bounded* (C, β), $\beta \geqslant 0$, *as* $s \to c$ *in any manner in* $\sigma > c > 0$, *and* $f(s) \to A(C, \kappa)$, $\kappa \geqslant 0$, *as* $s \to c$ *along one straight line in* $\sigma > c$, *then* (1) *is summable* (R, α) *to sum* A *at* $s = c$ *provided that* $\alpha > \beta$ *and* $\alpha \geqslant \gamma$.

The proofs of these theorems are very similar to those of their analogues for power series. We therefore give them in outline only. The details may readily be filled in by referring to Bosanquet's paper. We require the following lemmas.

LEMMA 5. *Let*

$$\phi(s) = \frac{f(s)}{s^{\alpha+1}}, \tag{15}$$

α *being a positive number. Then, if* $f(s) \to A(C, \beta)$, $\beta \geqslant 0$, *as* $s \to c$ *in any manner in* $\sigma > c$, *it follows that* $\phi(s) \to Ac^{-\alpha-1}(C, \beta)$ *as* $s \to c$ *in any manner.*

Proof. The lemma is obvious if $\beta = 0$. If $\beta > 0$, we have

$$\Gamma(\beta)\left\{\phi_\beta(s) - \frac{f_\beta(s)}{s^{\alpha+1}}\right\} = \int_c^s (s-u)^{\beta-1}\left(\frac{1}{u^{\alpha+1}} - \frac{1}{s^{\alpha+1}}\right) f(u)\,du.$$

Put

$$h(u) = (s-u)^{\beta-1}(u^{-\alpha-1} - s^{-\alpha-1})$$

and consider the case $0 < \beta < 1$. Then

$$h'(u) = -(\beta-1)(s-u)^{\beta-2}(u^{-\alpha-1} - s^{-\alpha-1}) - (\alpha+1)(s-u)^{\beta-1}u^{-\alpha-2}.$$

Now write $s-c = re^{i\theta}$ and $u-c = \rho e^{i\theta}$; then

$$|h'(u)| < M|s-u|^{\beta-1} = M(r-\rho)^{\beta-1}.$$

Hence, integrating by parts, we get

$$\Gamma(\beta)\left|\phi_\beta(s) - \frac{f_\beta(s)}{s^{\alpha+1}}\right| = \left|\int_c^s \Big[h(u)\,f_1(u)\Big] - \int_c^s h'(u) f_1(u)\,du\right|$$

$$\leqslant \int_c^s |h'(u)|\,|f_1(u)|\,|du| + o(1)$$

$$\leqslant M\int_0^r (r-\rho)^{\beta-1}\,d\rho = Mr^\beta,$$

and this tends to zero as $s \to c$ in any manner in $\sigma > c$. The case $\beta > 1$ can be reduced to this by integration by parts.

The next three lemmas are the analogues of Lemmas 4, 6, and 8 respectively of Bosanquet (**1**). Lemma 6 is similar to one given by Hardy and Littlewood (**5**, 80) and can be proved by their argument.

LEMMA 6. *If $f(s)$ satisfies the hypotheses of either Theorem 2 or 3 and if κ is a positive integer, then $\phi_\kappa(s)$ is regular for $\sigma > c$ and, for $c < \sigma < 2c$,*

$$\phi_\kappa(s) = \int_c^{2c} \phi_{\kappa-1}(u)\,du + \int_{2c}^{2c+it} \phi_{\kappa-1}(u)\,du - \int_s^{2c+it} \phi_{\kappa-1}(u)\,du,$$

the integrals being taken along straight lines.

LEMMA 7. *Under the hypotheses of either Theorem 2 or 3, if κ is a positive integer less than $\alpha+1$, then*

$$\phi_\kappa(s) = o\{(\sigma-c)^{\kappa-\alpha-1}\} \quad as \quad \sigma \to c.$$

Proof. Let C denote a constant. Then, using Lemma 6 and (5), we get, since $\alpha \geqslant \gamma$,

$$\phi_1(s) = C + \int_{2c}^{2c+it} \frac{du}{\Gamma(\alpha+1)} \int_0^\infty A^\alpha(x)\, e^{-ux}\,dx - \int_s^{2c+it} \frac{du}{\Gamma(\alpha+1)} \int_0^\infty A^\alpha(x)\, e^{-ux}\,dx$$

$$= C + \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \frac{A^\alpha(x)}{x} \{e^{-2cx} - e^{-(2c+it)x}\}\,dx$$

$$- \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \frac{A^\alpha(x)}{x} \{e^{-sx} - e^{-(2c+it)x}\}\,dx.$$

Hence

$$|\phi_1(s)| < |C| + o(1) \int_0^\infty x^{\alpha-1} \{e^{-cx} + e^{-(\sigma-c)x}\}\,dx = O(1) + o\{(\sigma-c)^{-\alpha}\}.$$

The required result follows by induction.

LEMMA 8. *Under the hypotheses of Theorem 2 with $A = 0$, if $0 < \xi < \eta \leqslant 2\xi$ and h is the greatest integer not greater than β, then*

$$|\phi_{h+1}(\sigma+i\eta) - \phi_{h+1}(\sigma+i\xi)| = o\{(\xi+\sigma-c)^\beta (\eta-\xi)^{h-\beta+1}\} \tag{16}$$

as $\xi \to 0$ and $\sigma \to c$.

When only the hypotheses of Theorem 3 are satisfied, we must replace o by O in (16).

Bosanquet and Miss Cartwright (**2**) have proved that $\phi_\beta(s)$ is regular for $\sigma > c$. Hence we may apply the argument of Bosanquet's Lemma 8.

Proof of Theorem 2. There is no loss of generality in supposing that $A = 0$. Then, writing $s = c + 1/\omega + it$, we have, by Theorem 1 and (15),

$$A^\alpha(c, \omega) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\sigma-i\delta}^{\sigma+i\delta} \phi(s)\, Q(s)\, e^{\omega(s-c)}\, ds + o(\omega^\alpha),$$

where δ is fixed but may be chosen as small as we please, $\alpha > \beta$, and $\alpha \geqslant \gamma$. Then, by h integrations by parts, where h is the greatest integer not greater than β,

$$\int_{\sigma-i\delta}^{\sigma+i\delta} \phi(s)\, Q(s)\, e^{\omega(s-c)}\, ds = \sum_{\nu=1}^{h} \Big[(-)^{\nu-1} \phi_\nu(s) \frac{d^{\nu-1}}{ds^{\nu-1}} \{Q(s)\, e^{\omega(s-c)}\} \Big]_{\sigma-i\delta}^{\sigma+i\delta} + (-)^h \int_{\sigma-i\delta}^{\sigma+i\delta} \phi_h(s) \frac{d^h}{ds^h} \{Q(s)\, e^{\omega(s-c)}\}\, ds.$$

The terms outside the integral are, by Lemma 7,

$$\sum_{\nu=1}^{h} o(\omega^{-\nu+\alpha+1})\, O(\omega^{\nu-1}) = o(\omega^\alpha).$$

Write

$$J(\omega) = \int_0^\delta \phi_h(s) \frac{d^h}{ds^h} \{Q(s)\, e^{\omega(s-c)}\}\, dt$$

$$= \int_0^{1/\omega} + \sum_{\nu=1}^{n-1} \int_{\nu/\omega}^{(\nu+1)/\omega} + \int_{n/\omega}^{\delta}$$

$$= J_0 + \sum_{\nu=1}^{n-1} J_\nu + J_n,$$

where n is the greatest integer not greater than $\delta\omega$. Then it is sufficient to show that $J(\omega) = o(\omega^\alpha)$.

Let δ be chosen so that, in virtue of Lemma 8, for $0 < \xi < \eta \leqslant 2\xi < \delta$ and $\omega > 1/\delta$, we have

$$|\phi_{h+1}(\sigma+i\eta) - \phi_{h+1}(\sigma+i\xi)| < \epsilon \{(\xi+1/\omega)^\beta (\eta-\xi)^{h+1-\beta}\}, \qquad (17)$$

where $\sigma = c + 1/\omega$ and ϵ is arbitrary. Then, by the argument used by Bosanquet (**1**, 764), it follows that

$$J_0 = o(\omega^\alpha)$$

and

$$J_\nu = o(\omega^\alpha)\, \nu^{\beta-\alpha-1} \quad (0 < \nu \leqslant \eta)$$

uniformly in ν. Hence

$$J(\omega) = o(\omega^\alpha) + o(\omega^\alpha) \sum_{\nu=1}^{n} \nu^{\beta-\alpha-1} \tag{18}$$
$$= o(\omega^\alpha),$$

since $\alpha > \beta$, and the proof is complete.

Proof of Theorem 3. Consider first the case $\kappa \leqslant \beta$. From the consistency relation (14) it follows that $f(s) \to A\ (C, \beta)$ along the line referred to in the hypothesis. We now proceed in the same way as in the proof of Theorem 2, but instead of (17) we have only

$$|\phi_h(\sigma+i\eta) - \phi_h(\sigma+i\xi)| < M\{(\xi+1/\omega)^\beta(\eta-\xi)^{h+1+\beta}\},$$

where M is a constant. This gives, in the place of (18),

$$|J(\omega)| < \sum_{\nu=0}^{K} |J_\nu| + M\omega^\alpha \sum_{\nu=K+1}^{n} \nu^{\beta-\alpha-1} = \Sigma_1 + \Sigma_2.$$

Now, since $\alpha > \beta$, we can, by choosing K great enough, make $\Sigma_2 = o(\omega^\alpha)$. K being fixed, the terms which give rise to Σ_1 correspond to values of s for which $-K/\omega < t < K/\omega$, *i.e.* for which $|\arg(s-c)| < \text{arc tan}\, K$, $\sigma > 0$. Now $f_\beta(s)$ is regular and bounded for $\sigma > c$ and so, by Montel's theorem (*cf.* **3**, 457), since $f_\beta(s) \to A$ as $s \to c$ along one line, it follows that $f_\beta(s) \to A$ uniformly in the angle just described. Hence, for $\nu \leqslant K$, (17) holds and therefore $\Sigma_1 = o(\omega^\alpha)$ by the argument used before. This completes the proof for the case $\kappa \leqslant \beta$.

Again, if $\kappa > \beta$, we have $f(s)$ bounded (C, κ) by consistency, and so, by the case already established, the series is summable (C, κ). Hence, by the analogue of Frobenius' theorem (**6**, 44), $f(s) \to A\ (C, 0)$ and so (C, β) along the line. The proof can now be completed as before.

4. *The converse theorem.*

THEOREM 4. *If the integral* (1) *is summable* (R, α), $\alpha \geqslant 0$, *at the point* $s = c$ *to the sum* A, *then* $f(s)$ *is regular in* $\sigma > c$ *and* $f(s) \to A\ (C, \beta)$ *as* $s \to c$ *in any manner in* $\sigma > c$ *provided that* $\beta > \alpha + 1$.

Proof. Here c may be positive, negative, or zero. Without loss of generality, we may take $A = 0$; then our hypothesis becomes

$$A^\alpha(c, \omega) = \int_0^\omega (\omega - x)^\alpha e^{-cx}\, dA(x) = o(\omega^\alpha). \tag{19}$$

On this hypothesis, Hardy and Riesz (**6**, 39) have shown that

$$f(s) = \frac{(s-c)^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty A^\alpha(c, x)\, e^{-x(s-c)}\, dx,$$

where the integral in the second member is uniformly convergent for all those values of s for which

$$|\arg(s-c)| \leqslant \tfrac{1}{2}\pi - \delta, \quad \delta > 0, \quad \sigma > c.$$

The function $f(s)$ so defined is regular in the half plane $\sigma > c$. Hence

$$\Gamma(\beta) f_\beta(s) = \int_c^s (s-u)^{\beta-1} f(u)\, du$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_c^s (s-u)^{\beta-1} (u-c)^{\alpha+1}\, du \int_0^\infty A^\alpha(c, x)\, e^{-x(u-c)}\, dx.$$

Therefore, in virtue of the uniform convergence of the infinite integral, we may invert the order of integration and obtain

$$\Gamma(\alpha+1)\,\Gamma(\beta)\, \frac{f_\beta(s)}{(s-c)^\beta} = \int_0^\infty A^\alpha(c, x)\, H_\beta(s, x)\, dx, \tag{20}$$

where

$$H_\beta(s, \omega) = \frac{1}{(s-c)^\beta} \int_c^s (s-u)^{\beta-1} (u-c)^{\alpha+1} e^{-\omega(u-c)}\, du.$$

The function $H_\beta(s, x)$ corresponds to Bosanquet's function $\Delta^\kappa G_\beta(n, z)$ (**1**, 765). Write $r = |s-c|^{-1}$. Then it is easy to construct proofs of the following relations:

$$\left.\begin{aligned} &|H_\beta(s, \omega)| \leqslant M r^{-\alpha-1}, \\ &|H_\beta(s, \omega)| \leqslant M r^{\beta-\alpha-1} \omega^{-\beta}, \quad 0 < \beta \leqslant \alpha+2, \end{aligned}\right\} \tag{21}$$

where M is a constant. The second member of (20) may be written

$$\sum_{\nu=0}^{\infty} \int_{\nu r}^{(\nu+1) r} A^\alpha(c, x)\, H_\beta(s, x)\, dx = \sum_{\nu=0}^{\infty} J_\nu.$$

By (19) and (21),

$$|J_0| < M r^{-\alpha-1} \int_0^r o(x^\alpha)\, dx = o(1).$$

Again, using $\alpha+1 < \beta < \alpha+2$, if $\nu \neq 0$, $r \geqslant r_0(\epsilon)$,

$$|J_\nu| = \int_{\nu r}^{(\nu+1) r} o(x^\alpha)\, r^{-\alpha+\beta-1} x^{-\beta}\, dx$$

$$= o\{r^{-\alpha+\beta} (\nu r)^{\alpha-\beta}\} = o(\nu^{\alpha-\beta}),$$

the o being uniform in ν. Hence, since $\beta > \alpha+1$,

$$\sum_{\nu=1}^{\infty} |J_\nu| = o(1) \sum_{\nu=1}^{\infty} \nu^{\alpha-\beta} = o(1)$$

as $s \to c$. This completes the proof.

References.

More complete lists are given in **1** and **7**.

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DIVISOR FUNCTIONS: FOURIER INTEGRAL THEOREMS

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1. *Introduction.*

In a recent paper*, Steen has introduced a class of functions which he terms divisor functions.

These are functions defined by the equation

$$Y(a_1, \ldots, a_\nu;\ \mu;\ z) = -\int_C z^s e^{\mu\pi i s}\,\Gamma(a_1-s)\ldots\Gamma(a_\nu-s)\,ds,$$

where C is a contour starting from $+\infty$, encircling all the poles of the integrand in a counter-clockwise direction, and returning to $+\infty$; μ is supposed real; the other arguments may assume complex values.

Steen considers the case when μ is an integer, but, since we evidently have

$$Y(a_1, \ldots, a_\nu;\ \mu_1+\mu_2;\ z) = Y(a_1, \ldots, a_\nu;\ \mu_1;\ ze^{\pi i\mu_2}),$$

this involves no loss of generality. In this paper it is more convenient to regard z as real, and allow μ to take non-integral values.

In this paper we consider theorems of the type of Fourier's integral theorems, where the kernels are divisor functions, or combinations of divisor functions.

In §§ 2–4 we consider theorems of the type

$$\int_0^\infty K(ux)\,du\int_0^\infty f(y)\,K'(uy)\,dy = f(x).$$

One such theorem, in which the kernels are combinations of divisor functions, is suggested by the following formal transformation.

* S. W. P. Steen, "Divisor functions: their differential equations and recurrence formulae", *Proc. London Math. Soc.* (2), 31 (1930), 47–80. This paper will be referred to as D 1. Further properties of divisor functions are given by Steen in the paper "Divisor functions: solutions of the differential equation", *Proc. London Math. Soc.* (2), 32 (1931), 356–368, which will be referred to as D 2.

Applying the Fourier cosine integral theorem ν times to the function $F(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\nu)$, and rearranging the orders of integration, we obtain

$$F(x_1, x_2, \ldots, x_\nu) = \left(\frac{2}{\pi}\right)^\nu \int_0^\infty \ldots \int_0^\infty \cos u_1 x_1 \ldots \cos u_\nu x_\nu \, du_1 \ldots du_\nu$$

$$\times \int_0^\infty \ldots \int_0^\infty F(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\nu) \cos u_1 \mathrm{v}_1 \ldots \cos u_\nu \mathrm{v}_\nu \, d\mathrm{v}_1 \ldots d\mathrm{v}_\nu;$$

in this result let

$$F(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\nu) = \mathrm{v}_1^{-a_1} \mathrm{v}_2^{-a_2} \ldots \mathrm{v}_{\nu-1}^{-a_{\nu-1}} f(\mathrm{v}_1 \mathrm{v}_2 \ldots \mathrm{v}_\nu),$$

and write

$$x_1 x_2 \ldots x_\nu = x;$$

then make the following substitutions.

In the inner integral,

$$\mathrm{v}_1 u_1 = y_1,$$

$$\mathrm{v}_2 u_2 = y_2,$$

$$\ldots \qquad \ldots$$

$$\mathrm{v}_{\nu-1} u_{\nu-1} = y_{\nu-1},$$

$$\mathrm{v}_\nu = y(y_1 y_2 \ldots y_{\nu-1})^{-1} (u_1 u_2 \ldots u_{\nu-1}).$$

The Jacobian is $(y_1 y_2 \ldots y_{\nu-1})^{-1}$.

In the outer integral,

$$u_1 x_1 = z_1,$$

$$u_2 x_2 = z_2,$$

$$\ldots \qquad \ldots$$

$$u_{\nu-1} x_{\nu-1} = z_{\nu-1},$$

$$u_\nu = z(z_1 z_2 \ldots z_{\nu-1})^{-1} (x_1 x_2 \ldots x_{\nu-1}).$$

The Jacobian is $(z_1 z_2 \ldots z_{\nu-1})^{-1}$.

We obtain the result

$$f(x) = \left(\frac{2}{\pi}\right)^\nu \int_0^\infty \ldots \int_0^\infty \cos z_1 \ldots \cos z_{\nu-1}$$

$$\times \cos\left(\frac{zx}{z_1 z_2 \ldots z_{\nu-1}}\right) z_1^{a_1-1} \ldots z_{\nu-1}^{a_{\nu-1}-1} \, dz_1 \ldots dz_{\nu-1} dz$$

$$\times \int_0^\infty \ldots \int_0^\infty f(y) \cos y_1 \ldots \cos y_{\nu-1}$$

$$\times \cos\left(\frac{yz}{y_1 y_2 \ldots y_{\nu-1}}\right) y_1^{-a_1-1} \ldots y_{\nu-1}^{-a_{\nu-1}-1} \, dy_1 \ldots dy_{\nu-1} dy.$$

That is to say,

$$f(x) = \left(\frac{2}{\pi}\right)^{\nu} \int_0^\infty K(zx)\,dz \int_0^\infty f(y)\,K'(yz)\,dy,$$

where

$$K(z) = \int_0^\infty \dots \int_0^\infty \cos t_1 \dots \cos t_{\nu-1} \cos\left(\frac{z}{t_1 \dots t_{\nu-1}}\right) t_1^{a_1-1} \dots t_{\nu-1}^{-a_\nu-1-1}\,dt_1 \dots dt_{\nu-1},$$

$$K'(z) = \int_0^\infty \dots \int_0^\infty \cos t_1 \dots \cos t_{\nu-1} \cos\left(\frac{z}{t_1 \dots t_{\nu-1}}\right) t_1^{-a_1-1} \dots t_{\nu-1}^{-a_\nu-1-1}\,dt_1 \dots dt_{\nu-1}.$$

Now let

$$g_\nu(z) = g_\nu(a_1, \dots, a_\nu;\ z)$$

$$= -\int_C \Gamma(a_1-s) \dots \Gamma(a_\nu-s) \cos\tfrac{1}{2}\pi(s-a_1) \dots \cos\tfrac{1}{2}\pi(s-a_\nu)\,z^s\,ds,$$

$$g_{\nu-1}(z) = g_{\nu-1}(a_1, \dots, a_{\nu-1};\ z).$$

Then*

$$\int_0^\infty t^{a_\nu-1} \cos t\, g_{\nu-1}\left(\frac{z}{t}\right) dt$$

$$= -\int_0^\infty t^{a_\nu-1} \cos t\,dt \int_C \Gamma(a_1-s) \dots \Gamma(a_{\nu-1}-s) \cos\tfrac{1}{2}\pi(s-a_1)$$

$$\dots \cos\tfrac{1}{2}\pi(s-a_{\nu-1})\left(\frac{z}{t}\right)^s ds$$

$$= -\int_C \Gamma(a_1-s) \dots \Gamma(a_{\nu-1}-s) \cos\tfrac{1}{2}\pi(s-a_1)$$

$$\dots \cos\tfrac{1}{2}\pi(s-a_{\nu-1})\,z^s\,ds \int_0^\infty t^{a_\nu-1-s} \cos t\,dt$$

$$= g_\nu(z).$$

By repeated applications of this result, we see that

$$g_\nu(a_1, \dots, a_\nu;\ z) = K(z),$$

and, similarly, $$g_\nu(-a_1, \dots, -a_\nu;\ z) = K'(z).$$

Thus $K(z)$ and $K'(z)$ can both be expressed as the sum of divisor functions.

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* The result $\int_0^\infty t^{a_\nu-1} \cos t\, g_{\nu-1}\left(\frac{z}{t}\right) dt = g_\nu(z)$ may be proved rigorously, for all values of the parameters for which the integral exists, by the same method as that by which formula (3.25) is proved in D 1.

More generally, if, instead of applying the Fourier cosine integral theorem ν times, we apply the cosine integral theorem r times and the sine integral theorem $\nu-r$ times, we obtain in a similar manner the result

$$f(x)=\left(\frac{2}{\pi}\right)^{\nu}\int_0^\infty K(zx)\,dz\int_0^\infty f(y)\,K'(yz)\,dy,$$

where the functions K and K' are now given by the formulae

$$K(z)=-\int_C \Gamma(a_1-s)\dots\Gamma(a_\nu-s)\cos\tfrac{1}{2}\pi(s-a_1)\dots\cos\tfrac{1}{2}\pi(s-a_r)\times\sin\tfrac{1}{2}\pi(s-a_{r+1})\dots\sin\tfrac{1}{2}\pi(s-a_\nu)\,z^s\,ds,$$

$$K'(z)=-\int_C \Gamma(-a_1-s)\dots\Gamma(-a_\nu-s)\cos\tfrac{1}{2}\pi(s+a_1)\dots\cos\tfrac{1}{2}\pi(s+a_r)\times\sin\tfrac{1}{2}\pi(s+a_{r+1})\dots\sin\tfrac{1}{2}\pi(s+a_\nu)\,z^s\,ds.$$

This result is proved (under certain restrictions) in Theorem II. The proof is similar to that of Hankel's theorem*.

This result is included in a formal result given by Hardy and Titchmarsh†. It is obtained, in fact, by putting in their result

$$k(s)=\pi^{-\frac{1}{2}\nu}2^{\frac{1}{2}\nu}\Gamma(a_1+s)\dots\Gamma(a_\nu+s)\cos\tfrac{1}{2}\pi(-s-a_1)\dots\cos\tfrac{1}{2}\pi(-s-a_r)\times\sin\tfrac{1}{2}\pi(-s-a_{r+1})\dots\sin\tfrac{1}{2}\pi(-s-a_\nu),$$

$$h(s)=\pi^{-\frac{1}{2}\nu}2^{\frac{1}{2}\nu}\Gamma(-a_1+s)\dots\Gamma(-a_\nu+s)\cos\tfrac{1}{2}\pi(-s+a_1)\dots\cos\tfrac{1}{2}\pi(-s+a_r)\times\sin\tfrac{1}{2}\pi(-s+a_{r+1})\dots\sin\tfrac{1}{2}\pi(-s+a_\nu).$$

For a certain range of values of the parameters Theorem II is included in the theorems proved in the paper cited. The connection is indicated in § 3, after the proof of Theorem II.

Theorem I is of a somewhat different type, being of the form

$$\int_0^\infty K(ux)\,du\int_0^\infty f(y)\,K'(uy)\,dy=f(x)+\int_0^\infty L(z)f(z)\,dz.$$

A theorem of this type, in which the kernels are Bessel functions, has been given by Watson‡. This theorem, as will be seen, is included in our Theorem I; and it is evident that this theorem itself includes Hankel's theorem.

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* Watson, *The theory of Bessel functions* (1922), 456.

† G. H. Hardy and E. C. Titchmarsh, "A class of Fourier kernels", *Proc. London Math. Soc.* (2), 35 (1933), 116–155 (118).

‡ *Loc. cit.*, 465.

Various other theorems of the type of Fourier's integral theorem in which the kernels are combinations of Bessel functions are known*; *e.g.* Weber's theorem. None of these are, however, included in the theorems of this paper.

Now there are two types of Fourier integral theorem in which the kernels are trigonometrical functions:

$$\int_0^\infty \cos ux\, du \int_0^\infty f(y) \cos uy\, dy = \tfrac{1}{2}\pi f(x)$$

and

$$\int_0^\infty du \int_{-\infty}^\infty f(y) \cos u(y-x)\, dy = \pi f(x).$$

These may, of course, be at once deduced from one another by means of the addition formulae.

When the kernels are divisor functions, we cannot deduce the theorems of the second type from those of the first type. It is, however, still possible to prove theorems of the second type, and these are considered in §§ 5–8. That is to say, we consider theorems of the form

$$\int_0^\infty du \int_{-\infty}^\infty f(y)\, K\{u^2(y-x)^2\}\, dy = f(x),$$

where the kernel K is the sum of divisor functions. This equation may be written in the form

$$\int_0^\infty du \int_0^\infty \phi(t)\, K(u^2 t^2)\, dt = \tfrac{1}{2}\phi(0),$$

where

$$\phi(t) = f(x+t)+f(x-t).$$

Our results will be stated in this form.

A theorem of this type is obtained in Theorem III.

For some kernels the theorem takes a different form. We have, in fact,

$$\int_0^\lambda du \int_0^\infty \phi(t)\, K(u^2 t^2)\, dt = \log \lambda \{c\phi(0)+o(1)\} \quad \text{as} \quad \lambda \to \infty,$$

where c is a constant.

This result is obtained in Theorem IV.

After the proof of these two theorems, some examples of special cases are given.

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* See E. C. Titchmarsh, "Weber's integral theorem", *Proc. London Math. Soc.* (2), 22 (1924), 15; R. J. Cooke, "The inversion formulae of Hardy and Titchmarsh", *Proc. London Math. Soc.* (2), 24 (1926), 381, and references there given.

In these theorems, we assume, as usual, that $\phi(t)$ is of bounded variation in the neighbourhood of the origin, and also that it satisfies certain conditions necessary to ensure convergence of the integral at infinity. If these latter conditions are not satisfied, it may still be possible to prove that the integrals are summable by Cesàro's methods. Results of this kind are included in Theorems III and IV.

It is also possible to relax the first condition. Paley* and Bosanquet† have shown that, if an integrable function $f(x)$ is summable (C, α) to s at the point a, *i.e.* if

$$\lim_{t\to 0} \frac{1}{t} \int_a^{a+t} f(x) \left(1-\frac{x-a}{t}\right)^{\alpha-1} dx = s,$$

then the Fourier series of $f(x)$ is summable $(C, \alpha+\delta)$ to s at a provided that $\delta > 0$. The special case of this result in which $\alpha < 1$ had previously been obtained by Hardy and Littlewood‡.

It is evident that the same result will hold for the Fourier repeated integral, provided that the usual conditions to ensure convergence at infinity are enforced.

Corresponding results may be proved for the integrals which we are considering. The results of this form corresponding to Theorems III and IV are obtained in Theorems V and VI. The proof is similar to Paley's.

In the papers cited Paley and Bosanquet have obtained a kind of converse to the first result; they prove that, if the Fourier series of $f(x)$ is summable (C, α) to s at a, then $f(x)$ is summable $(C, \alpha+\delta+1)$ to s at a for any $\delta > 0$. This result has also been obtained by Verblunsky§. The corresponding results do not hold for the theorems of this paper. We conclude this paper by showing that there are kernels $K(u^2t^2)$ of the kind considered in Theorems III-VI for which no sort of converse to Theorems V and VI is true.

I must express my thanks to Mr. S. W. P. Steen, who suggested this subject for research. To him the formal work in this paragraph is due.

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* R. E. A. C. Paley, "On the Cesàro summability of Fourier series and allied series", *Proc. Camb. Phil. Soc.*, 26 (1930), 173–203.

† L. S. Bosanquet, "On the summability of Fourier series", *Proc. London Math. Soc.* (2), 31 (1930), 144–163.

‡ G. H. Hardy and J. E. Littlewood, "Notes on the theory of series (III): On the summability of the Fourier series of a nearly continuous function", *Proc. Camb. Phil. Soc.*, 23 (1927), 681–684.

§ S. Verblunsky, "Note on the sum of an oscillating series", *Proc. Camb. Phil Soc.*, 26 (1930) 152–157.

2. *Notation.*

We now suppose that

$$\sum_{n=1}^{\nu} \alpha_n = 0, \quad |\mu| \leqslant \tfrac{1}{2}\nu. \tag{1}$$

We write

$$Y_n^{(1)}(\mu;\ z) = Y(a_1-1,\ \ldots,\ a_n-1,\ a_{n+1},\ \ldots,\ a_\nu;\ \mu;\ z), \tag{2}$$

$$Y_n^{(2)}(\mu;\ z) = Y(-a_1+1,\ \ldots,\ -a_n+1,\ -a_{n+1},\ \ldots,\ -a_\nu;\ \mu;\ z), \tag{3}$$

$$Y^{(1)}(\mu;\ z) = Y_0^{(1)}(\mu;\ z) = Y(a_1,\ \ldots,\ a_\nu;\ \mu;\ z), \tag{4}$$

$$Y^{(2)}(\mu;\ z) = Y_0^{(2)}(\mu;\ z) = Y(-a_1,\ \ldots,\ -a_\nu;\ \mu;\ z). \tag{5}$$

Then we have*,

$$\frac{d}{dz} z^{-a_n+1} Y_n^{(1)}(\mu;\ z) = -z^{-a_n} Y_{n-1}^{(1)}(\mu;\ z), \tag{6}$$

$$\frac{d}{dz} z^{a_{n+1}} Y_n^{(2)}(\mu;\ z) = -z^{a_{n+1}-1} Y_{n+1}^{(2)}(\mu;\ z). \tag{7}$$

Let $a = \min(\Re a_r)$, and let p be the greatest number of a_r's which are equal and whose real part is a.

Let $-b = \max(\Re a_r)$, and let q be the greatest number of a_r's which are equal and whose real part is $-b$.

Then we have†, for z small,

$$Y^{(1)}(\mu;\ z) = O(z^a |\log z|^{p-1}), \tag{8}$$

$$Y_n^{(2)}(\mu;\ z) = O(z^b |\log z|^{q-1}). \tag{9}$$

Of course, for $n > 0$ a stronger result than (9) is true, but (9) is all that is required.

We write $$\sigma = \frac{1}{\nu}, \quad k = -\tfrac{1}{2} + \tfrac{1}{2}\sigma.$$

We thus have $2k - \sigma = -1$.

Then the asymptotic expansions of $Y_n^{(1)}(\mu;\ z)$ and $Y_n^{(2)}(\mu;\ z)$ for z large are given by‡

$$Y_n^{(1)}(\mu;\ z) \sim M(ze^{\pi\mu i})^{k-n\sigma} \exp\{-\nu z^\sigma e^{\mu\sigma\pi i}\}\{1 + c(ze^{\mu\pi i})^{-\sigma} + \ldots\}, \tag{10}$$

$$Y_n^{(2)}(\mu;\ z) \sim M(ze^{\mu\pi i})^{k+n\sigma} \exp\{-\nu z^\sigma e^{\mu\sigma\pi i}\}\{1 + c(ze^{\mu\pi i})^{-\sigma} + \ldots\}, \tag{11}$$

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* D 1 (5.3).

† As in D 1 (2.50).

‡ D 2, 368. In the result given in D 2, $e^{2\mu\pi i}z$ should be replaced by $e^{\mu\pi i}z$, as is evident since

$$Y(a_1,\ \ldots,\ a_\nu;\ \mu;\ z) = Y(a_1,\ \ldots,\ a_\nu;\ 0;\ e^{\mu\pi i}z).$$

where the c's are (different) constants, and

$$M = (2\pi)^{\frac{1}{2}(\nu+1)}\nu^{-\frac{1}{2}}i. \tag{12}$$

We write

$$K(\mu_1;\ \mu_2;\ u;\ x,\ y) = u\sum_{n=1}^{\nu}(-1)^n\, Y_n^{(1)}(\mu_1;\ ux)\, Y_{n-1}^{(2)}(\mu_2;\ uy). \tag{13}$$

Then, by equations (6) and (7), and by D 1 (5.4),

$$\frac{d}{du}K(\mu_1;\ \mu_2;\ u;\ x,\ y) = \{1+(-1)^{\nu+1}(y/x)\,e^{(\mu_2-\mu_1)\pi i}\}\,Y^{(1)}(\mu_1;\ ux)\,Y^{(2)}(\mu_2;\ uy). \tag{14}$$

If $a+b>-1$, $Y^{(1)}(\mu_1;\ ux)\,Y^{(2)}(\mu_2;\ uy)$ is integrable in the neighbourhood of the origin, and hence $K(\mu_1;\ \mu_2;\ u;\ x,\ y)$ must tend to a finite limit as $u\to 0$.

We call this limit $\psi(\mu_1;\ \mu_2;\ x,\ y)$.

In writing all the above functions we omit the μ's whenever it is not necessary to specify their values.

3. *Proof of Theorem I.*

THEOREM I. *Suppose that*

(i) $$a+b>-1,\quad a+k+1>0,\quad x>0;$$

(ii) $f(y)\,y^k$ *is absolutely integrable in* $(0,\infty)$;

(iii) $f(y)\,y^b\,(\log y)^{q-1}$ *is integrable in* $(0,\ 1)$.

Then

(1) *If* $f(y)$ *is of bounded variation in some neighbourhood of* x, *then*

$$\int_0^\infty du\int_0^\infty f(y)\{Y^{(1)}(\tfrac{1}{2}\nu;\ ux)\,Y^{(2)}(-\tfrac{1}{2}\nu;\ uy)+Y^{(1)}(-\tfrac{1}{2}\nu;\ ux)\,Y^{(2)}(\tfrac{1}{2}\nu;\ uy)\}\,dy$$

$$= -\int_0^\infty \frac{f(y)\{\psi(\tfrac{1}{2}\nu,\ -\tfrac{1}{2}\nu;\ x,\ y)+\psi(-\tfrac{1}{2}\nu,\ \tfrac{1}{2}\nu;\ x,\ y)\}}{1-y/x}\,dy$$

$$-2^{\nu+1}\pi^{\nu+2}\{f(x+0)+f(x-0)\},$$

and, more generally, the repeated integral behaves, in respect of convergence or summability, in the same way as the Fourier repeated integral. Thus, for example, wherever $f(x+0)+f(x-0)$ *exists, it is summable* $(C,\ \delta)$, *for any* $\delta>0$, *to the expression given.*

(2) *If it is not the case* that* $\mu_1 = -\mu_2 = \pm\frac{1}{2}\nu$, *then*

$$\int_0^\infty Y^{(1)}(\mu_1;\ ux)\,du \int_0^\infty f(y)\,Y^{(2)}(\mu_2;\ uy)\,du = -\int_0^\infty \frac{f(y)\,\psi(\mu_1;\ \mu_2;\ x,\ y)}{1+(-1)^{\nu+1}(y/x)\,e^{(\mu_2-\mu_1)\pi i}}\,dy.$$

Before we prove Theorem I, we show that it includes the theorem due to Watson cited in § 1.

We consider the special case $\nu = 2$.

Write $\alpha_1 = \frac{1}{2}\xi,\ \alpha_2 = -\frac{1}{2}\xi$.

We suppose that $\xi > 0$.

Then†

$$Y^{(1)}(-1;\ z) = Y^{(2)}(-1;\ z) = -2\pi^2 e^{\frac{1}{2}\pi i\xi} H_\xi^{(1)}(2\sqrt{z}),$$

$$Y^{(1)}(1;\ z) = Y^{(2)}(1;\ z) = 2\pi^2 e^{-\frac{1}{2}\pi i\xi} H_\xi^{(2)}(2\sqrt{z}).$$

Further,

$$\psi(1,\ 1;\ x,\ y) = \psi(-1,\ -1;\ x,\ y) = -\frac{4\pi^3 x^{-1}}{\sin\pi\xi}\,\{x^{\frac{1}{2}\xi}y^{-\frac{1}{2}\xi} - x^{-\frac{1}{2}\xi}y^{\frac{1}{2}\xi}\},$$

$$\psi(1,\ -1;\ x,\ y) + \psi(-1,\ 1;\ x,\ y) = -8\pi^3 x^{-1}\cot\pi\xi\{x^{\frac{1}{2}\xi}y^{-\frac{1}{2}\xi} - x^{-\frac{1}{2}\xi}y^{\frac{1}{2}\xi}\}.$$

Hence, if we make the substitutions

$$u = \tfrac{1}{4}\mathrm{v}^2,\quad x = r^2,\quad y = R^2,\quad f(y) = F(R),$$

we see that equation (1) of Theorem I, and equation (2), taken for the two cases $\mu_1 = \mu_2 = 1$ and $\mu_1 = \mu_2 = -1$, reduce respectively to

$$(1)\quad -\int_0^\infty \mathrm{v}\,d\mathrm{v} \int_0^\infty RF(R)\{H_\xi^{(1)}(\mathrm{v}r)\,H_\xi^{(2)}(\mathrm{v}R) + H_\xi^{(2)}(\mathrm{v}r)\,H_\xi^{(1)}(\mathrm{v}R)\}\,dR$$
$$= \frac{4}{\pi}\cot\pi\xi\,I - 2\{F(r+0) + F(r-0)\},$$

$$(2)\qquad \int_0^\infty \mathrm{v}\,d\mathrm{v} \int_0^\infty RF(R)\,H_\xi^{(2)}(\mathrm{v}r)\,H_\xi^{(2)}(\mathrm{v}R)\,dR = \frac{2e^{\pi i\xi}}{\pi\sin\pi\xi}\,I,$$

$$(3)\qquad \int_0^\infty \mathrm{v}\,d\mathrm{v} \int_0^\infty RF(R)\,H_\xi^{(1)}(\mathrm{v}r)\,H_\xi^{(1)}(\mathrm{v}R)\,dR = \frac{2e^{-\pi i\xi}}{\pi\sin\pi\xi}\,I,$$

where
$$I = \int_0^\infty RF(R)\,\frac{r^\xi R^{-\xi} - r^{-\xi}R^\xi}{r^2 - R^2}\,dR.$$

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* So that either $\mu_1 = \mu_2 = \pm\frac{1}{2}\nu$, or one of $|\mu_1|$, $|\mu_2|$ is strictly less than $\frac{1}{2}\nu$.

† Watson, *loc. cit.*, 191.

Now let $$C_\xi(z) = \sigma\{\cos\alpha J_\xi(z) + \sin\alpha Y_\xi(z)\}$$
$$= \tfrac{1}{2}\sigma\{e^{-i\alpha}H_\xi^{(1)}(z) + e^{i\alpha}H_\xi^{(2)}(z)\}.$$

Then, multiplying equations (1), (2), and (3) by $-\frac{1}{4}\sigma^2$, $\frac{1}{4}\sigma^2 e^{2i\alpha}$, and $\frac{1}{4}\sigma^2 e^{-2i\alpha}$ respectively, and adding, we get

$$\int_0^\infty v\,dv \int_0^\infty RF(R)\,C_\xi(vr)\,C_\xi(vR)\,dR$$
$$= \tfrac{1}{2}\sigma^2\{F(r+0)+F(r-0)\} - \frac{2\sigma^2 \sin(\xi\pi+\alpha)\sin\alpha}{\pi\sin\xi\pi}\,I.$$

The conditions of the theorem reduce to (i) $\xi < 1$, (ii) $F(R)\,R^{\frac{1}{2}}$ is integrable in $(0, \infty)$, and (iii) $F(R)\,R^{1-\xi}$ is integrable in $(0, 1)$.

If $\xi \leqslant \frac{1}{2}$, the last condition is included in the second. We have thus obtained the theorem stated.

We now come to the proof of Theorem I. We require first a number of lemmas; in all these the conditions of the theorem are supposed satisfied.

LEMMA 1. *The integrals occurring on the right of the equations in Theorem I exist.*

It is evident that $K(u; x, y)$ can be expanded in the neighbourhood of the origin in an ascending series of powers of u and $\log u$.

$\psi(x, y)$ consists of those terms of the expansion which are independent of u.

Since $a+b > -1$, we have $\Re(a_r - a_s + 1) > 0$ for any r, s. Thus the only terms in the product $uY_n^{(1)}(ux)\,Y_{n-1}^{(2)}(uy)$ which are independent of u are of the form

$$cx^{a_r-1}y^{-a_r}(\log x)^\alpha(\log y)^\beta,$$

where c is a constant and α, β are integers.

We thus see that, for fixed x,

$$\psi(x, y) = O(y^b|\log y|^{q-1}) \text{ for } y \text{ small},$$

$$\psi(x, y) = O(y^{-a}|\log y|^{p-1}) \text{ for } y \text{ large}.$$

Since we suppose $f(y)\,y^b|\log y|^{q-1}$ to be integrable in the neighbourhood of the origin, it follows that

$$\frac{f(y)\,\psi(\mu_1, \mu_2; x, y)}{1+(-1)^{\nu+1}e^{(\mu_2-\mu_1)\pi i}\,y/x}$$

is integrable in the neighbourhood of the origin.

Further, since $f(y)y^k$ is absolutely integrable in $(1, \infty)$, and since $k > -a-1$, it follows that $f(y)y^{-a-1}|\log y|^{p-1}$, and hence the above expression, is integrable in the neighbourhood of infinity.

Hence, to complete the proof, it is sufficient to show that the expressions

$$\frac{\psi(\frac{1}{2}\nu, -\frac{1}{2}\nu;\ x, y)+\psi(-\frac{1}{2}\nu, \frac{1}{2}\nu;\ x, y)}{1-y/x} \tag{1}$$

and (if it is not the case that $\mu_1 = -\mu_2 = \pm\frac{1}{2}\nu$)

$$\frac{\psi(\mu_1, \mu_2;\ x, y)}{1+(-1)^{\nu+1}(y/x)\,e^{(\mu_2-\mu_1)\pi i}} \tag{2}$$

remain finite at the point $y = x$, even though the denominator may become infinite.

If $(-1)^\nu e^{(\mu_2-\mu_1)\pi i} = 1$ (the only case which we need consider), we have, by equation (14) of § 2,

$$\frac{d}{du}K(\mu_1, \mu_2;\ u;\ x, x) = 0.$$

Hence

$$\begin{aligned}\psi(\mu_1, \mu_2;\ x, x) &= \lim_{u\to\infty} K(\mu_1, \mu_2;\ u;\ x, x)\\ &= \lim_{u\to\infty} M^2 \exp\{-\nu(ux)^\sigma\,(e^{\mu_1\sigma\pi i}+e^{\mu_2\sigma\pi i})+k(\mu_1+\mu_2)\pi i\}\\ &\qquad \times x^{-1}\left\{\sum_{n=1}^{\nu}\exp\{-\mu_1 n\sigma\pi i+\mu_2(n-1)\sigma\pi i+\pi n i\}\right\}\end{aligned}$$

by equations (10) and (11) of § 2.

It follows at once that

$$\psi(\tfrac{1}{2}\nu, -\tfrac{1}{2}\nu;\ x, x)+\psi(-\tfrac{1}{2}\nu, \tfrac{1}{2}\nu;\ x, x) = 0.$$

Further, if either $|\mu_1|$ or $|\mu_2|$ is strictly less than $\frac{1}{2}\nu$, we have at once

$$\psi(\mu_1, \mu_2;\ x, x) = 0.$$

Finally, if $\mu_1 = \mu_2 = \pm\frac{1}{2}\nu$, we see that, when $(-1)^\nu e^{(\mu_2-\mu_1)\pi i} = 1$, ν must be even, and hence that

$$\sum_{n=1}^{\nu}\exp\{-\mu_1 n\sigma\pi i+\mu_2(n-1)\sigma\pi i+\pi n i\} = \mp i\sum_{n=1}^{\nu}(-1)^n = 0,$$

so that

$$\psi(\mu_1, \mu_2;\ x, x) = 0.$$

Thus, if the denominator of the expressions (1) or (2) becomes zero at $y = x$, the numerator also becomes zero. The numerators, being regular

functions of y and x (except possibly at the origin), must thus be $O(x-y)$ in the neighbourhood of $y=x$, and the result follows.

LEMMA 2. *If α, β are positive, then*

$$\int_{u\alpha}^{u\beta} Y_n^{(2)}(t)\, t^{-k-n\sigma}\, dt = o(u) \quad as \quad u\to\infty.$$

This result follows from equation (11) of § 2.

LEMMA 3. *Let $B>A\geqslant 0$ (B may be infinite). Suppose that $F(y)\, y^{k+n\sigma}$ is absolutely integrable in (A, B). If $A=0$, suppose further that $F(y)\, y^b |\log y|^{q-1}$ is integrable in the neighbourhood of the origin.*

Then $$\int_A^B F(y)\, Y_n^{(2)}(uy)\, dy = o(u^{k+n\sigma}) \quad as \quad u\to\infty.$$

We suppose first that $A>0$.

Then it is sufficient* to prove that $Y_n^{(2)}(uy)\,(uy)^{-k-n\sigma}$ is bounded for y large and u in (A, B), and that

$$\int_\alpha^\beta Y_n^{(2)}(uy)(uy)^{-k-n\sigma}\, dy \to 0 \quad \text{as} \quad u\to\infty,$$

where $B\geqslant\beta\geqslant\alpha\geqslant A$ (β finite).

The first result follows from equation (11) of § 2. The second follows from Lemma 2.

We now consider the case in which $A=0$.

We distinguish two cases.

Case I. $k+n\sigma\geqslant b$.

We have

$$Y_n^{(2)}(uy) = O(uy)^{k+n\sigma} \qquad \text{for } uy \text{ large},$$

$$Y_n^{(2)}(uy) = O(uy)^b |\log uy|^{q-1} \quad \text{for } uy \text{ small}.$$

It follows that, for u large and y small,

$$Y_n^{(2)}(uy) = O(u^{k+n\sigma} y^b |\log y|^{q-1}).$$

Hence

$$\left|\int_0^\eta F(y)\, Y^{(2)}(uy)\, dy\right| \leqslant c u^{k+n\sigma} \int_0^\eta |F(y)|\, y^b |\log y|^{q-1}\, dy,$$

* E. W. Hobson, *Functions of a real variable*, 2 (1926), 2 ed., 426.

where c is a constant, and this may be made less than $\epsilon u^{k+n\sigma}$ for any given ϵ by choosing η sufficiently small. But we have already proved that

$$\int_\eta^B F(y)\, Y_n^{(2)}(uy)\, dy = o(u^{k+n\sigma}),$$

and the result follows.

Case II. $k+n\sigma < b$.

We have

$$Y_n^{(2)}(uy) = O(uy)^{k+n\sigma} \qquad \text{for } uy \text{ large},$$

$$Y_n^{(2)}(uy) = O(uy)^b |\log uy|^{q-1} \qquad \text{for } uy \text{ small}.$$

Hence $$Y_n^{(2)}(uy) = O(uy)^{k+n\sigma} \quad \text{for all } uy.$$

Hence $$\left| \int_0^\eta F(y)\, Y_n^{(2)}(uy)\, dy \right| \leqslant cu^{k+n\sigma} \int_0^\eta |F(y)|\, y^{k+n\sigma}\, dy,$$

whence the result follows as before.

LEMMA 4.

$$\int_0^\infty du \int_0^\infty f(y)\, Y^{(1)}(ux)\, Y^{(2)}(uy)\, dy = \lim_{\lambda\to\infty} \int_0^\infty f(y)\, dy \int_0^\lambda Y^{(1)}(ux)\, Y^{(2)}(uy)\, du,$$

provided that the limit on the right exists.

Since $$Y^{(1)}(ux) = O(ux)^{a-\epsilon},$$

$$Y^{(2)}(uy) = O(uy)^{b-\epsilon},$$

for u, x, y small, it follows that the integrand is absolutely integrable in any finite interval.

Hence, if we write

$$I(\lambda) = \int_0^\lambda du \int_0^\infty f(y)\, Y^{(1)}(ux)\, Y^{(2)}(uy)\, dy - \int_0^\infty f(y)\, dy \int_0^\lambda Y^{(1)}(ux)\, Y^{(2)}(uy)\, dy,$$

we get

$$I(\lambda) = \int_0^\lambda du \int_A^\infty f(y)\, Y^{(1)}(ux)\, Y^{(2)}(uy)\, dy - \int_A^\infty f(y)\, dy \int_0^\lambda Y^{(1)}(ux)\, Y^{(2)}(uy)\, du.$$

If $b > k$, we have, as in Lemma 3, $Y^{(2)}(uy) = O(uy)^k$ for all u, y.

We also have $Y^{(1)}(ux) = O(u^{a-\epsilon})$ for all $u < \lambda$ and x, λ fixed.

Since $a+k > -1$, it follows that $I(\lambda)$ can be made arbitrarily small by taking A sufficiently large, and, since $I(\lambda)$ is independent of A, it follows that

$$I(\lambda) = 0.$$

If $b \leqslant k$, we have, as in Lemma 3, $Y^{(2)}(uy) = O(u^{b-\epsilon} y^k)$ for all relevant u, y and we deduce as before that $I(\lambda) = 0$.

When we make $\lambda \to \infty$, the result follows.

LEMMA 5. *Let $B > A \geqslant 0$ (B may be infinite). Then*

$$\int_A^B f(y)\,dy \int_0^\lambda Y^{(1)}(\mu_1;\ ux)\, Y^{(2)}(\mu_2;\ uy)\,du \to -\int_A^B F(x, y)\,\psi(\mu_1, \mu_2;\ x, y)\,dy \quad as \quad \lambda \to \infty,$$

where

$$F(x, y) = \frac{f(y)}{1+(-1)^{\nu+1}(y/x)\,e^{(\mu_2-\mu_1)\pi i}},$$

provided that $1+(-1)^{\nu+1}(y/x)\,e^{(\mu_2-\mu_1)\pi i} \neq 0$ *in* (A, B).

By equation (14) of § 2,

$$\int_A^B f(y)\,dy \int_0^\lambda Y^{(1)}(ux)\, Y^{(2)}(uy)\,du = \int_A^B F(x, y) \left\{ \lambda \sum_{n=1}^{\nu} (-1)^n\, Y_n^{(1)}(\lambda x)\, Y_{n-1}^{(2)}(\lambda y) - \psi(x, y) \right\} dy.$$

Since $f(y)\,y^k$ is absolutely integrable in (A, B), so is $F(x, y)\,y^{k+(n-1)\sigma}$, since $(n-1)\,\sigma < 1$. Also, since $f(y)\,y^b\,|\log y|^{q-1}$ is integrable in the neighbourhood of the origin, so is $F(x, y)\,y^b\,|\log y|^{q-1}$.

Hence, by Lemma 3,

$$\int_A^B F(x, y)\, Y_{n-1}^{(2)}(\lambda y)\,dy = o(\lambda^{k+(n-1)\sigma}) \quad \text{as} \quad \lambda \to \infty.$$

Since

$$Y_n^{(1)}(\lambda x) = O(\lambda^{k-n\sigma}), \quad 2k-\sigma = -1,$$

the result follows.

We are now in a position to prove Theorem I.

To prove the first part, we write

$$Y^{(1)}(\tfrac{1}{2}\nu;\ ux)\, Y^{(2)}(-\tfrac{1}{2}\nu;\ uy) + Y^{(1)}(-\tfrac{1}{2}\nu;\ ux)\, Y^{(2)}(\tfrac{1}{2}\nu;\ uy) = G(u, x, y).$$

Then, by Lemma 4,

$$(1) \qquad \int_0^\infty du \int_0^\infty f(y)\, G(u, x, y)\,dy = \lim_{\lambda \to \infty} \int_0^\infty f(y)\,dy \int_0^\lambda G(u, x, y)\,du,$$

provided that this limit exists.

By Lemma 5,

$$(2)\quad \lim_{\lambda\to\infty}\left\{\int_0^{x-\delta}+\int_{x+\delta}^{\infty}\right\} f(y)\,dy\int_0^{\lambda} G(u,x,y)\,dy$$

$$=-\left\{\int_0^{x-\delta}+\int_{x+\delta}^{\infty}\right\}\frac{f(y)}{1-y/x}\{\psi(\tfrac{1}{2}\nu,-\tfrac{1}{2}\nu;\,x,y)+\psi(-\tfrac{1}{2}\nu,\tfrac{1}{2}\nu;\,x,y)\}\,dy,$$

and this differs from the integral from 0 to ∞ (which exists by Lemma 1) by an amount arbitrarily small for sufficiently small δ.

It is evident that

$$\int_{x-\delta}^{x+\delta} f(y)\,dy\int_0^1 G(u,x,y)\,du$$

can be made arbitrarily small by taking δ sufficiently small, and we thus have only to consider

$$(3)\qquad \int_{x-\delta}^{x+\delta} f(y)\,dy\int_1^{\lambda} G(u,x,y)\,du.$$

It follows from equations (10) and (11) of § 2 that, for all relevant values of u, x, and y,

$$(4)\quad G(u,x,y)=2M^2u^{2k}x^ky^k\cos\nu u^\sigma(y^\sigma-x^\sigma)$$

$$+u^{-1}x^ky^k(cx^{-\sigma}+cy^{-\sigma})\sin\nu u^\sigma(y^\sigma-x^\sigma)$$

$$+O(u^{-1-\sigma}),$$

where the c's are (different) constants.

Now

$$\int_1^{\lambda}\frac{\sin\nu u^\sigma(y^\sigma-x^\sigma)}{u}\,du=\nu\int_{|y^\sigma-x^\sigma|}^{\lambda^\sigma|y^\sigma-x^\sigma|}\frac{\sin\nu u}{u}\,du$$

and is thus bounded. It therefore follows, from (4), that the integral (3) differs from

$$2M^2x^k\int_{x-\delta}^{x+\delta} f(y)\,y^k\,dy\int_0^{\lambda} u^{2k}\cos\nu u^\sigma(y^\sigma-x^\sigma)\,du,$$

that is to say from

$$(5)\qquad 2M^2x^k\int_0^{\lambda}u^{2k}\,du\int_{x-\delta}^{x+\delta} f(y)\,y^k\cos\nu u^\sigma(y^\sigma-x^\sigma)\,dy,$$

by an amount which is arbitrarily small for sufficiently small δ (and all λ).

If we write

$$\nu u^\sigma = U, \quad y^\sigma = Y, \quad x^\sigma = X, \quad (x+\delta)^\sigma = X+\delta_1, \quad (x-\delta)^\sigma = X-\delta_2,$$

$$f(y)\,y^{-k} = F(Y), \quad \nu\lambda^\sigma = \lambda',$$

the integral (5) becomes

$$2M^2\,\nu x^k \int_0^{\lambda'} dU \int_{x-\delta_2}^{x+\delta_1} F(Y)\cos U(Y-X)\,dY,$$

and the first part of the theorem follows at once.

For the second part, we see, as before, that

$$\int_0^\infty Y^{(1)}(\mu_1;\ ux)\,du \int_0^\infty Y^{(2)}(\mu_2;\ uy)\,du + \int_0^\infty \frac{f(y)\,\psi(\mu_1,\ \mu_2;\ x,\ y)}{1+(-1)^{\nu+1}(y/x)\,e^{(\mu_2-\mu_1)\,\pi i}}\,dy$$

differs from

$$\lim_{\lambda\to\infty} \int_{x-\delta}^{x+\delta} f(y)\,dy \int_1^\lambda Y^{(1)}(\mu_1;\ ux)\,Y^{(2)}(\mu_2;\ uy)\,du \tag{6}$$

by an amount arbitrarily small for sufficiently small δ.

If at least one of $|\mu_1|$, $|\mu_2|$ is less than $\frac{1}{2}\nu$, it follows from equations (10) and (11) of § 2 that $Y^{(1)}(\mu_1;\ ux)\ Y^{(2)}(\mu_2;\ uy)$ is absolutely integrable with respect to u in $(1,\ \infty)$. Hence the expression (6) is arbitrarily small for δ sufficiently small, and the result follows.

If $\mu_1 = \mu_2 = \pm\frac{1}{2}\nu$, we have

$$Y^{(1)}(\mu_1;\ ux)\,Y^{(2)}(\mu_2;\ uy)$$

$$= \{cu^{2k}x^k y^k + u^{-1}x^k y^k(cx^{-\sigma}+cy^{-\sigma})\}\exp\{\mp i\nu u^\sigma(y^\sigma+x^\sigma)\} + O(u^{-1-\sigma}),$$

where the c's are (different) constants.

Hence

$$\int_1^\lambda Y^{(1)}(\mu_1;\ ux)\,Y^{(2)}(\mu_2;\ uy)\,du$$

$$= \frac{cx^k y^k}{x^\sigma+y^\sigma}\left[\exp\{\mp i\nu\lambda^\sigma(y^\sigma+x^\sigma)\} - \exp\{\mp i\nu(y^\sigma+x^\sigma)\}\right]$$

$$+ x^k y^k(cx^{-\sigma}+cy^{-\sigma}) \int_{x^\sigma+y^\sigma}^{\lambda^\sigma(x^\sigma+y^\sigma)} u^{-1}\exp(\mp i\nu u)\,du + O\left\{\int_1^\lambda u^{-1-\sigma}\,du\right\}.$$

This expression is bounded for all x, y in any finite interval which does not include the origin. It follows that the expression (6) is arbitrarily small for δ sufficiently small, and the theorem follows.

4. *Proof of Theorem II.*

Some additional definitions are now necessary. Let

$$g_r(\beta_1, \ldots, \beta_r;\ \beta_{r+1}, \ldots, \beta_\nu;\ z)$$

$$= -\int_C \Gamma(\beta_1-s)\ldots\Gamma(\beta_\nu-s)\cos\tfrac{1}{2}\pi(s-\beta_1)\ldots\cos\tfrac{1}{2}\pi(s-\beta_r)$$
$$\times\sin\tfrac{1}{2}\pi(s-\beta_{r+1})\ldots\sin\tfrac{1}{2}\pi(s-\beta_\nu)z^s\,ds,$$

the contour C being the same as in the definition of $Y(\alpha_1\ldots\alpha_\nu;\ \mu;\ z)$.

We continue to suppose that $\sum_{n=1}^{\nu}\alpha_n = 0$.

Write

$$G^{(1)}(z) = g_r(\alpha_1, \ldots, \alpha_r;\ \alpha_{r+1}, \ldots, \alpha_\nu;\ z),$$

$$G^{(2)}(z) = g_r(-\alpha_1, \ldots, -\alpha_r;\ -\alpha_{r+1}, \ldots, -\alpha_\nu;\ z).$$

As in Theorem I, we write a for the real part of that pole (or poles) of the integrand in the integral defining $G^{(1)}(z)$ whose real part is smallest, and p for the greatest multiplicity of such poles. Of course, it is not now necessarily the case that $a = \min(\Re\alpha_n)$, for, if $n > r$, the pole of $\Gamma(\alpha_n-s)$ occurring at $s = \alpha_n$ is annihilated by the zero of $\sin\frac{1}{2}\pi(s-\alpha_n)$.

We similarly define b and q for the function $G^{(2)}(z)$.

We have, evidently, for small z,

$$G^{(1)}(z) = O(z^a|\log z|^{p-1}), \quad G^{(2)}(z) = O(z^b|\log z|^{q-1}).$$

We may now enunciate

THEOREM II. *Suppose that*

$$a+b > -1, \quad a+k+1 > 0, \quad x > 0,$$

and that $f(y)$ satisfies the conditions of Theorem I. Then, if $f(y)$ is of bounded variation in some neighbourhood of x,

$$\int_0^\infty G^{(1)}(ux)\,du\int_0^\infty f(y)\,G^{(2)}(uy)\,dy = -\pi^{\nu+2}2^{1-\nu}\{f(x+0)+f(x-0)\},$$

and, more generally, the integral behaves, in respect of convergence or summability, in the same way as the Fourier repeated integral.

It is convenient to describe the parameters $\beta_1, \ldots, \beta_r$ of the function $g_r(\beta_1, \ldots, \beta_r;\ \beta_{r+1}, \ldots, \beta_\nu;\ z)$ as "cosine parameters" and the parameters $\beta_{r+1}, \ldots, \beta_\nu$ as "sine parameters".

Consider the effect of applying the differential operator

$$z^{\beta_n+1}\frac{d}{dz}z^{-\beta_n}$$

to the function $g_r(\beta_1, \ldots, \beta_r;\ \beta_{r+1}, \ldots, \beta_\nu;\ z)$.

If β_n is a sine parameter, the result of the operation will be to replace it by a cosine parameter β_n+1, and to change the sign of the function.

If β_n is a cosine parameter, the result will be to replace it by a sine parameter β_n+1.

We now define the functions $G_n^{(2)}(z)$ for $n=0, 1, \ldots, 2\nu$, as follows:

$$G_0^{(2)}(z)=G^{(2)}(z),$$

$$G_n^{(2)}(z)=z^{-\alpha_n+1}\frac{d}{dz}\{z^{\alpha_n}G_{n-1}(z)\} \qquad (1\leqslant n\leqslant \nu),$$

$$G_n^{(2)}(z)=z^{-\alpha_n-\nu+2}\frac{d}{dz}\{z^{\alpha_n-\nu+1}G_{n-1}(z)\} \qquad (\nu+1\leqslant n\leqslant 2\nu).$$

The form of the functions $G_n^{(2)}(z)$ is evident from the preceding remarks, and evidently

$$G_{2\nu}^{(2)}(z)=(-1)^\nu g_r(-\alpha_1+2, \ldots, -\alpha_r+2;\ -\alpha_{r+1}+2, \ldots, -\alpha_\nu+2;\ z)$$
$$=(-1)^\nu z^2 G^{(2)}(z).$$

Again, let

$$G_{2\nu}^{(1)}(z)=(-1)^\nu z^{-2}G^{(1)}(z)$$
$$=(-1)^\nu g_r(\alpha_1-2, \ldots, \alpha_r-2;\ \alpha_{r+1}-2, \ldots, \alpha_\nu-2;\ z).$$

We define $G_n^{(1)}(z)$ by

$$G_{n-1}^{(1)}(z)=z^{\alpha_n-\nu-1}\frac{d}{dz}z^{-\alpha_n-\nu+2}G_n^{(1)}(z) \qquad (\nu+1\leqslant n\leqslant 2\nu);$$

$$G_{n-1}^{(1)}(z)=z^{\alpha_n}\frac{d}{dz}z^{-\alpha_n+1}G_n^{(1)}(z) \qquad (1\leqslant n\leqslant \nu).$$

Evidently $$G_0^{(1)}(z)=G^{(1)}(z).$$

Now let

$$K(u, x, y)=u\sum_{n=1}^{2\nu}(-1)^{n-1}G_n^{(1)}(ux)\,G_{n-1}^{(2)}(uy).$$

Then it follows from the above equations that

$$\frac{d}{du} K(u, x, y) = \left\{1 - \left(\frac{y}{x}\right)^2\right\} G^{(1)}(ux)\, G^{(2)}(uy).$$

If $a+b > -1$, $G^{(1)}(ux)\, G^{(2)}(uy)$ is integrable in the neighbourhood of the origin, so that $K(u, x, y)$ must tend to a finite limit as $u \to 0$. We now show that this limit is zero.

Consider first the case in which no two of $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ are equal or differ by an integer.

We show that no constant term can occur in the expansion of $uG_n^{(1)}(ux)\, G_{n-1}^{(2)}(uy)$ in ascending powers of u. The result will evidently follow. We take $n \leqslant \nu$; the case $n > \nu$ may be treated in a similar manner.

In the expansion of $g_r(\beta_1, \ldots, \beta_r; \beta_{r+1}, \ldots, \beta_\nu; z)$ in ascending powers of z, each cosine parameter β will give rise to a series beginning with z^{β_n}, and each sine parameter β_n to one beginning with z^{β_n+1}.

$G_n^{(1)}(z)$ is a g-function whose parameters are

$$\alpha_1 - 1, \ldots, \alpha_n - 1, \alpha_{n+1}, \ldots, \alpha_\nu.$$

$G_{n-1}^{(2)}(z)$ is a g-function whose parameters are

$$-\alpha_1 + 1, \ldots, -\alpha_{n-1} + 1, -\alpha_n, \ldots, -\alpha_\nu.$$

It follows that a constant term can arise in the product

$$uG_n^{(1)}(ux)\, G_{n-1}^{(2)}(uy)$$

only if $\alpha_n - 1$ is a cosine parameter in $G_n^{(1)}(z)$, and $-\alpha_n$ a cosine parameter in $G_n^{(2)}(z)$. It is, however, always the case that one of these is a sine parameter and the other a cosine parameter. The result follows.

For fixed x, y the product $G^{(1)}(ux)\, G^{(2)}(uy)$ is uniformly $O(u^{-c})$ as $u \to 0$ when the parameters are contained in any finite domain for which

$$a+b > -c > -1.$$

It follows that $K(u, x, y)$ tends to a limit as $x \to 0$ uniformly for all parameters in such a domain, and hence that $\lim_{u \to 0} K(u, x, y)$ is a continuous function of the parameters.

It follows that the result

$$\lim_{u \to 0} K(u, x, y) = 0$$

continues to hold when two of the parameters differ by an integer.

Thus, corresponding to the function $\psi(x, y)$ of Theorem I, we have zero in Theorem II.

Apart from the simplification produced by this fact, the proof is similar.

It is, perhaps, worth noting that, in the case $\nu = 1$, the theorem reduces to the Fourier integral theorem. For we must have either

$$G_1(z) = G_2(z) = -\int_C \Gamma(-s) \cos \tfrac{1}{2}\pi s \, z^s \, ds = 2\pi i \cos z,$$

or

$$G_1(z) = G_2(z) = -\int_C \Gamma(-s) \sin \tfrac{1}{2}\pi s \, z^s \, ds = -2\pi i \sin z.$$

We now indicate the connection between this theorem and the theorems proved by Hardy and Titchmarsh in the paper already cited. The kernels do not, as they stand, satisfy the conditions of Hardy and Titchmarsh's theorems, for we have (using their notation)

$$k(s) = \pi^{-\frac{1}{2}\nu} 2^{\frac{1}{2}\nu} L(-s);$$

$$h(s) = \pi^{-\frac{1}{2}\nu} 2^{\frac{1}{2}\nu} L'(-s),$$

where

$$L(s) = \Gamma(\alpha_1 - s) \dots \Gamma(\alpha_\nu - s) \cos \tfrac{1}{2}\pi(s-\alpha_1) \dots \cos \tfrac{1}{2}\pi(s-\alpha_r) \times \sin \tfrac{1}{2}\pi(s-\alpha_{r+1}) \dots \sin \tfrac{1}{2}\pi(s-\alpha_\nu),$$

$$L'(s) = \Gamma(-\alpha_1 - s) \dots \Gamma(-\alpha_\nu - s) \cos \tfrac{1}{2}\pi(s+\alpha_1) \dots \cos \tfrac{1}{2}\pi(s+\alpha_r) \times \sin \tfrac{1}{2}\pi(s+\alpha_{r+1}) \dots \sin \tfrac{1}{2}\pi(s+\alpha_\nu).$$

Thus, for t large and positive,

$$k(\sigma + it) = e^{\nu i (t \log t - t)} t^{\nu(\sigma - \frac{1}{2})} \left\{ \alpha + \frac{\beta}{t} + O\left(\frac{1}{t^2}\right) \right\}$$

with a corresponding result for t negative, and a similar result for k.

By writing the theorem in a slightly different form, we see that it is, nevertheless, of the type considered by Hardy and Titchmarsh. For Theorem II states that, under certain conditions,

$$\int_0^\infty G^{(1)}(xu)\, du \int_0^\infty G^{(2)}(uy) f(y)\, dy = -\pi^{\nu+2} 2^{2-\nu} f(x).$$

Hence

$$\int_0^\infty G^{(1)}\left(\frac{x^\nu u^\nu}{\nu^\nu}\right) u^{\nu-1}\, du \int_0^\infty G^{(2)}\left(\frac{u^\nu y^\nu}{\nu^\nu}\right) y^{\nu-1} f(y^\nu)\, dy = -\nu^{\nu-2} \pi^{\nu+2} 2^{2-\nu} f(x^\nu).$$

Now write $$F(x) = f(x^\nu)\, x^{\frac{1}{2}(\nu-1)},$$

$$K(z) = cG^{(1)}(z^\nu/\nu^\nu)\, z^{\frac{1}{2}(\nu-1)}, \quad H(z) = cG^{(2)}(z^\nu/\nu^\nu)\, z^{\frac{1}{2}(\nu-1)},$$

where $$c = -i\nu^{\frac{1}{2}(2-\nu)}\, \pi^{-\frac{1}{2}(\nu+2)}\, 2^{-\frac{1}{2}(2-\nu)}.$$

We obtain

$$\int_0^\infty K(xu)\, du \int_0^\infty H(uy)\, F(y)\, dy = F(x).$$

The functions $k(s)$ and $h(s)$ are now given by

$$k(s) = \pi^{-\frac{1}{2}\nu}\, 2^{\frac{1}{2}\nu}\, \nu^{s-\frac{1}{2}}\, L\left(\frac{\frac{1}{2}-s}{\nu} - \frac{1}{2}\right),$$

$$h(s) = \pi^{-\frac{1}{2}\nu}\, 2^{\frac{1}{2}\nu}\, \nu^{s-\frac{1}{2}}\, L'\left(\frac{\frac{1}{2}-s}{\nu} - \frac{1}{2}\right).$$

These satisfy equation (2.1.1) of the paper by Hardy and Titchmarsh, and (more generally) the equations

$$k(s) = k_1(s)\left\{\alpha + \frac{\beta}{s} + O\left(\frac{1}{|s|^2}\right)\right\},$$

with a similar result for $h(s)$, given by them in § 6.1.

The equations

$$K(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{-s}\, k(s)\, ds,$$

$$H(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^{-s}\, h(s)\, ds,$$

where the contour of integration is a straight line, will hold provided that $a > -\frac{1}{2}$, $b > -\frac{1}{2}$, where a and b are defined as in Theorem II of this paper. They will not hold if these conditions are not satisfied, since the contours must then be bent so as to pass to the right of the poles of the integrand.

We thus see that, provided $a > -\frac{1}{2}$, $b > -\frac{1}{2}$, Theorem II of this paper, so far as it concerns the case when $f(y)$ is of bounded variation in a neighbourhood of x, is practically included in Theorem I* of Hardy and Titchmarsh. Of course, the latter proves only a kind of summability, whereas the former proves convergence. Further, the restrictions imposed on $F(x)$ in the neighbourhood of the origin are less stringent in the former.

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* Theorem I is enunciated for the case in which the two kernels are the same, but, as the authors remark in § 6.8, a similar result will hold when they are different.

This arises from the fact that, whereas in the general case Hardy and Titchmarsh have only the result proved in their Lemma 5,

$$K(x) = O(x^{-\frac{1}{2}}) \text{ for } x \text{ small},$$

in our particular case we have the stronger result

$$K(x) = O(x^{\nu a+\frac{1}{2}(\nu-1)}|\log x|^{p-1}) \text{ for } x \text{ small},$$

with a similar result for $H(x)$ with a replaced by b.

For x large, $K(x)$ is $O(1)$ in our particular case; this is the same as the result obtained by Hardy and Titchmarsh in Lemma 5.

The poles of $k(s)$ are evidently the same as those of $L\{(\frac{1}{2}-s)/\nu-\frac{1}{2}\}$. Those lying furthest to the right have their real part given by

$$\Re\left(\frac{\frac{1}{2}-s}{\nu}-\frac{1}{2}\right) = a,$$

i.e.

$$-\Re s = \tfrac{1}{2}\nu-\tfrac{1}{2}+a\nu.$$

Thus, if $a > -\frac{1}{2}(1-\nu^{-1})$, $k(s)$ is regular in some half plane $\sigma_1 < \Re s$, where

$$\sigma_1 < 0.$$

Again, if $a = -\frac{1}{2}(1-\nu^{-1})$ and $p = 1$, then $k(s)$ is regular in this half plane apart from simple poles on the imaginary axis.

Thus, if either

$$a > -\tfrac{1}{2}(1-\nu^{-1})$$

or

$$a = -\tfrac{1}{2}(1-\nu^{-1}), \quad p = 1,$$

$k(s)$ belongs to the class termed $\mathbf{k}'$ in the paper by Hardy and Titchmarsh.

A corresponding result will hold for $h(s)$; thus in the cases in which (i)

$$a > -\tfrac{1}{2}(1-\nu^{-1}) \text{ or } a = -\tfrac{1}{2}(1-\nu^{-1}), \quad p = 1,$$

and (ii)

$$b > -\tfrac{1}{2}(1-\nu^{-1}) \text{ or } b = -\tfrac{1}{2}(1-\nu^{-1}), \quad q = 1,$$

we see that Theorem II of this paper, so far as it concerns the case when $f(y)$ of bounded variation in a neighbourhood of x, is included in Theorem IV of Hardy and Titchmarsh, except that the conditions imposed on $F(x)$ at the origin are, as before, less stringent.

5. *Notation.*

In the rest of this paper, we use a different notation. We write $G(z)$ for any function of the form

$$G(z) = -\int_C K(s) z^s \, ds,$$

where the contour C is the same as in the definition of $Y(\alpha_1, \ldots, \alpha_n; \mu; z)$, and where

$$K(s) = \Gamma(\alpha_1 - s) \ldots \Gamma(\alpha_\nu - s) \sum_{r=1}^{n} c_r e^{\mu_r \pi i s},$$

where $|\mu_r| \leqslant \frac{1}{2}\nu$, and the c_r's are constants. Thus

$$G(z) = \sum_{r=1}^{n} c_r Y(\alpha_1, \ldots, \alpha_\nu; \mu_r; z).$$

In particular, from the results

$$-2\pi^2 H_\xi^{(1)}(2\sqrt{z}) = e^{-\frac{1}{2}\pi i \xi} Y(-\tfrac{1}{2}\xi, \tfrac{1}{2}\xi; -1; z),$$

$$2\pi^2 H_\xi^{(2)}(2\sqrt{z}) = e^{\frac{1}{2}\pi i \xi} Y(-\tfrac{1}{2}\xi, \tfrac{1}{2}\xi; 1; z),$$

we see that $G(z)$ may be a Bessel function of argument $2\sqrt{z}$.

We no longer suppose that $\Sigma \alpha_r = 0$; we write

$$\Sigma \alpha_r = \alpha + i\beta.$$

We continue to write $\sigma = 1/\nu$, and write

$$k = -\tfrac{1}{2} + \sigma(\tfrac{1}{2} + \alpha).$$

We write $$K(-\tfrac{1}{2}) = K,$$

$$\tfrac{1}{2}\pi i \, \Gamma(\delta+1) \left[\frac{d}{ds} \frac{K(s)}{\Gamma(2s+\delta+2)}\right]_{s=-\frac{1}{2}} = L(\delta).$$

In the case $K = 0$,

$$L(\delta) = \tfrac{1}{2}\pi i \left[\frac{d}{ds} K(s)\right]_{s=-\frac{1}{2}}.$$

It is thus independent of δ, and we then call it L. We define a and p as before, so that

$$G(z) = O(z^a |\log z|^{p-1}) \quad \text{for } z \text{ small.}$$

We note that, if $|\mu_r| < \frac{1}{2}\nu$ for all r, or if

$$c > -\tfrac{1}{2} + \sigma(\alpha+1),$$

$G(z)$ may be expressed in the form*

$$G(z) = \int_{c-i\infty}^{c+i\infty} K(s)\, z^s\, ds,$$

the contour being bent if necessary, so that all the poles of the integrand lie on the right of it.

We note that†, if $|\mu_r| < \frac{1}{2}\nu$ for all r,

$$G(z) = O[\exp(-cz^\sigma)] \quad \text{for } z \text{ large},$$

where c is a positive constant.

If not, the asymptotic expansion of $G(z)$ is given by

$$G(z) \sim z^{k+i\beta\sigma}(c \sin z^\sigma + c \cos z^\sigma + cz^{-\sigma} \sin z^\sigma + cz^{-\sigma} \cos z^\sigma + \ldots),$$

where the c's are (different) constants.

In obtaining our various results, we have frequently to impose restrictions involving α. Thus in the second part of Theorem IV we have the condition that $t^{2k}\phi(t)$ should be absolutely integrable in (h, ∞); and k depends on α.

The necessity for such restrictions arises in two ways.

(1) We may require to express $G(z)$ in the form

$$\int_{c-i\infty}^{c+i\infty} K(s)\, z^s\, ds,$$

which (unless $|\mu_r| < \frac{1}{2}\nu$ for all r) necessitates $c > -\frac{1}{2} + \sigma(\alpha+1)$.

(2) We may require that an infinite integral containing $G(z)$ should be convergent.

Now in the special case in which $|\mu_r| < \frac{1}{2}\nu$ for all r, the restriction $c > -\frac{1}{2} + \sigma(\alpha+1)$ is no longer necessary in (1). Further, instead of only having $G(z) = O(z^k)$ for large z, we have $G(z) = O(z^{-\Delta})$ for any Δ. Thus α may be replaced by $-\Delta$ in the conditions imposed.

Since Δ is arbitrary, we see that

In all results which are proved subject to restrictions involving α, these restrictions may be removed in the special case in which $|\mu_r| < \frac{1}{2}\nu$ for all r.

We make this statement here to avoid constant repetition.

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* D 1, (1.22), (1.23), and (1.231).

† By D 2, 368, subject to the correction indicated in foot-note ‡, p. 167 *supra*.

6. *Proof of Theorems III and IV.*

We now require some further lemmas.

LEMMA 6. *Suppose that* $a > -\frac{1}{2}$, $\alpha < 1+\delta$, $\delta > -1$. *Let*

$$\mathrm{I}(\lambda) = \int_0^\eta dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy.$$

Then, for fixed η,

$$\mathrm{I}(\lambda) = \pi i K(\log \eta\lambda - \gamma) + L(\delta) + o(1),$$

where γ *is Euler's constant.*

We have

$$\mathrm{I}(\lambda) = \int_0^\eta dt \int_0^\lambda \left(1-\frac{y}{\lambda}\right)^\delta dy \int_{c-i\infty}^{c+i\infty} K(s)\, t^{2s} y^{2s}\, ds,$$

where

$$c > -\tfrac{1}{2} + \sigma(\alpha+1).$$

The contour may be chosen so that $\Re s \geqslant$ constant $> -\frac{1}{2}$ at all points of it; the triple integral is then absolutely convergent and we may therefore change the order of integration. Thus, performing the integrations with respect to t and y, we obtain

$$\mathrm{I}(\lambda) = \Gamma(\delta+1) \int_{c-i\infty}^{c+i\infty} \frac{K(s)\,\eta^{2s+1}\lambda^{2s+1}\,\Gamma(2s+1)\,ds}{(2s+1)\,\Gamma(2s+\delta+2)}.$$

We may now displace the contour to the straight line $(c'-i\infty, c'+i\infty)$, where

$$\alpha - 1 - \delta - \tfrac{1}{2}\nu < \nu c' < -\tfrac{1}{2}\nu.$$

The integral along this line is absolutely convergent and is $O(\lambda^{2c'+1})$; it is therefore $o(1)$.

Hence, since the residue of the pole at $s = -\frac{1}{2}$ is

$$\tfrac{1}{2}K\{\Gamma(\delta+1)\}^{-1}(\log \eta\lambda - \gamma) + \tfrac{1}{4}\left[\frac{d}{ds}\frac{K(s)}{\Gamma(2s+\delta+2)}\right]_{s=-\frac{1}{2}},$$

the lemma follows.

COROLLARY. *Under the same conditions, if* $\eta \geqslant \xi \geqslant 0$, *and* ξ *is allowed to vary with* λ,

$$\int_\xi^\eta dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy$$

is bounded as $\lambda \to \infty$ *provided that* $K = 0$, *and is* $O(\log \lambda)$ *in any case.*

For the expression is equal to

$$\mathrm{I}(\lambda)-\Gamma(\delta+1)\int_{c-i\infty}^{c+i\infty}\frac{K(s)\,\xi^{2s+1}\lambda^{2s+1}\,\Gamma(2s+1)\,ds}{(2s+1)\,\Gamma(2s+\delta+2)}.$$

If $\xi\lambda\leqslant 1$, this integral is evidently $O(1)$, since $\Re(2s+1)>0$ on the contour.

If $\xi\lambda>1$, we displace the contour to $(c'-i\infty,\ c'+i\infty)$ as before, and the result follows.

LEMMA 7. *Suppose that* $\delta>-1$, $r>0$, *and* h *is any positive constant. Let*

$$\mathrm{I}(\delta,\ s,\ u)=\int_h^u\left(1-\frac{x}{u}\right)^{\delta}x^s\exp(ix^r)\,dx.$$

Then, for large u,

$$\mathrm{I}(\delta,\ s,\ u)=c+O(u^{-1})+O(u^d),$$

where

$$d=\Re s+1-r(\delta+1)$$

and c *is a constant.*

The result holds uniformly for x *in any finite domain.*

We use c throughout to denote a constant; it may be a different constant at each occurrence.

We divide the integral into two parts:

$$\mathrm{I}(\delta,\ s,\ u)=\mathrm{I}_1(\delta,\ s,\ u)+\mathrm{I}_2(\delta,\ s,\ u), \tag{1}$$

where

$$\begin{aligned}\mathrm{I}_1(\delta,\ s,\ u)&=\int_{u-u^{1-r}}^{u}\left(1-\frac{x}{u}\right)^{\delta}x^s\exp(ix^r)\,dx\\&=O\left\{u^{\Re s}\int_{u-u^{1-r}}^{u}\left(1-\frac{x}{u}\right)^{\delta}dx\right\}\\&=O(u^d),\end{aligned} \tag{2}$$

$$\begin{aligned}\mathrm{I}_2(\delta,\ s,\ u)&=\int_h^{u-u^{1-r}}\left(1-\frac{x}{u}\right)^{\delta}x^s\exp(ix^r)\,dx\\&=-\frac{i}{r}\left[\left(1-\frac{x}{u}\right)^{\delta}x^{s-r+1}\exp(ix)^r\right]_h^{u-u^{1-r}}\\&\qquad+\frac{i}{r}\int_h^{u-u^{1-r}}\exp(ix^r)\left[\frac{d}{dx}\left(1-\frac{x}{u}\right)^{\delta}x^{s-r+1}\right]dx\\&=c+O(u^{-1})+O(u^d)+cu^{-1}\mathrm{I}_2(\delta-1,\ s+1-r,\ u)+c\mathrm{I}_2(\delta,\ s-r,\ u).\end{aligned}$$

Repeating this process, we get

$$(3)\quad I_2(\delta, s, u) = c + \sum_{m=0}^{n} cu^{-m} I_2(\delta - m, s + m - nr, u) + O(u^{-1}) + O(u^d).$$

Now

$$(4)\quad I_2(\Delta, t, u) = \int_{\hbar}^{\frac{1}{2}u} x^t \exp(ix^r)\, dx + O\left(\frac{1}{u}\int_{\hbar}^{\frac{1}{2}u} x^{\Re t+1}\, dx\right)$$

$$+ O\left\{u^{\Re t}\int_{\frac{1}{2}u}^{u-u^{1-r}} \left(1-\frac{x}{u}\right)^{\Delta} dx\right\}$$

$$= c + O(u^{\Re t+1}) + O(u^{-1}) + O(u^{\Re t+1-r(\Delta+1)}).$$

By choosing n sufficiently large and using equations (3) and (4), we see that

$$(5)\quad I_2(\delta, t, u) = c + O(u^{-1}) + O(u^d).$$

The lemma follows from equations (1), (2), and (5).

LEMMA 8. *If $\eta > 0$, then $\delta > -1$,*

$$\int_{\eta}^{y} \left(1-\frac{x}{y}\right)^{\delta} G(x^2)\, x^{\rho}\, dx = c + O(y^{-1}) + O(y^d) \quad as \quad y \to \infty,$$

where c is a constant, $d = 2k + \mu + 1 - 2\sigma(\delta+1) = \sigma(2a - 2\delta - 1) + \mu$, and $\mu = \Re\rho$.

The result holds uniformly for ρ in any finite domain, and it continues to hold when $\eta = 0$, provided $2a + \mu > -1$.

We use c throughout to denote a constant; it may be a different constant at each occurrence.

We have

$$G(x^2) = x^{2(k+i\beta\sigma)} \sum_{r=0}^{n-1} x^{-2r\sigma}(c \sin x^{2\sigma} + c \cos x^{2\sigma}) + R_n(x^2),$$

where

$$R_n(x^2) = O(x^{2k-2n\sigma}).$$

It follows from Lemma 7 that, for any $r \geqq 0$,

$$\int_{\eta}^{y} \left(1-\frac{x}{y}\right)^{\delta} x^{2(k+i\beta\sigma-r\sigma)+\rho}(c \sin x^{2\sigma} + c \cos x^{2\sigma})\, dx = c + O(y^{-1}) + O(y^d).$$

Further, for n sufficiently large, we have

$$\int_\eta^y \left(1-\frac{x}{y}\right)^\delta R_n(x^2)\,x^\rho\,dx$$

$$=\int_\eta^\infty R_n(x^2)\,x^\rho\,dx-\int_{\frac{1}{2}y}^\infty R_n(x^2)\,x^\rho\,dx$$

$$+O\left\{\frac{1}{y}\int_\eta^{\frac{1}{2}y} x\,|R_n(x^2)x^\rho|\,dx\right\}+O\left\{\int_{\frac{1}{2}y}^y \left(1-\frac{x}{y}\right)^\delta |R_n(x^2)\,x^\rho|\,dx\right\}$$

$$=c+O(y^{-1}),$$

and the first part follows.

For the second part, we have, for any fixed positive η,

$$\int_0^\eta \left(1-\frac{x}{y}\right)^\delta G(x^2)\,x^\rho\,dx=\int_0^\eta G(x^2)\,x^\rho\,dx+O(y^{-1})$$

$$=c+O(y^{-1});$$

the integral on the left exists, since $G(x^2)=O(x^{2a}|\log x|^{p-1})$ for x small.

By the first part,

$$\int_\eta^y \left(1-\frac{x}{y}\right)^\delta G(x^2)\,x^\rho\,dx=c+O(y^{-1})+O(y^d),$$

and the result follows.

LEMMA 9. *Let* $a>-\frac{1}{2}$, $\delta>-1$.

Suppose $\phi(t)(t^{-1}+t^{2k})$ *to be absolutely integrable in* $(1,\infty)$; *in the case* $a=-\frac{1}{2}$, *i.e.* $2k=-1$, *suppose* $\phi(t)\,t^{-1}\,(1+\log t)$ *to be absolutely integrable in* $(1,\infty)$. *Also suppose* $\phi(t)\,t^{2a}(|\log t|^{p-1}+1)$ *to be integrable in* $(0,1)$. *Then*

$$\int_0^\infty \phi(t)\,dt\int_0^\lambda G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy=\int_0^\lambda \left(1-\frac{y}{\lambda}\right)^\delta dy\int_0^\infty \phi(t)\,G(t^2y^2)\,dt.$$

It follows at once, from the conditions imposed, that the double integral

$$\int_0^A |\phi(t)|\,dt\int_0^\lambda \left|G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^\delta\right| dy$$

exists, and hence the order of integration over this domain may be inverted, so that we have only to prove that

$$\mathrm{I}(A)=\int_0^\lambda \left(1-\frac{y}{\lambda}\right)^\delta dy\int_A^\infty \phi(t)\,G(t^2y^2)\,dt\to 0 \quad \text{as} \quad A\to\infty.$$

We divide $\mathrm{I}(A)$ into two parts, $\mathrm{I}_1(A)$ and $\mathrm{I}_2(A)$, where

$$\mathrm{I}_1(A) = \int_0^{1/A} \left(1-\frac{y}{\lambda}\right)^\delta dy \int_A^{1/y} \phi(t)\, G(t^2y^2)\, dt$$

$$= O\left\{\int_A^\infty |\phi(t)|\, dt \int_0^{1/t} (ty)^{2\alpha-\epsilon}\, dy\right\}$$

$$= O\left\{\int_A^\infty |\phi(t)|\, t^{-1}\, dt\right\} \to 0 \quad \text{as} \quad A \to \infty,$$

$$\mathrm{I}_2(A) = \int_0^{1/A} \left(1-\frac{y}{\lambda}\right)^\delta dy \int_{1/y}^\infty \phi(t)\, G(t^2y^2)\, dt + \int_{1/A}^\lambda \left(1-\frac{y}{\lambda}\right)^\delta dy \int_A^\infty \phi(t)\, G(t^2y^2)\, dt$$

$$= O\left\{\int_A^\infty |\phi(t)|\, dt \int_{1/t}^\lambda (ty)^{2k} \left(1-\frac{y}{\lambda}\right)^\delta dy\right\}.$$

If $2k > -1$, this is

$$O\left(\int_A^\infty |\phi(t)|\, t^{2k}\, dt\right).$$

If $2k = -1$, it is

$$O\left(\int_A^\infty |\phi(t)|\, t^{-1} \log t\, dt\right).$$

If $2k < -1$, it is

$$O\left(\int_A^\infty |\phi(t)|\, t^{-1}\, dt\right).$$

In any case the result follows.

THEOREM III. *Let* $K = 0$, $\alpha > -\frac{1}{2}$, $\delta > -1$, $\alpha \leqslant \frac{1}{2}+\delta$.

Suppose $\phi(t)$ *to be such that, for some* $h > 0$, $\phi(t)\, t^{-1}$ *is absolutely integrable in* (h, ∞), *and such that* $\phi(t)$ *is of bounded variation in* $(0, h)$. *Then*

$$\lim_{\lambda\to\infty} \int_0^\infty \phi(t)\, dt \int_0^\lambda G(t^2y^2) \left(1-\frac{y}{\lambda}\right)^\delta dy = L\phi(+0).$$

If, further, $t^{2k}\phi(t)$ *is absolutely integrable in* (h, ∞), *and, in the case* $\alpha = -\frac{1}{2}$, $\phi(t)\, t^{-1} \log t$ *is absolutely integrable in* (h, ∞), *then*

$$\int_0^\infty dy \int_0^x \phi(t)\, G(t^2y^2)\, dt$$

is summable (C, δ) *to* $L\phi(+0)$.

It is to be noted that, in the case $\alpha < \frac{1}{2}+\delta$, this theorem is included in Theorem V.

It follows from Lemma 6 that, if B and A are fixed, and $B > A > 0$,

$$\int_A^B dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{1}$$

Now, by Lemma 8,

$$t\int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy \tag{2}$$

is bounded for λ large, and $t \geqslant \eta > 0$. Hence, so also is

$$\int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy. \tag{3}$$

By the "general convergence theorem*" it follows, from (1) and (3), that

$$\int_A^B t\,dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy \to 0 \quad \textit{as} \quad \lambda \to \infty. \tag{4}$$

Since $\phi(t)t^{-1}$ is absolutely integrable in (η, ∞) for any $\eta > 0$, we have, by (2) and (4), applying the "general convergence theorem†" to the infinite interval (η, ∞),

$$\int_\eta^\infty \phi(t)\,dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{5}$$

By Lemma 6 and its corollary, it follows that

$$\overline{\lim_{\lambda\to\infty}} \left| \int_0^\eta \phi(t)\,dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy - L\phi(+0) \right| \tag{6}$$

is arbitrarily small for η sufficiently small, and the first part of the theorem follows.

The second part follows by Lemma 9.

THEOREM IV. *Let* $a > -\frac{1}{2}$, $\delta > -1$, $a \leqslant \frac{1}{2}+\delta$.

Suppose $\phi(t)$ *to be such that, for some* $h > 0$, $\phi(t)t^{-1}$ *is absolutely integrable in* (h, ∞), *and such that* $\phi(t)$ *is of bounded variation in* $(0, h)$. *Then*

$$\int_0^\infty \phi(t)\,dt \int_0^\lambda G(t^2 y^2)\left(1-\frac{y}{\lambda}\right)^\delta dy = \log\lambda\{\pi i K\phi(+0)+o(1)\} \quad \textit{as} \quad \lambda \to \infty.$$

* Hobson, *loc. cit.*, 423.

† *Loc. cit.*, 426.

If, further, $t^{2k}\phi(t)$ is absolutely integrable in (h, ∞), and, in the case $\alpha = -\frac{1}{2}$, $\phi(t)t^{-1}\log t$ is absolutely integrable in (h, ∞), then

$$\int_0^\lambda \left(1-\frac{y}{\lambda}\right)^\delta dy \int_0^\infty \phi(t)\, G(t^2y^2)\, dt = \log\lambda\{\pi i K\phi(+0)+o(1)\} \quad as \quad \lambda\to\infty.$$

It is to be noted that, in the case $\alpha < \frac{1}{2}+\delta$, this theorem is included in Theorem VI.

The proof of this theorem is similar to that of Theorem III, except that L is replaced by $\pi i K$, and the expressions on the left of equations (1), (2), (3), (4), (5), and (6) are divided by $\log\lambda$.

We now give some examples of particular cases of Theorems III and IV. In each case the result holds provided that $\phi(t)$ is of bounded variation in some neighbourhood of the origin, and satisfies certain conditions regarding integrability.

If we take

$$K(s) = \Gamma(\alpha-s)\cos\tfrac{1}{2}\pi(s-\tfrac{1}{2}) \quad (\alpha > -\tfrac{1}{2}),$$

we have

$$G(z) = 2\pi i z^\alpha \cos(z+\tfrac{1}{4}\pi-\tfrac{1}{2}\pi\alpha),$$

$$K = 0, \quad L = \tfrac{1}{4}\pi^2 i\Gamma(\alpha+\tfrac{1}{2}).$$

Thus, by Theorem III, we see that

$$\int_0^\infty y^{2\alpha}\, dy \int_0^\infty \phi(t)\, t^{2\alpha}\cos(t^2y^2+\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha\pi)\, dt$$

is summable $(C, \alpha-\frac{1}{2})$ to $\frac{1}{8}\pi\Gamma(\alpha+\frac{1}{2})\phi(+0)$.

Changing the notation, we see that, if $\alpha > -1$,

$$\int_0^\infty y^\alpha\, dy \int_0^\infty \phi(t)\, t^\alpha \cos(ty-\tfrac{1}{2}\alpha\pi)\, dt$$

is summable (C, α) to $\frac{1}{2}\pi\,\Gamma(\alpha+1)\phi(+0)$.

If we take $\alpha = 0$, we get the ordinary Fourier integral theorem.

Again, if we take $\alpha = 1$, and write the result in the form in which it is given by the first part of Theorem III, we get,

$$\lim_{\lambda\to\infty}\int_0^\infty \phi(t)\, t\, dt \int_0^\lambda y\left(1-\frac{y}{\lambda}\right)\cos(ty-\tfrac{1}{2}\pi)\, dy = \tfrac{1}{2}\pi\phi(+0).$$

When we evaluate the inner integral, this becomes

$$\lim_{\lambda\to\infty}\int_0^\infty \phi(t)\left[-\frac{1}{t}\sin\lambda t+\frac{2}{t^2\lambda}(1-\cos\lambda t)\right] dt = \tfrac{1}{2}\pi\phi(+0).$$

If we take $$K(s) = \Gamma(\alpha - s),$$

we have $$G(z) = 2\pi i z^{\alpha} e^{-z}, \quad K = \Gamma(\alpha + \tfrac{1}{2}),$$

so that, by Theorem IV, if $\alpha > -\frac{1}{2}$,

$$\int_0^{\lambda} y^{2\alpha}\, dy \int_0^{\infty} \phi(t)\, t^{2\alpha} \exp(-t^2 y^2)\, dt \sim \tfrac{1}{2}\Gamma(\alpha+\tfrac{1}{2}) \log \lambda\, \phi(+0) \quad \text{as} \quad \lambda \to \infty.$$

Changing the notation, we see that, if $\alpha > -1$,

$$\int_0^{\lambda} y^{\alpha}\, dy \int_0^{\lambda} \phi(t)\, t^{\alpha} e^{-ty}\, dt \sim \Gamma(\alpha+1) \log \lambda\, \phi(+0) \quad \text{as} \quad \lambda \to \infty.$$

We have here taken $\delta = 0$. This is legitimate, since, by the result obtained at the end of § 5, the condition $\alpha \leqslant \frac{1}{2} + \delta$ may in this case be removed.

If we take $$K(s) = \Gamma(\alpha - s) \cos \tfrac{1}{2}\pi(\alpha - s),$$

we have

$$G(z) = 2\pi i z^{\alpha} \cos z, \quad K = \Gamma(\alpha+\tfrac{1}{2}) \cos \tfrac{1}{2}\pi(\alpha+\tfrac{1}{2}),$$

so that, by Theorem IV, if $\alpha > -\frac{1}{2}$,

$$\int_0^{\lambda} y^{2\alpha} \left(1 - \frac{y}{\lambda}\right)^{\alpha - \frac{1}{2}} dy \int_0^{\infty} \phi(t)\, t^{2\alpha} \cos(t^2 y^2)\, dt$$
$$\sim \tfrac{1}{2}\Gamma(\alpha+\tfrac{1}{2}) \cos \tfrac{1}{2}\pi(\alpha+\tfrac{1}{2}) \log \lambda\, \phi(+0) \quad \text{as} \quad \lambda \to \infty.$$

Changing the notation, we see that, if $\alpha > -1$,

$$\int_0^{\lambda} y^{\alpha} \left(1 - \sqrt{\frac{y}{\lambda}}\right)^{\alpha} dy \int_0^{\infty} \phi(t)\, t^{\alpha} \cos ty\, dt$$
$$\sim \Gamma(\alpha+1) \cos \tfrac{1}{2}\pi(\alpha+1) \log \lambda\, \phi(+0) \quad \text{as} \quad \lambda \to \infty,$$

from which we may deduce that

$$\int_0^{\lambda} y^{\alpha} \left(1 - \frac{y}{\lambda}\right)^{\alpha} dy \int_0^{\infty} \phi(t)\, t^{\alpha} \cos ty\, dt$$
$$\sim \Gamma(\alpha+1) \cos \tfrac{1}{2}\pi(\alpha+1) \log \lambda\, \phi(+0) \quad \text{as} \quad \lambda \to \infty.$$

A similar result holds when the cosines are replaced by sines.

If we take

$$K(s) = -i\, \Gamma(\beta + \nu + \tfrac{1}{2} - s)\, \Gamma(\beta + \tfrac{1}{2} - s) \sin \pi(s + \tfrac{1}{2}),$$

we have

$$G(z) = \pi^2 z^{\beta+\frac{1}{2}\nu+\frac{1}{2}} [e^{\pi i\beta} H_\nu^{(2)}(2\sqrt{z}) + e^{-\pi i\beta} H_\nu^{(1)}(2\sqrt{z})],$$

and, provided that neither β nor $\beta+\nu$ is a negative integer,

$$K = 0, \quad L = \tfrac{1}{2}\pi^2 \Gamma(\beta+\nu+1)\,\Gamma(\beta+1).$$

Thus, by Theorem III, provided that $\beta+1>0$ and that $\beta+\nu+1>0$, we see that

$$\int_0^\infty y^{2\beta+\nu+1}\,dy \int_0^\infty \phi(t)\, t^{2\beta+\nu+1} [e^{\pi i\beta} H_\nu^{(2)}(2ty) + e^{-\pi i\beta} H_\nu^{(1)}(2ty)]\,dt$$

is summable $(C,\ 2\beta+\nu+\frac{1}{2})$ to $\frac{1}{2}\Gamma(\beta+\nu+1)\,\Gamma(\beta+1)\,\phi(+0)$.

If in this result we take $\nu = -\frac{1}{2}$, we get the first example.

Again, if we take $\beta = n$, where n is a positive integer or zero, provided that $n+\nu+1>0$, we see that

$$\int_0^\infty y^{2n+\nu+1}\,dy \int_0^\infty \phi(t)\, t^{2n+\nu+1} J_\nu(2ty)\,dt$$

is summable $(C,\ 2n+\nu+\frac{1}{2})$ to $(-1)^n \tfrac{1}{4}\Gamma(n+\nu+1)\,\Gamma(n+1)\,\phi(+0)$.

If we take

$$K(s) = \Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha-s)\,\Gamma(-\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha-s)\exp[-\pi i(s-\tfrac{1}{2}\alpha)],$$

we have

$$G(z) = -2\pi^2 e^{\frac{1}{2}\pi i\nu} z^{\frac{1}{2}\alpha} H_\nu^{(1)}(2\sqrt{z}),$$

$$K = i\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})\exp(\tfrac{1}{2}\pi i\alpha);$$

hence, provided that $\nu+\alpha+1>0$ and that $-\nu+\alpha+1>0$, we deduce from Theorem IV that

$$\int_0^\lambda y^\alpha \left(1-\frac{y}{\lambda}\right)^{\alpha-\frac{1}{2}} dy \int_0^\infty \phi(t)\, t^\alpha H_\nu^{(1)}(2ty)\,dt$$

$$\sim \frac{1}{2\pi}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})\exp[\tfrac{1}{2}\pi i(\alpha-\nu)]\log\lambda\,\phi(+0) \text{ as } \lambda\to\infty.$$

Similarly, under the same conditions, we have

$$\int_0^\lambda y^\alpha \left(1-\frac{y}{\lambda}\right)^{\alpha-\frac{1}{2}} dy \int_0^\infty \phi(t)\, t^\alpha H_\nu^{(2)}(2ty)\,dt$$

$$\sim \frac{1}{2\pi}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}\nu+\tfrac{1}{2}\alpha+\tfrac{1}{2})\exp[-\tfrac{1}{2}\pi i(\alpha-\nu)]\log\lambda\,\phi(+0) \text{ as } \lambda\to\infty.$$

In these two integrals the term $(1-y/\lambda)^{\alpha-\frac{1}{2}}$ must be replaced by $(1-y/\lambda)^{\delta}$, where $\delta > -1$, if $\alpha-\frac{1}{2} \leqslant -1$.

On adding the last two equations, we obtain the corresponding result for Bessel functions of the first kind. This result holds provided that $\nu+\alpha+1>0$; it is not necessary to suppose that $-\nu+\alpha+1>0$, as is seen when we obtain the result directly from Theorem IV.

7. *Proof of Theorems V and VI.*

We now require some further lemmas.

LEMMA 10. *Let* $s=\sigma+it.$

Suppose that $\eta>0, \quad \delta>-1, \quad \sigma\neq-1.$

Then, for fixed η, δ, σ,

$$\int_{\eta}^{y}\left(1-\frac{x}{y}\right)^{\delta}x^{s}\,dx=-\frac{\eta^{s+1}}{s+1}+O(y^{\sigma+1})+O(y^{-1}),$$

uniformly in t *as* $y\to\infty$.

If $\delta=0$, the result is obvious, and, if $\delta>0$, it follows at once on integrating by parts, integrating x^s.

If $\delta<0$, we divide the integral into two parts.

$$\int_{\frac{1}{2}y}^{y}\left(1-\frac{x}{y}\right)^{\delta}x^{s}\,dx=O\left[y^{\sigma}\int_{\frac{1}{2}y}^{y}\left(1-\frac{x}{y}\right)^{\delta}dx\right]=O(y^{\sigma+1}),$$

$$\int_{\eta}^{\frac{1}{2}y}\left(1-\frac{x}{y}\right)^{\delta}x^{s}\,dx=\left[\frac{x^{s+1}}{s+1}\left(1-\frac{x}{y}\right)^{\delta}\right]_{\eta}^{\frac{1}{2}y}+\frac{\delta}{s+1}\,y^{-1}\int_{\eta}^{\frac{1}{2}y}x^{s+1}\left(1-\frac{x}{y}\right)^{\delta-1}dx$$

$$=-\frac{\eta^{s+1}}{s+1}+O(y^{\sigma+1})+O(y^{-1}).$$

The result follows.

LEMMA 11. *Suppose that*

$$\delta>-\tfrac{1}{2}+\alpha, \quad \delta>-1, \quad \alpha>-\tfrac{1}{2}.$$

Then $$\lim_{y\to\infty}\int_{0}^{y}\left(1-\frac{x}{y}\right)^{\delta}G(x^{2})\,dx=\pi iK.$$

Let ρ be a complex number and write $\mu=\Re\rho$.

We suppose first that we can choose c so that

$$-\tfrac{1}{2}+\sigma(1+\alpha)<c, \tag{1}$$

$$2c<-(\mu+1). \tag{2}$$

This necessitates

$$\mu<-2\sigma(1+\alpha). \tag{3}$$

Since equation (1) is satisfied, we then have

$$\int_\eta^y \left(1-\frac{x}{y}\right)^\delta G(x^2)\,x^\rho\,dx=\int_\eta^y \left(1-\frac{x}{y}\right)^\delta x^\rho\,dx\int_{c-i\infty}^{c+i\infty} K(s)x^{2s}\,ds.$$

The double integral is absolutely convergent. We may, therefore, invert the order of integration, and we then obtain

$$\int_{c-i\infty}^{c+i\infty} K(s)\,ds\int_\eta^y x^{2s+\rho}\left(1-\frac{x}{y}\right)^\delta dx.$$

By Lemma 10 and equation (2) we see that

$$\lim_{y\to\infty}\int_\eta^y \left(1-\frac{x}{y}\right)^\delta G(x^2)\,x^\rho\,dx=-\int_{c-i\infty}^{c+i\infty} K(s)(2s+\rho+1)^{-1}\eta^{2s+\rho+1}\,ds. \tag{4}$$

The contours in these integrals are bent, if necessary, so that the poles of the integrand lie to the right of them.

Now the integral on the left of (4) is evidently an analytic function of ρ. Further, by Lemma 8, it tends to a limit as $y\to\infty$, uniformly for all ρ in any finite domain contained in

$$\mu\leqslant\text{constant}<-\sigma(2\alpha-2\delta-1).$$

Hence, by analytic continuation, the equation (4) holds for all ρ satisfying

$$\mu<-\sigma(2\alpha-2\delta-1)$$

(provided, of course, that c is chosen so that the integral on the right exists).

Now suppose that

$$\alpha+\tfrac{1}{2}\rho>-\tfrac{1}{2}.$$

Then the integral on the right of (4) is equal to

$$\pi iK(-\tfrac{1}{2}-\tfrac{1}{2}\rho)-\int_{c-i\infty}^{c+i\infty} K(s)(2s+\rho+1)^{-1}\eta^{2s+\rho+1}\,ds,$$

where $s = -\frac{1}{2} - \frac{1}{2}\rho$ now lies on the left of the contour, but the remaining poles of the integrand continue to lie on the right of it.

We can arrange the contour so that

$$\Re(2s+\rho+1) \geqslant d > 0$$

at all points on the contour.

We see, then, that the integral is $O(\eta^d) = o(1)$ as $\eta \to 0$.

Hence, by making $\eta \to 0$, and then taking the special case $\rho = 0$, the result follows.

LEMMA 12. *Suppose that* $a > -\frac{1}{2}$ *and that*

$$R_\beta(y) = \frac{1}{\Gamma(\beta+1)} \int_0^y (y-x)^\beta G(x^2)\, dx \quad (\beta > -1).$$

Evidently $\dfrac{dR_{\beta+1}(y)}{dy} = R_\beta(y)$; *we define* $R_\beta(y)$ *for* $\beta \leqslant -1$ *by means of this relation.*

Then

(1) *For y large,*

$$\Gamma(\beta+1)\, R_\beta(y) = \pi i K y^\beta + O(y^{\beta+\sigma(-1+2a-2\beta)}) + O(y^{\beta-1}).$$

(2) *For y small,*

$$R_\beta(y) = O(y^{\beta+1+2a} |\log y|^{p-1}).$$

(3) *If* $\beta > 0$, $\delta > -1$, *then*

$$R_{\beta+\delta}(y) = \frac{1}{\Gamma(\beta)} \int_0^y (y-x)^{\beta-1} R_\delta(x)\, dx.$$

(4) *If* $\beta > 0$, $\delta > -2$, *then*

$$R_{\beta+\delta}(y) = \frac{1}{\Gamma(\beta-1)} \int_0^{\frac{1}{2}y} (y-x)^{\beta-2} R_{\delta+1}(x)\, dx + \frac{1}{\Gamma(\beta)} \int_{\frac{1}{2}y}^y (y-x)^{\beta-1} R_\delta(x)\, dx + \frac{1}{\Gamma(\beta)} (\tfrac{1}{2}y)^{\beta-1} R_{\delta+1}(\tfrac{1}{2}y).$$

(1) If $\beta > -1$, the result follows at once from Lemma 8 and Lemma 11.

Now suppose that $-1 \geqslant \beta > -2$. Then

$$R_{\beta+1}(y) = \frac{y^{\beta+2}}{\Gamma(\beta+2)} \int_0^1 (1-x)^{\beta+1} G(x^2 y^2)\, dx.$$

Hence

$$R_\beta(y) = \frac{(\beta+2)\,y^{\beta+1}}{\Gamma(\beta+2)} \int_0^1 (1-x)^{\beta+1} G(x^2y^2)\,dx$$
$$+ \frac{y^{\beta+2}}{\Gamma(\beta+2)} \int_0^1 (1-x)^{\beta+1} \left(\frac{d}{dy} G(x^2y^2)\right) dx$$
$$= I_1 + I_2, \quad \text{say.}$$

$$I_1 = (\beta+2)\,\frac{R_{\beta+1}(y)}{y} = \frac{\beta+2}{\Gamma(\beta+2)}\,\pi i y^\beta K + O(y^{\beta+\sigma(-3+2\alpha-2\beta)}) + O(y^{\beta-1}),$$

by the result for $\beta+1$.

Now

$$\frac{d}{dy} G(x^2y^2) = -\frac{d}{dy} \int_C K(s)\,x^{2s}y^{2s}\,ds$$
$$= -y^{-1} \int_C 2sK(s)\,x^{2s}y^{2s}\,ds$$
$$= -y^{-1} \left(2\alpha_r \int_C K(s)\,x^{2s}y^{2s}\,ds - 2\int_C (\alpha_r - s)\,K(s)\,x^{2s}y^{2s}\,ds\right)$$
$$= y^{-1}\{2\alpha_r G(x^2y^2) - 2G'(x^2y^2)\},$$

where G' is the function obtained by replacing $\Gamma(\alpha_r - s)$ by $\Gamma(\alpha_r+1-s)$ in the expression defining $K(s)$; it is thus a function of the same type as G, but the value of α corresponding to it is greater by 1 than that corresponding to G.

Thus

$$I_2 = \frac{y^{\beta+1}}{\Gamma(\beta+2)} \int_0^1 (1-x)^{\beta+1} \{2\alpha_r G(x^2y^2) - 2G'(x^2y^2)\}\,dx$$
$$= \frac{\pi i y^\beta}{\Gamma(\beta+2)} \{2\alpha_r K - 2(\alpha_r + \tfrac{1}{2})\,K\} + O(y^{\beta+\sigma(-1+2\alpha-2\beta)}) + O(y^{\beta-1})$$

by the result for $\beta+1$.

The result follows from the expressions for I_1 and I_2.

By repeating the process we may obtain the result for any negative β.

(2) If $\beta > -1$, $R_\beta(y)$ is the sum of an infinite series of terms of the form

$$cy^{\beta+1} \int_0^1 (1-x)^\beta (xy)^d |\log(xy)|^n\,dx,$$

i.e. of the form $$cy^{\beta+1+d} |\log y|^n,$$

where the c's are (different) constants.

The term which is of the highest order for y small is given by $d = 2a$, $n = p-1$; hence the result follows at once for $\beta > -1$; the result for $\beta < -1$ follows by repeated differentiation.

(3) We substitute for $R_\delta(x)$ the integral defining it, and invert the order of integration; the result follows.

(4) By (3),

$$\Gamma(\beta) R_{\beta+\delta+1}(y) = y^\beta \int_0^1 (1-x)^{\beta-1} R_{\delta+1}(xy)\, dx$$

$$= (\beta-1)\, y^{\beta-1} \int_0^{\frac{1}{2}} (1-x)^{\beta-2} R_{\delta+2}(xy)\, dx$$

$$+ y^\beta \int_{\frac{1}{2}}^1 (1-x)^{\beta-1} R_{\delta+1}(xy)\, dx + (\tfrac{1}{2}y)^{\beta-1} R_{\delta+2}(\tfrac{1}{2}y),$$

by a partial integration of the portion of the integral from 0 to $\frac{1}{2}$.

Differentiating with respect to y, we have

$$\Gamma(\beta)\, R_{\beta+\delta}(y) = (\beta-1)^2 y^{\beta-2} \int_0^{\frac{1}{2}} (1-x)^{\beta-2} R_{\delta+2}(xy)\, dx$$

$$+ (\beta-1)\, y^{\beta-1} \int_0^{\frac{1}{2}} (1-x)^{\beta-2} x R_{\delta+1}(xy)\, dx$$

$$+ \beta y^{\beta-1} \int_{\frac{1}{2}}^1 (1-x)^{\beta-1} R_{\delta+1}(xy)\, dx$$

$$+ y^\beta \int_{\frac{1}{2}}^1 (1-x)^{\beta-1} x R_\delta(xy)\, dx$$

$$+ \tfrac{1}{2}(\beta-1)(\tfrac{1}{2}y)^{\beta-2} R_{\delta+2}(\tfrac{1}{2}y) + \tfrac{1}{2}(\tfrac{1}{2}y)^{\beta-1} R_{\delta+1}(\tfrac{1}{2}y).$$

Integrate the first and third integrals by parts, integrating $(1-x)^{\beta-2}$ and $(1-x)^{\beta-1}$ respectively, and the result follows.

THEOREM V. *Let*

$$K = 0; \quad a > -\tfrac{1}{2}; \quad \beta \geqq 0; \quad \delta > -\tfrac{1}{2} + a; \quad \beta + \delta > -1.$$

Suppose $\phi(t)\, t^{-1}$ *to be absolutely integrable in* $(\frac{1}{2}, \infty)$, *and*

$$\phi(t)\, t^{2a} |\log t|^{p-1}$$

to be integrable in $(0, \frac{1}{2})$. *If* $a > 0$, *suppose* $\phi(t)$ *to be integrable in* $(0, 1)$.

Suppose $\phi(t)$ *to be summable* (C, β) *to* s *as* $t \to 0$.

Then

$$\lim_{\lambda\to\infty}\int_0^\infty \phi(t)\,dt\int_0^\lambda G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^{\beta+\delta}dy = Ls.$$

If, further, $t^{2k}\phi(t)$ *is absolutely integrable in* $(1,\infty)$ *and, in the case* $\alpha=-\frac{1}{2}$, $\phi(t)t^{-1}\log t$ *is absolutely integrable in* $(1,\infty)$, *then*

$$\int_0^\infty dy\int_0^\infty \phi(t)\,G(t^2y^2)\,dt$$

is summable $(C,\beta+\delta)$ *to* Ls.

It is necessary to prove only the first part; the second part then follows at once by Lemma 9.

Let $$\phi_r(t)=\frac{1}{\Gamma(r)}\int_0^t\phi(u)(t-u)^{r-1}du.$$

Let k be the integer satisfying

$$k+1>\beta\geqslant k.$$

We first show that it is sufficient to prove the theorem for the special case in which

$$s=0.$$

We have $$\phi_{k+1}(t)=\phi_k(t)=\ldots=\phi_1(t)=\phi(t)=0$$

for all sufficiently large t.

For let $\Phi(t)$ be such that

$$\Phi(t)=\phi(t)-s \quad \text{for} \quad t\leqslant\eta,$$

where η is any positive constant; then

$$\Phi_{k+1}(t)=0$$

for t sufficiently large, and $\Phi(t)$ is integrable in any finite range.

Let $$\psi(t)=\phi(t)-\Phi(t).$$

Since $\psi(t)=s$ in the neighbourhood of the origin, the result for $\psi(t)$ follows by Theorem III, so that it is sufficient to prove the theorem for $\Phi(t)$, which has the properties indicated.

It follows that all the apparently infinite integrals which we use are really only finite, so that all the inversions in the order of integration are

permissible. We choose b, d, so that

$$b+d=\beta+\delta,$$

$$k+1>b>\beta,$$

$$\delta>d>-\tfrac{1}{2}+\alpha.$$

Case 1. $d>-1$. Then

$$\int_0^\infty \phi(t)\,dt\int_0^\lambda G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^{\beta+\delta}dy$$

$$=\Gamma(\beta+\delta+1)\,\lambda\int_0^\infty \phi(t)\,R_{\beta+\delta}(\lambda t)\,(\lambda t)^{-(\beta+\delta+1)}\,dt$$

$$=\frac{\Gamma(\beta+\delta+1)}{\Gamma(b)}\,\lambda^{-d}\int_0^\infty \phi(t)\,t^{-(\beta+\delta+1)}\,dt\int_0^t (t-y)^{b-1}\,R_d(y\lambda)\,dy$$

[by Lemma 12 (3)]

$$=\frac{\Gamma(\beta+\delta+1)}{\Gamma(b)}\,\lambda^{-d}\int_0^\infty R_d(y\lambda)\,y^{1-d}\,\chi(y)\,dy,$$

where $$\chi(y)=y^{1+d}\int_y^\infty (t-y)^{b-1}\,\phi(t)\,t^{-(b+d+1)}\,dt.$$

Now, by Lemma 12 (1) and (2), $R_d(y)\,y^{1-d}$ is absolutely integrable in $(0,\infty)$. It has been proved by Paley* that $\chi(y)\to 0$ as $y\to 0$, and $\chi(y)$ is evidently bounded; the result follows†.

Case 2. $d\leqslant -1$.

If $a\leqslant 0$,

$$|\phi_1(u)|\leqslant u^{-2a}|\log u|^{-(p-1)}\int_0^u |\phi(t)|\,t^{2a}|\log t|^{p-1}\,dt$$

$$=o(u^{-2a}|\log u|^{-(p-1)})\quad\text{as}\quad u\to 0,$$

since $\phi(t)\,t^{2a}(\log t)^{p-1}$ is integrable in $(0,1)$.

Hence $$\phi_2(u)=o\int_0^u (t^{-2a}|\log t|^{-(p-1)})\,dt$$

$$=o(t^{1-2a}|\log t|^{-(p-1)}),$$

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* Paley, *loc. cit.*, 181–183.

† Paley, *loc. cit.*, Lemma 2.2, p. 179.

and, more generally,

$$\phi_r(u) = o(t^{r-1-2a}|\log t|^{-(p-1)}) \quad \text{as} \quad t \to 0. \tag{1}$$

This result continues to hold when $a > 0$; indeed, since we are supposing $\phi(t)$ integrable, we have the stronger result

$$\phi_r(u) = o(t^{r-1}).$$

We may evidently suppose that $-d$ is not an integer.

Let $$-n > d > -(n+1).$$

Then, since $$b+d = \beta+\delta > -1,$$

we must have $$b > n-1.$$

We consider first the case $b > n$.

We adopt the notation

$$R_\beta(u)\, u^{-1-\beta} = \gamma_\beta(u),$$

$$\frac{d^m}{du^m}\gamma_\beta(u) = \gamma_\beta^{(m)}(u).$$

Then

$$\int_0^\infty \phi(t)\,dt \int_0^\lambda G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^{\beta+\delta} dy = \Gamma(\beta+\delta+1)\,\lambda \int_0^\infty \phi(t)\,\gamma_{\beta+\delta}(\lambda t)\,dt. \tag{2}$$

Now

$$\int_\eta^\infty \phi(t)\,\gamma_{\beta+\delta}(\lambda t)\,dt = -\phi_1(\eta)\,\gamma_{\beta+\delta}(\lambda\eta) - \lambda \int_\eta^\infty \phi_1(t)\,\gamma_{\beta+\delta}^{(1)}(\lambda t)\,dt.$$

By Lemma 12 (2) and equation (1),

$$\phi_1(\eta)\,\gamma_{\beta+\delta}(\lambda\eta) \to 0 \quad \text{as} \quad \eta \to 0.$$

Thus

$$\int_0^\infty \phi(t)\,\gamma_{\beta+\delta}(\lambda t)\,dt = \lim_{\eta\to 0}\left(-\lambda \int_\eta^\infty \phi_1(t)\,\gamma_{\beta+\delta}^{(1)}(\lambda t)\,dt\right).$$

Repeating the process, we see that the expression on the left of equation (2) is equal to

$$\lim_{\eta\to 0}\left((-1)^n\,\Gamma(\beta+\delta+1)\,\lambda^{n+1} \int_\eta^\infty \phi_n(t)\,\gamma_{\beta+\delta}^{(n)}(\lambda t)\,dt\right). \tag{3}$$

Now, by Lemma 12 (3),

$$\gamma_{\beta+\delta}(\lambda t) = \frac{1}{\Gamma(b-n)}\, t^{-(\beta+\delta+1)} \int_0^t (t-y)^{b-n-1} y^{1+n+d}\, \gamma_{n+d}(y\lambda)\, dy.$$

Differentiating n times with respect to λ, we see that

$$\gamma^{(n)}_{\beta+\delta}(\lambda t) = \frac{1}{\Gamma(b-n)}\, t^{-(\beta+\delta+n+1)} \int_0^t (t-y)^{b-n-1} y^{1+2n+d}\, \gamma^{(n)}_{n+d}(y\lambda)\, dy.$$

Substituting this result, we see that the expression (3) becomes

$$\lim_{\eta\to 0}\Big((-1)^n \frac{\Gamma(\beta+\delta+1)}{\Gamma(b-n)}\, \lambda^{n+1} \int_\eta^\infty \phi_n(t)\, t^{-(\beta+\delta+n+1)}\, dt \times \int_0^t (t-y)^{b-n-1} y^{1+2n+d}\, \gamma^{(n)}_{n+d}(y\lambda)\, dy\Big)$$

$$= \lim_{\eta\to 0} (-1)^n \frac{\Gamma(\beta+\delta+1)}{\Gamma(b-n)} \Big(\lambda^{n+1} \int_0^\infty \gamma^{(n)}_{n+d}(y\lambda)\, y^n \chi(y)\, dy - \lambda^{n+1} \int_0^\eta \gamma^{(n)}_{n+d}(y\lambda)\, y^{1+2n+d}\, dy \int_y^\eta \phi_n(t)\, t^{-(\beta+\delta+n+1)} (t-y)^{b-n-1}\, dt\Big)$$

$$= (-1)^n \frac{\Gamma(\beta+\delta+1)}{\Gamma(b-n)}\, (I_1 + I_2) \quad \text{(say)},$$

where $$\chi(y) = y^{1+n+d} \int_y^\infty \phi_n(t)\, t^{-(\beta+\delta+n+1)} (t-y)^{b-n-1}\, dt$$

(provided, of course, that these integrals exist).

Now $\gamma^{(n)}_{n+d}(y)\, y^n$ is [by Lemma 12 (1) and (2)] absolutely integrable in $(0, \infty)$, and, by employing Paley's argument again, we see that $\chi(y) \to 0$ as $y \to 0$. It follows that I_1 exists, and tends to 0 as $\lambda \to 0$.

It thus only remains to prove that, for fixed λ, $I_2 \to 0$ as $\eta \to 0$.

Integrating by parts $k-n$ times and using the result (1), we see that

$$(4) \quad \int_y^\eta \phi_n(t)\, t^{-(\beta+\delta+n+1)} (t-y)^{b-n-1}\, dt = o\{\eta^{-2\alpha-\beta-\delta-2+k-n} |\log \eta|^{-(p-1)} (\eta-y)^{b-k}\} + (-1)^{k-n} \int_y^\eta \phi_k(t)\, \frac{d^{k-n}}{dt^{k-n}} \{t^{-(\beta+\delta+n+1)} (t-y)^{b-n-1}\}\, dt.$$

Integrating by parts again, we see that the integral on the right of (4) is equal to

$$(5)\quad \left[\int_y^t \phi_k(u)\,du\,\frac{d^{k-n}}{dt^{k-n}}\left\{t^{-(\beta+\delta+n+1)}(t-y)^{b-n-1}\right\}\right]_{t=y}^{t=\eta} - \int_y^\eta \left(\int_y^t \phi_k(u)\,du\right)\frac{d^{k-n+1}}{dt^{k-n+1}}\left\{t^{-(\beta+\delta+n+1)}(t-y)^{b-n-1}\right\}dt,$$

provided that this exists.

Now* $$\int_y^t \phi_k(u)\,du = o\{t^\beta(t-y)^{k+1-\beta}\}.$$

Hence the expression (5) exists and is

$$o\{\eta^{-\delta-n-1}(\eta-y)^{b-\beta}\}+o\int_y^\eta t^{-\delta-n-1}(t-y)^{b-\beta-1}\,dy = o(y^{-\delta-n-1}\eta^{b-\beta}).$$

Thus, from equation (4),

$$\int_y^\eta \phi_n(t)\,t^{-(\beta+\delta+n+1)}\,dt = o(\eta^{-2a-d-2-n}|\log\eta|^{-(p-1)})+o(y^{-\delta-n-1}\eta^{b-\beta}).$$

Hence, using Lemma 12 (2), we have

$$I_2 = o\left(\eta^{-2a-d-2-n}|\log\eta|^{-(p-1)}\int_0^\eta y^{2a+1+n+d}|\log y|^{p-1}\,dy + \eta^{b-\beta}\int_0^\eta y^{2a+d-\delta}|\log y|^{p-1}\,dy\right).$$

We may evidently suppose that $2a+d-\delta > -1$, and we therefore have

$$I_2 \to 0 \quad \text{as} \quad \eta \to 0.$$

The result follows.

Now let $n > b > n-1$.

We have, as before,

$$\int_0^\infty \phi(t)\,dt\int_0^\lambda G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^{\beta+\delta}dy = \lim_{\eta\to 0}\left\{(-1)^{n-1}\Gamma(\beta+\delta+1)\,\lambda^n\int_\eta^\infty \phi_{n-1}(t)\,\gamma_{\beta+\delta}^{(n-1)}(\lambda t)\,dt\right\}.$$

* Paley, *loc. cit.*, 183 [equation (3.4)]; if in Paley's equation ξ^a is replaced by η^a it is unnecessary to take $\eta \leqslant 2\xi$.

As before, but with the use of Lemma 12 (4) in place of Lemma 12 (3), we see that this is equal to

$$\lim_{\eta\to 0}\left\{(-1)^{n-1}\frac{\Gamma(\beta+\delta+1)}{\Gamma(b-n+1)}\left[(b-n)I_1+I_2+2^{-(\beta+\delta+n)}I_3\right]\right\},$$

where

$$I_1=\lambda^n\int_\eta^\infty \phi_{n-1}(t)\,t^{-(\beta+\delta+n)}\,dt\int_0^{\frac{1}{2}t}(t-y)^{b-n-1}y^{d+2n}\gamma_{d+n}^{(n-1)}(y\lambda)\,dy,$$

$$I_2=\lambda^n\int_\eta^\infty \phi_{n-1}(t)\,t^{-(\beta+\delta+n)}\,dt\int_{\frac{1}{2}t}^{t}(t-y)^{b-n}y^{d+2n-1}\gamma_{d+n-1}^{(n-1)}(y\lambda)\,dy,$$

$$I_3=\lambda^n\int_\eta^\infty \phi_{n-1}(t)\,\gamma_{d+n}^{(n-1)}(\tfrac{1}{2}\lambda t)\,dt.$$

Now

$$I_1=\lambda^n\int_0^\infty \gamma_{n+d}^{(n-1)}(y\lambda)\,y^{n-1}\chi_1(y)\,dy$$

$$-\lambda^n\int_0^{\frac{1}{2}\eta}\gamma_{n+d}^{(n-1)}(y\lambda)\,y^{d+2n}\,dy\int_{2y}^{\eta}\phi_{n-1}(t)\,t^{-(\beta+\delta+n)}(t-y)^{b-n-1}\,dt$$

$$=I_{11}-I_{12}\quad\text{(say)},$$

where $$\chi_1(y)=y^{d+n+1}\int_{2y}^\infty \phi_{n-1}(t)(t-y)^{b-n-1}t^{-(\beta+\delta+n)}\,dt,$$

provided that these integrals exist.

Now $y^{n-1}\gamma_{n+d}^{(n-1)}(y)$ is absolutely integrable in $(0,\infty)$, and $\chi_1(y)\to 0$ as $y\to 0$, as is seen on integrating by parts, integrating $\phi_{n-1}(t)$, and using the result $\phi_n(t)=o(t^n)$, which holds because $n>\beta$. Thus I_{11} exists and tends to zero as $\lambda\to\infty$.

Again, on integrating by parts, we see that

$$\int_{2y}^{\eta}\phi_{n-1}(t)\,t^{-(\beta+\delta+n)}(t-y)^{b-n-1}\,dt=o(y^{-(d+n+1)}),$$

so that, for fixed λ, $I_{12}\to 0$ as $\eta\to 0$.

$$I_2=\lambda^n\int_{\frac{1}{2}\eta}^\infty \gamma_{d+n-1}^{(n-1)}(y\lambda)\,y^{n-1}\chi_2(y)\,dy$$

$$-\lambda^n\int_{\frac{1}{2}\eta}^{\eta}\gamma_{d+n-1}^{(n-1)}(y\lambda)\,y^{d+2n-1}\,dy\int_y^\eta\phi_{n-1}(t)(t-y)^{b-n}t^{-(\beta+\delta+n)}\,dt$$

$$=I_{21}-I_{22}\quad\text{(say)},$$

where $$\chi_2(y)=y^{d+n}\int_y^{2y}\phi_{n-1}(t)(t-y)^{b-n}t^{-(\beta+\delta+n)}\,dt.$$

Now $y^{n-1}\gamma_{d+n-1}^{(n-1)}(y)$ is absolutely integrable in $(0, \infty)$ and $\chi_2(y) \to 0$ as $y \to 0$, as may be shown by integrating by parts and using the result

$$\int_y^t \phi_{n-1}(u)\,du = o\{y^\beta (t-y)^{n-\beta}\}.$$

Hence $I_{21} \to 0$ uniformly in η as $\lambda \to \infty$.

Again,

$$\int_y^\eta \phi_{n-1}(t)(t-y)^{b-n} t^{-(\beta+\delta+n)}\,dt = o\{y^{-\delta-n}(\eta-y)^{b-\beta}\},$$

so that, for fixed λ,

$$I_{22} \to 0 \quad \text{as} \quad \eta \to 0.$$

Again,

$$I_3 = -\lambda^n \phi_n(\eta)\,\gamma_{d+n}^{(n-1)}(\tfrac{1}{2}\lambda\eta) - \tfrac{1}{2}\lambda^{n+1}\int_\eta^\infty \phi_n(t)\,\gamma_{d+n}^{(n)}(\tfrac{1}{2}\lambda t)\,dt.$$

Since $\phi_n(t) = o(t^n)$ as $t \to 0$, it follows that

$$\begin{aligned}\lim_{\eta\to 0} I_3 &= -\tfrac{1}{2}\lambda^{n+1}\int_0^\infty \phi_n(t)\,\gamma_{d+n}^{(n)}(\tfrac{1}{2}\lambda t)\,dt \\ &= o\left(\lambda^{n+1}\int_0^{1/\lambda} t^n(\lambda t)^{2a-e-n}\,dt + \lambda^{n+1}\int_{1/\lambda}^\infty t^n(\lambda t)^{-n-1+\sigma(-1+2a-2d)}\,dt\right) \\ &\to 0 \text{ as } \lambda \to \infty.\end{aligned}$$

The theorem follows.

THEOREM VI. *Let* $a > -\frac{1}{2}$, $\beta \geqslant 0$, $\delta > -\frac{1}{2}+a$, $\beta+\delta > -1$. *Suppose* $\phi(t)t^{-1}$ *to be absolutely integrable in* $(\frac{1}{2}, \infty)$, *and*

$$\phi(t)\,t^{2a}|\log t|^{p-1}$$

to be integrable in $(0, \frac{1}{2})$. *If* $a > 0$, *suppose* $\phi(t)$ *to be integrable in* $(0, 1)$.

Suppose $\phi(t)$ *to be summable* (C, β) *to* s *as* $t \to 0$.

Then

$$\int_0^\infty \phi(t)\,dt \int_0^\lambda G(t^2y^2)\left(1-\frac{y}{\lambda}\right)^{\beta+\delta} dy = \log\lambda\{\pi i K s + o(1)\} \quad \text{as} \quad \lambda \to \infty.$$

If, further, $t^{2k}\phi(t)$ *is absolutely integrable in* $(1, \infty)$, *and, in the case* $a = -\frac{1}{2}$, $t^{-1}\phi(t)\log t$ *is absolutely integrable in* $(1, \infty)$, *then*

$$\int_0^\lambda \left(1-\frac{y}{\lambda}\right)^{\beta+\delta} dy \int_0^\infty \phi(t)\,G(t^2y^2)\,dy = \log\lambda\{\pi i K s + o(1)\} \text{ as } \lambda \to \infty.$$

It follows from Theorem IV, in the same way as the corresponding result in the proof of Theorem V follows from Theorem III, that it is sufficient to prove that

$$I(\lambda)\equiv\int_0^\infty \phi(t)\,dt\int_0^\lambda G(t^2y^2)\Big(1-\frac{y}{\lambda}\Big)^{\beta+\delta}dy=o(\log\lambda)\quad\text{as}\quad\lambda\to\infty,$$

where $$\phi_{k+1}(t)=\phi_k(t)=\ldots=\phi_1(t)=\phi(t)=0$$

for all sufficiently large t, and where

$$\phi(t)\to 0\,(C,\beta)\quad\text{as}\quad t\to 0.$$

We define b, d as in Theorem V.

First let $d>-1$.

As in Theorem V,

$$I(\lambda)=c\lambda^{-d}\int_0^\infty R_d(y\lambda)\,y^{1-d}\chi(y)\,dy,$$

where c is a constant, and $\chi(y)$ is bounded, is zero for all sufficiently large y, and tends to zero as $y\to 0$.

By Lemma 12 (2),

$$\lambda^{-d}\int_0^{1/\lambda}R_d(y\lambda)\,y^{1-d}\chi(y)\,dy=o\Big(\lambda^{-d}\int_0^{1/\lambda}(y\lambda)^{d+1+2a-\epsilon}y^{1-d}\,dy\Big)$$

$$\to 0\quad\text{as}\quad\lambda\to\infty.$$

By Lemma 12 (1),

$$\lambda^{-d}\int_{1/\lambda}^\infty R_d(y\lambda)\,y^{1-d}\chi(y)\,dy=O\Big(\int_{1/\lambda}^\infty y^{-1}|\chi(y)|\,dy\Big)$$

$$=o(\log\lambda)\quad\text{as}\quad\lambda\to\infty,$$

and the result follows.

The case $d\leqslant -1$ is dealt with in the same way as in Theorem V.

8. *The absence of a converse to Theorems V and VI.*

We now show that there are functions $G(z)$ of the type considered for which no sort of converse to Theorems V and VI is true. That is to say that, given β, there is no δ depending on β such that the following theorem is true.

If (a) the function $\phi(t)$ satisfies those conditions of Theorem V (or, what is the same thing, Theorem VI) which concern integrability, and either (b) $K=0$ and

$$\int_0^\infty dy\int_0^\infty \phi(t)\,G(t^2y^2)\,dt$$

is summable (C, β) *to* Ls, *or* (*c*) $K \neq 0$ *and*

$$\int_0^\lambda \left(1-\frac{y}{\lambda}\right)^\beta dy \int_0^\infty \phi(t)\, G(t^2y^2)\, dt = \log\lambda\{\pi i K s + o(1)\}.$$

Then $\phi(t)$ *is summable* (C, δ) *to* s *as* $t \to 0$.

That no such theorem is true is obvious in the case

$$G(z) = \sum_r c_r\, Y(a_1, \ldots, a_\nu;\ \mu_r;\ z),$$

where $|\mu_r| < \frac{1}{2}\nu$ for all r.

For, if $\phi(t)$ is any function satisfying condition (*a*), and such that

$$\phi(t) \to s\ (C, \delta') \quad \text{as} \quad t \to 0$$

for any δ', however large, it follows, from Theorem V (if $K = 0$) or Theorem VI (if $K \neq 0$), and the result obtained at the end of § 5, that condition (*b*) or (*c*) must be satisfied for any $\beta > -1$. Hence, if such a theorem was true, the condition

$$\phi(t) \to s \quad (C, \delta')$$

would imply that

$$\phi(t) \to s \quad (C, \delta).$$

Even if we suppose that

$$G(z) = \sum_r c_r\, Y(a_1, \ldots, a_\nu;\ \mu_r;\ z),$$

where, for one or more values of r, the sum contains terms such that

$$\mu_r = \pm\tfrac{1}{2}\nu,$$

we can find functions $G(z)$ for which no such theorem is true.

This can be seen by considering the function

$$\phi(t) = \begin{cases} t^{-\rho} & (t < 1) \\ 0 & (t > 1), \end{cases}$$

where $\rho > 0$. It is evident that

$$\phi(t) \to \infty \quad \text{as} \quad t \to 0.$$

but, nevertheless, if ρ and $G(z)$ are suitably chosen, the integral

$$\int_0^\infty dy \int_0^\infty \phi(t)\, G(t^2y^2)\, dt$$

converges; in other words

$$\int_0^1 t^{-\rho}\,dt \int_0^\lambda G(t^2 y^2)\,dy$$

tends to a finite limit as $\lambda \to \infty$.

For, provided that $2\alpha - \rho > -1$ and that $\alpha < 1$, it can be shown, by a method similar to the proof of Lemma 6, that the last integral is equal to

$$\frac{\pi i}{\rho}\,[\lambda^\rho K(-\tfrac{1}{2}+\tfrac{1}{2}\rho) - K] + o(1) \quad \text{as} \quad \lambda \to \infty.$$

Thus, if we choose $G(z)$ so that $K(-\frac{1}{2}+\frac{1}{2}\rho) = 0$, we have an example of the kind required.

We note that, if $G(z)$ is a single divisor function, *i.e.* if

$$G(z) = Y(a_1, \ldots, a_\nu;\ \mu\ ;\ z),$$

the condition $K(-\frac{1}{2}+\frac{1}{2}\rho) = 0$ cannot be satisfied; for we then have

$$K(s) = \Gamma(a_1 - s) \ldots \Gamma(a_\nu - s)\, e^{u\pi i s}.$$

We can, however, evidently arrange that $K(-\frac{1}{2}+\frac{1}{2}\rho) = 0$ by taking $G(z)$ to be the sum of a suitable combination of divisor functions.

ON REGULAR LOGARITHMIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Introduction.

Let $x = 0$ be a typical isolated singular point of the equation

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p_n(x)\,y = 0. \tag{1}$$

If the coefficients are single-valued functions of the complex variable x in the vicinity of $x = 0$, there are canonical fundamental systems of solutions near $x = 0$ whose members are arranged in "Hamburger sets". If the solutions of such a set are analytically continued round a simple circuit about the origin, the new forms are connected with the old by relations of the type

$$\Phi_1(x) = \lambda'\phi_1(x), \quad \Phi_2(x) = \lambda'[\phi_1(x)+\phi_2(x)],$$

$$\Phi_3(x) = \lambda'[\phi_2(x)+\phi_3(x)], \quad \ldots, \quad \Phi_e(x) = \lambda'[\phi_{e-1}(x)+\phi_e(x)], \tag{2}$$

where $(\lambda-\lambda')^e$ is an invariant factor of the "characteristic determinant" $\Delta(\lambda)$, of order n. It can be proved that the last function is of the form

$$\phi_e(x) \equiv x^{\rho'}\left[\sum_{i=1}^{e} f_i(x)\,(\log x)^{e-i}\right], \tag{3}$$

where $2\pi i\rho' = \log\lambda'$, and where $\{f_i(x)\}$ are single valued near $x = 0$. The other functions of the set are seen from (2) to be the successive differences of $\phi_e(x)$, with respect to $(\log x)/2\pi i$, regarded as a variable independent of x.

In general, the uniform functions $\{f_i(x)\}$ are Laurent series; but in special cases finite values can be assigned to (ρ') which render all the auxiliary functions holomorphic. The singularity is then said to be "regular", and (by Fuchs's fundamental theorem) a necessary and

sufficient condition for this is that it should be possible to write the equation in the form

$$x^n \frac{d^n y}{dx^n} + P_1(x) x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n(x) y = 0, \tag{4}$$

where the functions $\{P_i(x)\}$ are holomorphic at $x = 0$.

The appropriate values of (ρ') are roots of the "indicial equation", of order n; after solving this, we can find as many terms as we please of the series by elementary methods. When, however, the indicial equation has multiple roots, or roots differing by integers, these calculations are very laborious, because we do not at first know the structure of the Hamburger canonical sets. Leaving aside methods only suitable for proving existence theorems, we may follow the procedure worked out by Heffter, which consists in finding in turn all solutions free from $\log x$, then those of the first degree, and so on. In the first part of this paper, we suggest an alternative arrangement of this analysis, which appears simpler and more direct in the critical stages, before the structure of the solution has been elucidated.

In the second part of the paper, we re-examine the beautiful method of Frobenius and suggest an amendment which appears to render it easier, and to obviate the necessity for the adjustments hitherto required when applying the method to any special case.

Direct method.

Let us put $$t \equiv \log x, \quad \delta \equiv \frac{d}{dt} \equiv x \frac{d}{dx}. \tag{5}$$

Using the identity

$$x^m \frac{d^m y}{dx^m} \equiv \delta(\delta - 1) \ldots (\delta - m + 1) y,$$

we may rearrange (4) in the form

$$Fy \equiv F(x, \delta) y \equiv \delta^n y + Q_1(x) \delta^{n-1} y + \ldots + Q_n(x) y = 0, \tag{6}$$

where $\{Q_i(x)\}$ are holomorphic at $x = 0$.

We now propose to group together all the solutions belonging to Hamburger sets which have the same multiplier λ'. The most general solution built up from these will be of the form

$$y = x^\rho \sum_{i=0}^{\infty} u_i x^i, \tag{7}$$

where (u_i) are polynomials in $t = \log x$, with constant coefficients, whose degree cannot exceed $n-1$; we suppose further that u_0 does not vanish identically. If we have

$$Q_i(x) = \sum_{j=0}^{\infty} Q_{ij} x^j, \tag{8}$$

and if we write

$$\left.\begin{aligned} F_0(\delta) &\equiv \delta^n + Q_{10}\delta^{n-1} + Q_{20}\delta^{n-2} + \ldots + Q_{n0}, \\ \text{and, if } j > 0, \\ F_j(\delta) &\equiv \quad Q_{1j}\delta^{n-1} + Q_{2j}\delta^{n-2} + \ldots + Q_{nj}, \end{aligned}\right\}, \tag{9}$$

we may write (6) in the form

$$\sum_{j=0}^{\infty} x^j F_j(\delta) y = 0; \tag{10}$$

formal substitution of (7) gives

$$\begin{aligned} F(x, \delta) y &\equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^{\rho+i+j} F_j(\delta+\rho+i) u_i \\ &\equiv \sum_{r=0}^{\infty} x^{\rho+r} \left[\sum_{i=0}^{r} F_{r-i}(\delta+\rho+i) u_i \right], \end{aligned} \tag{11}$$

by virtue of the identity $\delta^m(x^\theta u) \equiv x^\theta(\delta+\theta)^m u$.

If $|x|$ is smaller than the radii of convergence of (7) and (8), then (11) is convergent. By a known theorem†, a convergent series $\Sigma x^{\rho+r} v_r$, whose coefficients are polynomials in $\log x$ of bounded degree K, cannot vanish identically unless $v_r \equiv 0$. We must therefore have, for a formal solution, the relations

$$F_0(\delta+\rho) u_0 = 0 \tag{12}$$

and

$$\sum_{i=0}^{r} F_{r-1}(\delta+\rho+i) u_i = 0 \quad (r = 1, 2, \ldots). \tag{13}$$

The former is the generalization of the indicial equation.

The auxiliary system.

We have to discuss the equations (12) and (13), which are a system of linear differential equations with constant coefficients in the independent variable t. But it is important to observe that what is required is not the most general solution, but the most general solution in *polynomials*. If

† Heffter, 237.

$f(\delta)$ is an operator with constant coefficients and u is a polynomial in t, then, if $f(0) \neq 0$, $f(\delta)\,u$ is a polynomial of the same degree as u and cannot vanish identically unless $u \equiv 0$. For the same reason, if v is a polynomial, and $f(0) \neq 0$, the equation $f(\delta)\,u = v$ admits a unique polynomial solution of the same degree as v; it is well known† that this can be found in the form $u = [f(\delta)]^{-1}v$ by expanding the inverse operator in powers of δ. Accordingly, the generalized indicial equation cannot be satisfied by u_0 (different from zero) unless

$$F_0(\rho) = 0, \tag{14}$$

which is the ordinary indicial equation.

We shall suppose that the n roots of (14) are collected in groups, and that all the roots of one group are either equal to one another or differ by integers. We select one of the groups, consisting of h distinct roots $\rho_1, \rho_2, \ldots, \rho_h$, of respective multiplicities $k_1, k_2, \ldots, k_h$, where

$$k_1 + k_2 + \ldots + k_h = k \leqslant n.$$

By the theory of the Hamburger sets, $k-1$ is an upper bound of the degree of the corresponding polynomials $u_i(t)$ in (7). It is convenient to assume that the roots (ρ_i) are arranged in order of *ascending* real parts. To obtain the most general solution belonging to this group, we choose the root of (14) having the smallest real part, say $\rho = \rho_1$, since the other roots of the group will appear in due course. We now find, from (12),

$$F_0(\delta+\rho_1)\,u_0 \equiv g(\delta)\,\delta^{k_1}\,u_0 = 0, \tag{15}$$

where $g(0) \neq 0$. Since u_0 is restricted to be a polynomial, we can replace (15) by

$$\delta^{k_1}\,u_0 = 0, \tag{16}$$

which is the first equation of the reduced system. This implies, of course, that u_0 is an arbitrary polynomial of degree k_1-1, or a constant if $k_1 = 1$.

By a procedure exactly analogous to Heffter's, we shall now eliminate the coefficients of the intermediate powers of x between $x^{\rho_1}, x^{\rho_2}, \ldots, x^{\rho_h}$, and obtain a reduced system of equations for these critical coefficients. If $\rho_2 - \rho_1 > 1$, we have $F_0(\rho_1+1) \neq 0$, so that u_1 is determined as the unique polynomial solution of

$$F_0(\delta+\rho_1+1)\,u_1 = -F_1(\delta+\rho_1)\,u_0,$$

i.e.

$$u_1 = -\frac{F_1(\delta+\rho_1)}{F_0(\delta+\rho_1+1)}\,u_0 = L_{10}(\delta)\,u_0, \tag{17}$$

† For a rigorous discussion, see de la Vallée Poussin, II, 213, or Jordan, III, 171.

where $L_{10}(\delta)$ is an operator of order not greater than k_1-1, and where u_1 is a polynomial of degree not exceeding that of u_0. We may now eliminate u_1 from the equations (13) and merge the corresponding terms with those in u_0, putting everywhere $\delta^{k_1} u_0 = 0$. We can now solve for $u_2, u_3, \ldots, u_{\rho_2-\rho_1-1}$, and eliminate these variables of the auxiliary system, which are given explicitly by

$$u_2 = L_{20}(\delta)\, u_0, \quad u_3 = L_{30}(\delta)\, u_0, \quad \ldots, \quad u_{\rho_2-\rho_1-1} = L_{\rho_2-\rho_1-1,\,0}(\delta)\, u_0. \tag{18}$$

When, however, we come to the equation giving $u_{\rho_2-\rho_1}$, we have $F_0(\rho_2) = 0$. We cannot therefore express $u_{\rho_2-\rho_1}$ explicitly in terms of u_0; but, by using the fact that $u_{\rho_2-\rho_1}$ is a polynomial, we can reduce the equation which it satisfies to the form

$$f_{21}(\delta)\, u_0 + \delta^{k_2} u_{\rho_2-\rho_1} = 0, \tag{19}$$

which will be associated with (16) as the second equation of the reduced system. We can easily see in this way that every coefficient u_i for which $F_0(\rho_1+i) \neq 0$ is explicitly given in terms of the critical coefficients, thus

$$u_i = L_{i0}(\delta)\, u_0 + L_{i,\,\rho_2-\rho_1}(\delta)\, u_{\rho_2-\rho_1} + \ldots + L_{i,\,\rho_h-\rho_1}(\delta)\, u_{\rho_h-\rho_1}, \tag{20}$$

while the critical coefficients are the solutions of the reduced system with constant coefficients:

$$\left.\begin{aligned}
&\delta^{k_1} u_0 = 0,\\
&f_{21}(\delta)\, u_0 + \delta^{k_2} u_{\rho_2-\rho_1} = 0,\\
&f_{31}(\delta)\, u_0 + f_{32}(\delta)\, u_{\rho_2-\rho_1} + \delta^{k_3} u_{\rho_3-\rho_1} = 0,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&f_{h1}(\delta)\, u_0 + f_{h2}(\delta)\, u_{\rho_2-\rho_1} + \ldots + \delta^{k_h} u_{\rho_h-\rho_1} = 0.
\end{aligned}\right\} \tag{21}$$

Reduced system. Hamburger sets.

The polynomials satisfying (21) are given by elementary quadratures; it is easily seen that the degree of $u_{\rho_s-\rho_1}$ cannot exceed $k_1+k_2+\ldots+k_s-1$ and may in special cases be smaller. If ρ' is a root of the indicial equation of multiplicity k', we can construct solutions whose leading terms are $x^{\rho'}$, $x^{\rho'} \log x$, ..., $x^{\rho'} (\log x)^{k'-1}$. The labour of calculation may be shortened by calculating any solution of the highest possible degree in $\log x$, and obtaining other solutions from it by formal differentiation with respect to $\log x$ (treated as a separate variable)†.

† See, for instance, Heffter, 105; Forsyth, IV, 62–66.

But the structure of the fundamental system of solutions at a singularity will not have been fully elucidated unless we are able to resolve it into Hamburger sets of solutions. By taking any fundamental system at an ordinary point near the singularity and continuing it analytically round a closed circuit about the singularity, we can in theory construct the characteristic equation; we then know that the several Hamburger sets are in one-to-one correspondence with the invariant factors of the characteristic determinant. If the solutions of the fundamental system are all known in the form (3), the calculation of the characteristic equation is not impracticable, though very laborious†.

We shall now prove, however, that the Hamburger sets also correspond to the invariant factors of the matrix of operators

$$\begin{pmatrix} \delta^{k_1}, & 0, & 0, & \ldots, & 0 \\ f_{21}(\delta), & \delta^{k_2}, & 0, & \ldots & \ldots \\ f_{31}(\delta), & f_{32}(\delta), & \delta^{k_3}, & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_{h1}(\delta), & f_{h2}(\delta), & f_{h3}(\delta), & \ldots, & \delta^{k_h} \end{pmatrix}. \tag{22}$$

We have seen that these operators can all be found by elementary calculations (as soon as the indicial equation is solved) and without as yet knowing the forms (3) of the solution.

These invariant factors are of the form δ^{e_i}, where $\Sigma e_i = \Sigma k_i = k$, and where certain of the e_i may be zero. They can always be found by a finite sequence of elementary reversible steps‡. We thus obtain a set of variables (v_i) connected with $(u_{\rho_i - \rho_1})$ by relations of the type which may be written symbolically

$$\mathbf{u} = \mathbf{A}\mathbf{v}, \quad \mathbf{v} = \mathbf{A}^*\mathbf{u}, \quad \mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A} = \mathbf{1}, \tag{23}$$

where $\mathbf{A}$, $\mathbf{A}^*$ are square matrices of h^2 linear operators, whose determinant is unity. The equations satisfied by (v_i) are then

$$\delta^{e_i} v_i = 0 \quad (i = 1, 2, \ldots, h), \tag{24}$$

so that (v_i) are arbitrary and mutually independent polynomials of respective degrees $e_i - 1$. If all except one are made zero, it is easy to see that at least one of the $(u_{\rho_i - \rho_1})$ is of as high degree as that one. For if not, every element of one of the columns of the matrix $\mathbf{A}$ would be divisible by δ, and the relations (23) could not hold.

† Heffter, 133–141; Schlesinger, I, 190–194; Forsyth, IV, 112–113.

‡ Jordan, III, 176.

In general, certain of the equations (24) reduce to the form $v_i = 0$, and the corresponding variable may then be simply erased. But for every invariant factor with $e_i \neq 0$, we can construct a solution in which every coefficient u_j is an explicit function of v_i. If we assign to v_i the form of a polynomial whose degree is actually $e_i - 1$, this solution will be the last solution of a Hamburger set, the other members being readily derived from it by forming differences. It is sufficient to choose

$$v_i \equiv t^{e_i-1} \equiv (\log x)^{e_i-1}.$$

The condition that the solutions of the group should all be free from $\log x$ is that the invariant factors should all be linear, *i.e.* $e_i = 1$. This includes the condition that a singularity where all the exponents are positive integers should be an apparent singularity.

If we do not require the values of the intermediate coefficients, the system of equations (21) can be constructed directly by elimination. But the labour and the chance of error in the evaluation of the determinants is perhaps greater than in the method suggested. The coefficients of the series coming after that of x^{ρ_h} may be calculated in several ways. Knowing the degree in t or $\log x$, we may introduce undetermined coefficients and identify powers of t in (21); this brings us back to the classical method and leads to the same series. The classical proof of convergence, by means of dominant functions, remains valid, and need not be repeated. We may also express these coefficients explicitly in terms of the critical coefficients, in the same way as the intermediate coefficients. The coefficients of the leading terms of the inverse operator $[F_0(\rho_1+i+\delta)]^{-1}$ are rational functions of i, whose form can be calculated once and for all by Taylor's theorem, provided that $F_0(\rho_1+i) \neq 0$. This method appears to be more compact, since the arbitrary constants occurring in u_0, $u_{\rho_2-\rho_1}$, ... need not be introduced until the final stage.

A modification of Frobenius's method†.

We retain the notation of (6), (8), and (9), and we suppose that the indicial equation at $x = 0$ is

$$F_0(\rho) \equiv \prod_{i=1}^{s} (\rho-\rho_i)^{k_i} = 0, \tag{25}$$

† Forsyth, IV, 78; Schlesinger, I, 155; Ince, 396-408.

where (ρ_i) are the distinct roots and $\sum_{i=1}^{s} k_i = n$. We shall again suppose that the first h roots belong to a single group of k roots differing by integers or zero. We introduce the function

$$\Psi(\rho) \equiv \prod_{i=1}^{s} [\Gamma(\rho-\rho_i)]^{k_i}, \tag{26}$$

and we observe that $1/\Psi(\rho)$ is an integral function of ρ, since the reciprocal of the gamma function is defined by

$$\frac{1}{\Gamma(z)} \equiv e^{\gamma z} z \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}. \tag{27}$$

Let α be any number, which we suppose in the first instance not to be equal to any of the ρ_i nor to be of the form $\rho_i - m$, where m is a positive integer; and let us determine a series of the form

$$y \equiv \phi(x, \alpha) \equiv x^{\alpha}[c_0 + c_1 x + \ldots], \tag{28}$$

satisfying the non-homogeneous equation

$$F(x, \delta)\, y = x^{\alpha}/\Psi(\alpha). \tag{29}$$

Operating on both sides with $\delta - \alpha$, we see that y satisfies a homogeneous equation of order $n+1$, with a regular singularity at $x = 0$, and the indicial equation $(\rho-\alpha) F_0(\rho) = 0$. Since, by our restrictions, α is an isolated root, there is a well-determined solution of the form (28), with constant values of (c_i). On expanding (29), we have, therefore,

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^{\alpha+i+j} F_j(\alpha+i)\, c_i = \frac{x^{\alpha}}{\Psi(\alpha)}, \tag{30}$$

i.e.

$$\sum_{r=0}^{\infty} x^{\alpha+r} \left[\sum_{i=0}^{r} F_{r-i}(\alpha+i)\, c_i \right] = \frac{x^{\alpha}}{\Psi(\alpha)}. \tag{31}$$

We have, on identifying coefficients,

$$F_0(\alpha)\, c_0 = 1/\Psi(\alpha), \tag{32}$$

$$\sum_{i=0}^{r} F_{r-i}(\alpha+i)\, c_i = 0 \quad (r = 1, 2, \ldots). \tag{33}$$

This system determines the c_r uniquely, because $F_0(\alpha+\nu)$ does not vanish for any positive integral or zero value of r.

Now every coefficient c_r is an integral function of α. For, on factorizing $F_0(\alpha)$ and using the multiplicative property of $\Gamma(z)$, we have, from (32),

$$c_0 = 1/\Psi(\alpha+1). \tag{34}$$

Again, we find that

$$\frac{c_r}{c_0} = \frac{(-)^r h_r(\alpha)}{F_0(\alpha+1)\, F_0(\alpha+2) \ldots F_0(\alpha+r)},$$

or, by similar reasoning,

$$c_r = (-)^r h_r(\alpha)/\Psi(\alpha+r+1), \tag{35}$$

where $h_r(\alpha)$ is the polynomial

$$h_r(\alpha) \equiv \begin{vmatrix} F_1(\alpha), & F_0(\alpha+1), & 0, & \ldots, & 0 \\ F_2(\alpha), & F_1(\alpha+1), & F_0(\alpha+2), & \ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{r-1}(\alpha), & F_{r-2}(\alpha+1), & F_{r-3}(\alpha+2), & \ldots, & F_0(\alpha+r-1) \\ F_r(\alpha), & F_{r-1}(\alpha+1), & F_{r-2}(\alpha+2), & \ldots, & F_1(\alpha+r-1) \end{vmatrix}. \tag{36}$$

Since $\Psi(z)$ is the reciprocal of an integral function, (35) shows that c_r is an integral function of α.

We shall now construct a dominant polynomial for $h_r(\alpha)$. We shall suppose that the equation (1) has no other singularity than $x = 0$ in the interior or on the boundary of the circle $|x| \leqslant R$. The functions (6) then admit a dominant function

$$Q_i(x) \ll \frac{M}{1-xR^{-1}}, \tag{37}$$

and we then have, from (9), the relations

$$\left.\begin{aligned} F_0(\alpha) &\ll \alpha^n + M\psi(\alpha), \\ F_j(\alpha) &\ll M R^{-j} \psi(\alpha), \end{aligned}\right\} \tag{38}$$

where

$$\psi(\alpha) \equiv \alpha^{n-1} + \alpha^{n-2} + \ldots + 1. \tag{39}$$

Finally, we note that the determinant

$$\begin{vmatrix} a_1, & -a_2, & 0, & 0, & \ldots & \ldots & \ldots \\ b_1, & b_2, & -b_3, & 0, & \ldots & \ldots & \ldots \\ & & & & & & \\ j_1, & j_2. & j_3, & j_4, & \ldots, & j_{n-1}, & -j_n \\ k_1, & k_2, & k_3, & k_4, & \ldots, & k_{n-1}, & k_n \end{vmatrix}$$

is a polynomial in all its arguments with positive coefficients. We thus find a dominant polynomial for $h_r(\alpha)$ of the form

$$H_r(\alpha)\equiv\begin{vmatrix} R^{-1}\psi(\alpha), & -\chi(\alpha+1), & 0, & \dots, & 0 \\ R^{-2}\psi(\alpha), & R^{-1}\psi(\alpha+1), & -\chi(\alpha+2), & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ R^{1-r}\psi(\alpha), & R^{2-r}\psi(\alpha+1), & R^{3-r}\psi(\alpha+2), & \dots, & -\chi(\alpha+r-1) \\ R^{-r}\psi(\alpha), & R^{1-r}\psi(\alpha+1), & R^{2-r}\psi(\alpha+2), & \dots, & R^{-1}\psi(\alpha+r-1) \end{vmatrix}, \tag{40}$$

where $\chi(\alpha)\equiv\alpha^n+M\psi(\alpha)$. But (40) can be evaluated in finite terms

$$H_r(\alpha)\equiv R^{-r}\psi(\alpha)\prod_{j=1}^{r-1}\{\chi(\alpha+j)+\psi(\alpha+j)\}, \tag{41}$$

and with a suitable constant A we have

$$\chi(\alpha)+\psi(\alpha)\equiv\alpha^n+2M(\alpha^{n-1}+\alpha^{n-2}+\dots+1)\ll(\alpha+A)^n, \tag{42}$$

while

$$\psi(\alpha)\equiv M(\alpha^{n-1}+\alpha^{n-2}+\dots+1)\ll M(\alpha+1)^{n-1}. \tag{43}$$

Hence

$$H_r(\alpha)\ll\frac{M(\alpha+1)^{n-1}}{R^r}\prod_{j=1}^{r-1}(\alpha+A+j)^n. \tag{44}$$

To show that the series (28) converges uniformly with respect to α, we observe that, if $|\alpha|<K$, we have

$$|c_r|\leqslant\frac{M(K+1)^{n-1}[\Gamma(K+A+r)]^n.}{R^r[\Gamma(K+A+1)]^n|\Psi(\alpha+r+1)|}. \tag{45}$$

If $|\alpha|<K$ and $r<K'$, this expression has a fixed bound depending on K and K'. But, as $r\to\infty$, we have

$$\Gamma(K+A+r)\sim r^{K+A}\Gamma(r), \tag{46}$$

$$\Psi(\alpha+r+1)\sim r^{\Sigma(\alpha-\rho'+1)}[\Gamma(r)]^n, \tag{47}$$

where the summation is over the roots of the indicial equation. Hence

$$|c_r|\leqslant\frac{G(K+1)^{n-1}r^{2nK+B}}{R^r[\Gamma(K+A+1)]^n}, \tag{48}$$

where B is independent of K, so that

$$|c_r|\leqslant H(K)r^{2nK+B}/R^r, \tag{49}$$

where $H(K)$ is independent of r.

It follows that $\phi(x, \alpha) = \Sigma c_r x^{\alpha+r}$ is uniformly convergent for $|\alpha| < K$ and $|x| \leqslant R-\epsilon < R$. We may now choose K so that all the roots of the indicial equation lie within the circle $|\rho_i| < K$.

Now consider the identity (29). For a fixed value of x, other than zero, where $|x| < R-\epsilon$, this is an identity between integral functions of α. It will still hold as α tends to one of the special values lying within the domain of absolute convergence.

If we now consider our set of roots differing by integers, and allow α to approach ρ_h, the root with the greatest real part, we see that the right-hand member is divisible by $(\alpha-\rho_h)^{k_h}$, but that c_0 remains finite. If we put $\alpha = \rho_h$ in $\phi(x, \alpha)$ and in its first k_h-1 derivatives with respect to α, we have the relations

$$F(x, \delta)\left[\frac{\partial^s}{\partial\alpha^s}\phi(x, \alpha)\right]_{\alpha=\rho_h} = 0 \qquad (50)$$

$$[s = 0, 1, \ldots, (k_h-1)],$$

giving k_h formal solutions of $Fy = 0$. These solutions are necessarily independent, since they are of degree 0, 1, ..., k_h-1 respectively in $\log x$, owing to the results of differentiating x^α.

Next, suppose that α approaches the next highest root ρ_{h-1}. The right-hand side is divisible by $(\alpha-\rho_{h-1})^{k_{h-1}+k_h}$, so that $k_{h-1}+k_h$ relations similar to (50) can be deduced by differentiation with respect to α.

We have, therefore, the solutions

$$y = \left[\frac{\partial^s}{\partial\alpha^s}\phi(x, \alpha)\right]_{\alpha=\rho_{h-1}} \qquad (51)$$

$$(s = 0, 1, \ldots, k_{h-1}+k_h-1).$$

But the first k_h of these are found to be merely combinations of (50), because the first $\rho_h-\rho_{h-1}$ coefficients are divisible by $(\alpha-\rho_{h-1})^{k_h}$.

In the same way, $(\alpha-\rho_{h-2})^{k_{h-2}+k_{h-1}+k_h}$ is a factor of the right-hand member, and gives solutions

$$y = \left[\frac{\partial^s}{\partial\alpha^s}\phi(x, \alpha)\right]_{\alpha=\rho_{h-2}} \qquad (52)$$

$$(s = 0, 1, \ldots, k_{h-2}+k_{h-1}+k_h-1).$$

But here again the first k_h are combinations of (50), and the next k_{h-1} of (50) and (51), so that only the last k_{h-2} are new.

By proceeding systematically through all the roots of the indicial equation, we can construct all the solutions as the limits of the derivatives

of a single expression $\phi(x, \alpha)$. In consequence of the properties of the gamma function, no multiplication by a polynomial, as in the original form of the method, is needed to remove singularities.

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PLANES, POINTS, AND SURFACES ASSOCIATED WITH A CUBIC SURFACE

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1. In two recent papers* the three Schur quadrics and the three quadrics through six lines, which belong to three associated double sixes of a cubic surface, were investigated. A single point X, defined algebraically, was proved to have certain geometric properties†, and a quartic surface $Q \equiv T_2 T_3 + T_3 T_1 + T_1 T_2 = T_3^2 - S_1 S_2$ was proved to contain the parabolic curve of the cubic, and to be related to the cubic and its Hessian by the identity $H = CE + 4Q$, where E is the polar plane of X with respect to the cubic. Later it was shown that the surface Q, so derived from each of the three sets of three associated double sixes of three associated Steiner trihedral pairs, is the same for all three‡.

Some time before, by a completely different approach §, certain points F_{123}, ..., had been discovered and, later‖, Dixon's X point was shown to be an F point.

In this paper further properties of three associated Steiner trihedral pairs are given, and relations between the 120 X points of the cubic and their forty polar planes E are found.

The notation used is that of (D) 2, for the associated loci and envelopes, and that of (M and M) for the triple tangent planes. A modified Schläfli notation is used for the lines of the surface.

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* (D) 1 and 2. For explanation of these references, see the bibliography at the end of the paper.

† (D) 2, 354, 355.

‡ (M and M), 188; (B), 224.

§ (M) 1, 136; (B), 244.

‖ (M) 3, 248.

When the equation of the surface is written $UVW = U'V'W'$, the pair of trihedra U, V, W and U', V', W' are a Steiner trihedral pair.

We take $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6, \\ 1' & 2' & 3' & 4' & 5' & 6' \end{matrix}$

as the notation for a double six, and denote by 12 the line which meets the four 1, 2, 1′, 2′; three associated trihedral pairs are then given by the three groups of lines

$$\begin{matrix} \text{I} \\ \left\{\begin{matrix} 14, & 15, & 16 \\ 24, & 25, & 26 \\ 34, & 35, & 36 \end{matrix}\right\} \end{matrix} \qquad \begin{matrix} \text{II} \\ \left\{\begin{matrix} 4', & 4, & 56 \\ 5', & 5, & 64 \\ 6', & 6, & 45 \end{matrix}\right\} \end{matrix} \qquad \begin{matrix} \text{III} \\ \left\{\begin{matrix} 1, & 2, & 3 \\ 1', & 2', & 3' \\ 23, & 31, & 12 \end{matrix}\right\} \end{matrix}.$$

U, V, W, U', V', W' are the triple tangent planes containing the lines given by the diagonals of the array I, according to the scheme:

$$U \equiv 14,\ 25,\ 36; \quad V \equiv 15,\ 26,\ 34; \quad W \equiv 16,\ 24,\ 35;$$

$$U' \equiv 14,\ 26,\ 35; \quad V' \equiv 15,\ 24,\ 36; \quad W' \equiv 16,\ 25,\ 34.$$

S_1, S_2, S_3, T_1, T_2, T_3 refer to I, that is they are determined by the three associated double sixes whose lines appear in II and III, so that T_3 is the Schur quadric which is associated with the given double six*.

Affixes (1), (2), (3) are used, where necessary, to distinguish loci associated with the different groups. Thus, for example, T_1 and $T_1^{(1)}$ represent the same quadric. Such a point as (1, 2′) is called a point of III.

2. *The surface* $Q \equiv T_2T_3 + T_3T_1 + T_1T_2 = 0$.

It is clear that the eight intersections of T_1, T_2, T_3 are nodes on Q. If we put† $T_r \equiv a_r x^2 + b_r y^2 + c_r z^2 + d_r t^2$, with the condition

$$u_1^{-1} + u_2^{-1} + u_3^{-1} = 0 \quad (u = a, b, c, d),$$

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* This notation now agrees, line by line, with the notation used in (M and M), 178, and also with that in (D) 1.

† (D) 1, 173.

then the equation of Q becomes

$$Q \equiv q_{ab}x^2y^2+q_{bc}y^2z^2+q_{ca}z^2x^2+q_{ad}x^2t^2+q_{bd}y^2t^2+q_{cd}z^2t^2=0,$$

where $$q_{ab}=a_1b_2+a_2b_1+a_2b_3+a_3b_2+a_3b_1+a_1b_3.$$

Thus four more nodes are apparent.

If now we put

$$X=(q_{ab}q_{ac}q_{ad})^{\frac{1}{2}}x, \quad Y=(q_{ab}q_{bc}q_{bd})^{\frac{1}{2}}y, \quad \ldots,$$

we obtain for $(q_{ab}q_{ac}q_{ad}q_{bc}q_{cd}q_{bd})^{\frac{1}{2}}Q$ the expression

$$(q_{bc}q_{ad})^{\frac{1}{2}}(Y^2Z^2+X^2T^2)+(q_{ca}q_{bd})^{\frac{1}{2}}(Z^2X^2+Y^2T^2)+(q_{ab}q_{cd})^{\frac{1}{2}}(X^2Y^2+Z^2T^2).$$

Further, we have

$$(q_{bc}q_{ad})^{\frac{1}{2}}+(q_{ca}q_{bd})^{\frac{1}{2}}+(q_{ab}q_{cd})^{\frac{1}{2}}=0,$$

for, when rationalized, this is

$$\begin{vmatrix} 0, & q_{ab}, & q_{ac}, & q_{ad} \\ q_{ab}, & 0, & q_{bc}, & q_{bd} \\ q_{ac}, & q_{bc}, & 0, & q_{cd} \\ q_{ad}, & q_{bd}, & q_{cd}, & 0 \end{vmatrix} = 0,$$

the left-hand side of which is the formal product of the two arrays

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3, & b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3, & c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3, & d_1, & d_2, & d_3 \end{vmatrix}, \qquad \begin{vmatrix} a_2, & a_3, & a_1, & a_3, & a_1, & a_2 \\ b_2, & b_3, & b_1, & b_3, & b_1, & b_2 \\ c_2, & c_3, & c_1, & c_3, & c_1, & c_2 \\ d_2, & d_3, & d_1, & d_3, & d_1, & d_2 \end{vmatrix},$$

and therefore vanishes.

The surface Q thus belongs to a pencil of quartic surfaces

$$\lambda(Y^2Z^2+X^2T^2)+\mu(Z^2X^2+Y^2T^2)+\nu(X^2Y^2+Z^2T^2)=0,$$

where $\lambda+\mu+\nu=0$, each of which passes through the sixteen lines $X^2=Y^2=Z^2$, ..., and has its twelve nodes at the eight points

$(1, \pm 1, \pm 1, \pm 1)$, and the four points

$$(1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1).$$

These nodes are therefore the eight points of intersection* of T_1, T_2, T_3 together with the four vertices of their common self-conjugate tetrahedron.

Three members of the pencil ($\lambda = 0$, $\mu = 0$, $\nu = 0$) reduce to four planes and form a triad of desmic tetrahedra† with the equations

$$(Y^2-Z^2)(X^2-T^2) = 0; \quad (Z^2-X^2)(Y^2-T^2) = 0; \quad (X^2-Y^2)(Z^2-T^2) = 0.$$

Further, with the same eighteen edges, in different arrangement, another triad of desmic tetrahedra is obtained, one member of which is the self-conjugate tetrahedron of T_1, T_2, T_3, while the other two are the tetrahedra whose edges are opposite diagonals of opposite faces of the octad $(1, \pm 1, \pm 1, \pm 1)$.

These two tetrahedra are therefore common self-conjugate tetrahedra to the other two triads of Schur quadrics from which the same quartic surface Q may be derived‡. We have thus the following result§:

Associated with each trihedral pair of three associated Steiner trihedral pairs on the cubic surface there are three Schur quadrics and three quadrics through six lines. The twelve such quadrics associated with any two of the trihedral pairs have four common points which are the vertices of the common self-conjugate tetrahedron of the six quadrics of the remaining pair.

This result gives another proof that sets of Schur quadrics are apolar in pairs.

These tetrahedra are also the bases of certain tetrahedral complexes associated with lines and edges of the cubic surface as follows.

With the notation of (M and M) for the quadrics of the three Steiner pairs, the equations of $S_1^{(3)}$, $S_2^{(3)}$, $S_3^{(3)}$ are

$$x_1\xi_2 - x_2\xi_1 = 0, \quad y_1\eta_2 - y_2\eta_1 = 0, \quad z_1\zeta_2 - z_2\zeta_1 = 0.$$

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* This implies the curious relation that the square of $\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$ is equal to $q_{bc}q_{ca}q_{ab}$ when $u_1^{-1}+u_2^{-1}+u_3^{-1} = 0$ $(u = a, b, c)$.

† *Vide* R. W. H. T. Hudson, *Kummer's quartic surface* (Cambridge, at the University Press, 1905), p. 3 and the references there given.

‡ (M and M), 188.

§ (B), 246. Bishara's proof is on different lines.

If we write

$$x_2 = \lambda x_1, \quad \xi_2 = \lambda \xi_1, \quad y_2 = \mu y_1, \quad \eta_2 = \mu \eta_1,$$

and substitute in the equation of $S_3^{(3)}$, we find that either $S_1^{(1)} = 0$ or $\mu = \lambda$.

The second alternative thus leads to the four points common to all the quadrics with affixes (2) and (3). The condition that the four linear equations then hold together is a quartic equation in λ, whose roots thus give the four common points*. Each point therefore satisfies nine equations,

$$x_2 = \lambda_r x_1, \quad \xi_2 = \lambda_r \xi_1, \quad X_2 = \lambda_r X_1,$$

$$y_2 = \lambda_r y_1, \quad \eta_2 = \lambda_r \eta_1, \quad Y_2 = \lambda_r Y_1,$$

$$z_2 = \lambda_r z_1, \quad \zeta_2 = \lambda_r \zeta_1, \quad Z_2 = \lambda_r Z_1,$$

for $r = 1, 2, 3, 4$.

It follows that the nine lines of a trihedral pair belong to a tetrahedral complex of which the tetrahedron associated with that pair is the base.

Moreover, each $S^{(2)}$ meets each $S^{(3)}$ in one of these lines and also in a space cubic curve, of which the line is a chord, and which contains the vertices of the tetrahedron. Thus all chords of the cubic curves belong to the tetrahedral complex and, in particular, sixty edges† of the cubic surface belong also to the complex.

When the quadrics of the first trihedral pair are represented by their equations in the variables U, V, W, U', V', W', the three desmic tetrahedra have simple equations. It is easy to show that the six equations

$$\lambda_p \lambda_q S_1 + (\lambda_p + \lambda_q) T_3 + S_2 = 0,$$

where λ_p and λ_q are two different roots of the equation‡

$$P_1\lambda^4 - 2P_1\lambda^3 + (P_1 + P_2 + P_3)\lambda^2 - 2P_2\lambda + P_2$$

$$= \frac{P_1}{a_2 b_2 c_2 d_2} (a_1 + \lambda a_2)(b_1 + \lambda b_2)(c_1 + \lambda c_2)(d_1 + \lambda d_2) = 0,$$

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* This quartic equation is the equation in (D) 1, 173, which gives a verification that the four points form the common self-conjugate tetrahedron of $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$.

† Those of types (*a*) and (*b*) through the points of intersection of lines of the trihedral pair. *Vide* (M and M), 180.

‡ *Cf.* (D) 1, 173.

represent pairs of planes, which clearly contain the eight intersections of S_1, T_3, S_2.

It follows that the three desmic tetrahedra are represented by the three equations

$$\text{(i)} \qquad \{\lambda_p \lambda_q S_1 + (\lambda_p + \lambda_q) T_3 + S_2\}\{\lambda_s \lambda_t S_1 + (\lambda_s + \lambda_t) T_3 + S_2\} = 0,$$

where p, q, s, t are 1, 2, 3, 4 in some order.

The sixteen straight lines which lie on the quartic surface Q are then given by the intersections of the desmic tetrahedra and thus have the equations

$$\lambda_r^2 S_1 = -\lambda_r T_3 = S_2 \quad (r = 1, 2, 3, 4).$$

The equation (i) reduces to

$$P_1 S_2 S_3 + P_2 S_3 S_1 + P_3 S_1 S_2 + P_1(\lambda_p + \lambda_q)(\lambda_s + \lambda_t) Q = 0,$$

and thus the surface

$$R \equiv P_1 S_2 S_3 + P_2 S_3 S_1 + P_3 S_1 S_2 = 0$$

is also a member of the desmic pencil.

3. *The surface* $R \equiv P_1 S_2 S_3 + P_2 S_3 S_1 + P_3 S_1 S_2 = 0$.

It is evident that R passes through the thirty-six points of the second and third trihedral pairs, for each point lies on two of the three quadrics S. Thus, $R^{(1)}$ and $R^{(2)}$ have common points not on the sixteen lines of Q, and so, since $R^{(1)}$ and $R^{(2)}$ belong to the desmic pencil, the members of which have only the sixteen lines common, it follows that $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ are all the same surface*. The surface R therefore passes through the fifty-four points of the three trihedral pairs.

If we denote by C the expression $UVW - U'V'W'$, then

$$P_r C = A_r U Y_r \zeta_r - A_r' U' Z_r \eta_r \quad (r = 1, 2, 3)$$

$$= Z_r x_r \eta_r - A_r U S_r,$$

together with alternative forms†.

* This result gives a second quadratic identity between the Schur quadrics of two associated pairs.

† (M and M), 182.

Thus

$$C(P_1 S_2 S_3 + P_2 S_3 S_1 + P_3 S_1 S_2)$$

$$= Z_1 x_1 \eta_1 S_2 S_3 + Z_2 x_2 \eta_2 S_3 S_1 + Z_3 x_3 \eta_3 S_1 S_2$$

$$= \Sigma Z_1 x_1 \eta_1 (x_2 \eta_2 - y_2 \xi_2)(x_3 \eta_3 - y_3 \xi_3),$$

where the summation is for cyclic interchange, which reduces to

$$\Sigma Z_1 (x_1 \eta_1 y_2 \xi_2 y_3 \xi_3 + y_1 \xi_1 x_2 \eta_2 x_3 \eta_3),$$

where the summation is for three terms. From this equation, and its various alternative forms, it is clear that R passes through the thirty-six trihedral summits of type $x_1 = \eta_2 = Z_3$. We have then the following result:

The fifty-four points of the three trihedral pairs and the thirty-six trihedral summits on those edges of the cubic surface, each of which contains one point of each pair, all lie on the quartic surface R.

If we consider the cubics of the pencil $UVW = kU'V'W'$, that is cubics through the lines of the first trihedral pair, then $S_r^{(2)}$, $S_r^{(3)}$ are the same for all. Thus $R^{(2)}$ must belong to the desmic pencil for all and must, therefore, be the same surface for all. It follows that R is the locus of the curves of intersection of pairs of S_1, S_2, S_3 for members of this pencil of cubics and also for members of the pencils associated with the second and third trihedral pairs.

The equation of R may be written

$$P_2 P_3 S_1^2 = (P_2 S_1 + P_1 S_2)(P_3 S_1 + P_1 S_3).$$

R therefore envelops $P_2 S_1 + P_1 S_2$ along the curve $S_1 = S_2 = 0$, and is the envelope of $P_2 S_1 + P_1 S_2$ for the members of the pencil

$$UVW = kU'V'W'.$$

Further, since $P_2 S_1 + P_1 S_2$ is the reciprocal of S_3 with respect to T_3, and since the line which joins (2, 3′), (2′, 3) is the reciprocal of the line 23, which is a generator of S_3, it follows that the quadric $P_2 S_1 + P_1 S_2$ is the quadric through the lines (2, 3′), (2′, 3); (3, 1′), (3′, 1); (1, 2′), (1′, 2). Thus, since (2, 3′) and (2′, 3) lie on the curve $S_1 = S_2 = 0$, the line (2, 3′), (2′, 3) is a bitangent to R at the points (2, 3′) and (2′, 3). The twenty-seven lines of the type (2, 3′), (2′, 3) derived from the three

trihedral pairs are therefore bitangents of R at the corresponding two points.

4. *The X points and the plane through the trihedral summits of the three associated trihedral pairs.*

Using Dixon's notation, A_1, B_1, C_1, etc., we denote the six expressions

$$B_1C_2-B_2C_1, \quad C_1A_2-C_2A_1, \quad A_1B_2-A_2B_1,$$

$$B_1'C_2'-B_2'C_1', \quad C_1'A_2'-C_2'A_1', \quad A_1'B_2'-A_2'B_1'$$

by a, b, c, a', b', c', and the six expressions

$$B_1C_1+B_2C_2+B_3C_3, \quad C_1A_1+C_2A_2+C_3A_3, \quad \ldots$$

by α, β, γ, α', β', γ'.

Then P and P', the trihedral summits

$$U'=V'=W'=0, \quad U=V=W=0,$$

are the points whose coordinates are

$$U=a, \quad V=b, \quad W=c, \quad U'=V'=W'=0,$$

and

$$U=V=W=0, \quad U'=a', \quad V'=b', \quad W'=c'.$$

In the notation of (M and M) the point P thus satisfies the equations

$$x_1=\eta_1=Z_1, \quad y_1=\zeta_1=X_1, \quad z_1=\xi_1=Y_1,$$

$$x_2=\eta_2=Z_2, \quad y_2=\zeta_2=X_2, \quad z_2=\xi_2=Y_2.$$

Let the trihedral summits of the second and third trihedral pairs be Q, Q' and R, R'.

By use of the analytical transformation which interchanges the pairs, we find that Q satisfies the equations

$$x_1=y_2=z_3, \quad x_2=y_3=z_1, \quad x_3=y_1=z_2,$$

$$\xi_1=\eta_2=\zeta_3, \quad \xi_2=\eta_3=\zeta_1, \quad \xi_3=\eta_1=\zeta_2.$$

If we now write $A_1U-A_1'U'$ for x_1, ..., it is easy to deduce that Q

satisfies the equations

$$A_1U+B_3V+C_2W=0,\quad A_2U+B_1V+C_3W=0,\quad A_3U+B_2V+C_1W=0,$$

$$A_1'U'+B_3'V'+C_2'W'=0,\quad A_2'U'+B_1'V'+C_3'W'=0,$$

$$A_3'U'+B_2'V'+C_1'W'=0.$$

The point Q is therefore the point*

$$U=B_1C_1-B_2C_3,\qquad V=C_1A_1-C_2A_3,\qquad W=A_1B_1-A_2B_3,$$

$$U'=B_1'C_1'-B_2'C_3',\quad V'=C_1'A_1'-C_2'A_3',\quad W'=A_1'B_1'-A_2'B_3',$$

which, as is easily seen, is the point

$$(\alpha-a,\ \beta-b,\ \gamma-c,\ \alpha'-a',\ \beta'-b',\ \gamma'-c').$$

By similar processes we find that the points Q', R, and R' are respectively

$$(\alpha+a,\ \beta+b,\ \gamma+c,\ \alpha'+a',\ \beta'+b',\ \gamma'+c'),$$

$$(\alpha+a,\ \beta+b,\ \gamma+c,\ \alpha'-a',\ \beta'-b',\ \gamma'-c'),$$

$$(\alpha-a,\ \beta-b,\ \gamma-c,\ \alpha'+a',\ \beta'+b',\ \gamma'+c').$$

The six points thus form the vertices of a complete quadrilateral whose diagonal points X_1, X_2, X_3 are

$$(\alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma'),\quad (a,\ b,\ c,\ -a',\ -b',\ -c'),\quad (a,\ b,\ c,\ a',\ b',\ c').$$

We thus have the result:

The six trihedral summits of three associated trihedral pairs are coplanar and lie by threes on four straight lines.

We will call this plane a trihedral summit plane.

It may be verified by direct substitution that the equation of this trihedral summit plane is, in Dixon's notation,

$$P_1(L_2-L_3)+P_2(L_3-L_1)+P_3(L_1-L_2)=0.$$

There will be forty such planes determined by a cubic surface.

* $B_1C_1-B_2C_3=B_2C_2-B_3C_1=B_3C_3-B_1C_2,\ \ldots$

The point X_1 has been proved to possess various properties, its characteristic property being that it is the point whose polar quadric, with respect to the cubic, is the quadric $T_1+T_2+T_3=0$. By the use of the forms given in (M and M), we find

$$T_1^{(2)}+T_2^{(2)}+T_3^{(2)}=M+M',$$

$$T_1^{(3)}+T_2^{(3)}+T_3^{(3)}=-M+M',$$

where M and M' are the cones

$$M=aVW+bWU+cUV,$$

$$M'=a'V'W'+b'W'U'+c'U'V'.$$

Since M and M' are the polar quadrics of P and P', it follows that the X points of the second and third trihedral pairs* are

$$(a, b, c, -a', -b', -c') \quad \text{and} \quad (a, b, c, a', b', c'),$$

namely the points X_2 and X_3 which, with X_1, are the three diagonal points of the quadrilateral formed by P, P', Q, Q', R, R'.

The polar plane with respect to the cubic surface of all three X points is the same plane†, namely the plane E, a further verification, from the identity $H=CE+4Q$, that Q is the same surface for the three pairs.

To each trihedral summit plane corresponds one E plane. The polar plane, with respect to the cubic, of every point on the line PP' passes through a line which lies in the E plane, since two of the poles of the E plane lie on PP'. The two lines similarly derived from QQ' and RR' lie also in the E plane. The E plane can thus be defined as the plane of these three lines.

Further, it is clear that the three lines so obtained lie on the polar cubic of the trihedral summit plane. Thus the E plane is the unique triple tangent plane of this polar cubic. In fact, if we write down the polar plane of the general point in the trihedral summit plane, its envelope is easily found to be

$$p(p'^2-x'^2)=p'(p^2-x^2),$$

---

* *Cf.* (M) 1, 139, where it is noted that "F_{123} and F_{456} harmonically separate H and K", *i.e.*, in our notation, X_2 and X_3 harmonically separate P and P'.

† *Cf.* (D) 2, 361.

where p and p' are the polar planes of P and P', and x and x' are the mixed polars of P, X_1 and P', X_1. Besides the three lines of this cubic surface in the E plane, $p = p'$, there are the six which join the four nodes. Two of these, $p = x = 0$ and $p' = x' = 0$, pass through P and P'. It follows then that the six edges of the tetrahedron formed by the nodes of the polar cubic of the trihedral summit plane meet this plane in the six trihedral summits*.

It is evident, from the identity $H = CE + 4Q$, that the sixteen lines which lie on Q are trisecant chords of the parabolic curve and meet the Hessian further in sixteen points which are coplanar. The E plane is thus also the plane of these points.

5. *Trihedral pairs which have a common Schur quadric.*

We consider the trihedral pair I and the pair derived from it by an interchange of the lines 3, 3′ with 4, 4′, and so by an interchange of the numbers 3 and 4 wherever they occur in the notation for the lines. The double six

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1' & 2' & 3' & 4' & 5' & 6' \end{pmatrix}$$

belongs to both pairs. Thus, in the notation of (D) 2, the Schur quadric T_3, the Milne quadric ϕ_3, and the envelope Γ_3 belong to both. We can thus apply the reciprocation method of (D) 2 simultaneously to both.

Using dashes, where necessary, for the second pair, we thus have the following transformation from the first to the second pair; the constants $s, t, p, \ldots$, are so arranged for convenience or generality:

$$(1) \qquad \begin{pmatrix} C, & S, & T, & \Gamma_3, & \phi_3, & \xi, & \Gamma, & \Sigma_1, & \Sigma_2 \\ C, & sS, & tT, & p\Gamma_3, & q\phi_3, & (p/q)\xi', & p\Gamma', & q\Sigma_1', & q\Sigma_2' \end{pmatrix}.$$

We have now the following identities†

$$3\Gamma = 3\Gamma_3 + 2\xi\phi_3, \qquad 3\Gamma' = 3\Gamma_3 + 2\xi'\phi_3,$$

whence

$$(2) \qquad 3(\Gamma - \Gamma') = 2\phi_3(\xi - \xi').$$

$$\phi_3 = \xi^2 + \Sigma_1 - \Sigma_2, \qquad q\phi_3 = \frac{p^2}{q^2}\xi'^2 + q(\Sigma_1' - \Sigma_2'),$$

---

* The six edges touch the Hessian at these summits.

† (D) 2, 356, *et seq.*

whence

$$q^3\xi^2-p^2\xi'^2+q^3(\Sigma_1-\Sigma_2-\Sigma_1'+\Sigma_2')=0. \tag{3}$$

$$S-\phi_3^2=3(\xi\Gamma_3+\Sigma_1\Sigma_2), \qquad sS-q^2\phi_3^2=3\left(\frac{p^2}{q}\xi'\Gamma_3+q^2\Sigma_1'\Sigma_2'\right),$$

whence

$$q(q^2-s)S=3\Gamma_3(q^3\xi-p^2\xi')+3q^3(\Sigma_1\Sigma_2-\Sigma_1'\Sigma_2'). \tag{4}$$

$$T=27\Gamma^2-4\phi_3(\phi_3+3\Sigma_2)(\phi_3-3\Sigma_1),$$

$$=8\phi_3^3-12\xi^2\phi_3^2+36\Sigma_1\Sigma_2\phi_3+27\Gamma^2,$$

since $\Sigma_1-\Sigma_2=\phi_3-\xi^2$,

$$tT=8q^3\phi_3^3-12p^2\xi'^2\phi_3^2+36q^3\Sigma_1'\Sigma_2'\phi_3+27p^2\Gamma'^2,$$

whence

$$(q^3-t)T=12\phi_3^2(p^2\xi'^2-q^3\xi^2)+36q^3\phi_3(\Sigma_1\Sigma_2-\Sigma_1'\Sigma_2')+27(q^3\Gamma^2-p^2\Gamma'^2). \tag{5}$$

Now Γ and Γ' have as common tangent planes the four tangent planes to C at the parabolic points on 12 and 56. These planes do not touch ϕ_3, for the locus of the points of contact on C of the common tangent planes of ϕ_3 and C lies on the surface* $T_3^2+2S_1S_2=0$, which does not pass through the parabolic points on 12 and 56. Thus, since the four planes meet at (12, 56), this point, from (2), must be the point $\xi-\xi'=0$. We can thus state the result:

The line which joins the two X points of the two trihedral pairs passes through the point (12, 56).

Further, Σ_1, Σ_2, Σ_1', Σ_2' have the four common tangent planes (1, 2′, 12), (1′, 2, 12), (5, 6′, 56), (5′, 6, 56) which meet at (12, 56). It follows from (3) that $p^2=q^3$. Also, since these planes do not touch S, it follows from (4) that $s=q^2$. Thus (3) and (4) may now be written

$$\xi^2-\xi'^2+\Sigma_1-\Sigma_2-\Sigma_1'+\Sigma_2'=0, \tag{6}$$

$$\Gamma_3(\xi-\xi')+\Sigma_1\Sigma_2-\Sigma_1'\Sigma_2'=0. \tag{7}$$

---

* (D) 2, 356.

With $t = q^3$, (5) can now be derived from the previous equations, as is to be expected. From (2) it is clear that Γ and Γ' have the same enveloping cone from (12, 56), and that their remaining common developable envelopes ϕ_3.

Reciprocate (7) now with respect to T_3 and we have the identity

$$\alpha_1 \alpha_2 S_1 S_2 - \alpha_1' \alpha_2' S_1' S_2' + CY = 0, \tag{8}$$

where α_1, α_2, α_1', α_2' are constants, and Y is the polar plane of (12, 56) with respect to T_3. But*

$$H - EC = 4(T_3^2 - S_1 S_2),$$

$$H - E'C = 4(T_3^2 - \beta_1 \beta_2 S_1' S_2'),$$

and so

$$C(E - E') = 4(S_1 S_2 - \beta_1 \beta_2 S_1' S_2'). \tag{9}$$

By a comparison of (8) and (9), we find that $\alpha_1' \alpha_2' = \alpha_1 \alpha_2 \beta_1 \beta_2$ and that $E - E'$ and Y must be the same plane.

Thus $E - E'$ is the polar plane of (12, 56) with respect to T_3.

6. *The Cremona representation of the surface.*

A cubic surface has thirty-six Cremona representations, to each of which corresponds one of the thirty-six double sixes, and therefore also one of the thirty-six Schur quadrics. The Cremona representation is particularly suitable for the investigation of properties of the ten trihedral pairs, all of which have a given double six associated with them. We give here a method which enables the forms associated with all ten to be written down from those of one, and we consider especially the equation of the common Schur quadric itself.

The equations

$$\sum_{p=1}^{6} X_p^3 = 0, \qquad \sum_{p=1}^{6} X_p = 0, \qquad \sum_{p=1}^{6} r_p X_p = 0$$

are taken to be associated with the standard double six for which 1 and 1′ are conjugate lines.

---

* (D) 2, 359. The multipliers of T_3^2 must be the same in the two identities because C, S_1, S_2, S_1', S_2' have common points not on H.

This representation and the trihedral representation I are taken to be connected by the equations

$$X_2+X_3=U, \qquad X_3+X_1=V, \qquad X_1+X_2=W,$$

$$X_5+X_6=-U', \qquad X_6+X_4=-V', \qquad X_4+X_5=-W',$$

$$A_1=\lambda_1-(r_2+r_3-r_1), \qquad B_1=\lambda_1-(r_3+r_1-r_2), \qquad C_1=\lambda_1-(r_1+r_2-r_3),$$

$$A_1'=\lambda_1-(r_5+r_6-r_4), \qquad B_1'=\lambda_1-(r_6+r_4-r_5), \qquad C_1'=\lambda_1-(r_4+r_5-r_6),$$

$$A_2=r_2+r_3-r_1-\lambda_2, \qquad B_2=r_3+r_1-r_2-\lambda_2, \qquad C_2=r_1+r_2-r_3-\lambda_2,$$

$$A_2'=r_5+r_6-r_4-\lambda_2, \qquad B_2'=r_6+r_4-r_5-\lambda_2, \qquad C_2'=r_4+r_5-r_6-\lambda_2,$$

$$A_3=B_3=C_3=A_3'=B_3'=C_3'=\lambda_2-\lambda_1,$$

where λ_1 and λ_2 are the roots of the equation

$$(r_2+r_3-r_1-\lambda)(r_3+r_1-r_2-\lambda)(r_1+r_2-r_3-\lambda)$$
$$=(r_5+r_6-r_4-\lambda)(r_6+r_4-r_5-\lambda)(r_4+r_5-r_6-\lambda).$$

By interchange of suffixes in these equations, we can obtain just the ten trihedral representations which are associated with the given double six. Thus, having obtained algebraical forms for one representation, we can write down the corresponding forms for any other of the ten, by suitable permutation.

Since T_3 is common to all the ten pairs, it is of interest to find a form for its equation which is unaltered by permutation.

$T_3\equiv\Sigma A_1A_2U^2-\Sigma A_1'A_2'U'^2$ (where the summations are for three terms)

$$\equiv-\Sigma\{\lambda_1-(r_2+r_3-r_1)\}\{\lambda_2-(r_2+r_3-r_1)\}(X_2+X_3)^2$$
$$+\Sigma\{\lambda_1-(r_5+r_6-r_4)\}\{\lambda_2-(r_5+r_6-r_4)\}(X_5+X_6)^2.$$

Thus $(r_1+r_2+r_3-r_4-r_5-r_6)\,T_3$ may be written

$$\Sigma(R_1-r_5-r_6+r_4)(R_1-r_6-r_4+r_5)(R_1-r_4-r_5+r_6)$$
$$\times(X_2+X_3)(X_1+X_4+X_5+X_6)$$
$$+\Sigma(R_4'-r_2-r_3+r_1)(R_4'-r_3-r_1+r_2)(R_4'-r_1-r_2+r_3)$$
$$\times(X_5+X_6)(X_1+X_2+X_3+X_4),$$

where $R_1=r_2+r_3-r_1, \ldots, \quad R_4'=r_5+r_6-r_4, \ldots.$

We write down the ten forms of this type obtained by associating the suffix 1 with all pairs chosen from 2, 3, 4, 5, 6 and add the right-hand sides, thus obtaining a symmetrical form for T_3.

The term X_1X_2 occurs only twice in each of the four forms associated with the groupings 123, 124, 125, 126, and the coefficients of X_1X_2 in these forms add up directly to

$$8(r_3+r_4-r_5-r_6)(r_3+r_5-r_6-r_4)(r_3+r_6-r_4-r_5).$$

The term X_1X_2 occurs four times in each of the remaining six forms and the sum of the coefficients is easily seen to be

$$12(r_1-r_2)^2(2r_1+2r_2-r_3-r_4-r_5-r_6)$$
$$+12(r_3+r_4-r_5-r_6)(r_3+r_5-r_6-r_4)(r_3+r_6-r_4-r_5).$$

By squaring the identity $\sum_{p=1}^{6} r_p X_p = 0$ and making replacements of the type $-X_1(X_2+X_3+X_4+X_5+X_6)$ for X_1^2, we find that

$$\Sigma(r_1-r_2)^2 X_1X_2 \equiv 0 \quad \text{(fifteen terms)},$$

and by a similar operation upon the identity

$$\left(\sum_{p=1}^{6} r_p X_p\right)\left(\sum_{p=1}^{6} r_p^2 X_p\right) \equiv 0,$$

we find that $\Sigma(r_1+r_2)(r_1-r_2)^2 X_1X_2 \equiv 0.$

Thus the symmetrical form of the equation of T_3 reduces to

$$\Sigma(r_3+r_4-r_5-r_6)(r_3+r_5-r_6-r_4)(r_3+r_6-r_4-r_5)X_1X_2 = 0,$$

the summation being for fifteen terms.

The equations of the lines pq are, agreeing with previous notation,

$$23, \quad X_1+X_4 = X_2+X_5 = X_3+X_6 = 0,$$
$$31, \quad X_1+X_6 = X_2+X_4 = X_3+X_5 = 0,$$
$$12, \quad X_1+X_5 = X_2+X_6 = X_3+X_4 = 0,$$
$$56, \quad X_1+X_4 = X_2+X_6 = X_3+X_5 = 0,$$
$$64, \quad X_1+X_6 = X_2+X_5 = X_3+X_4 = 0,$$
$$45, \quad X_1+X_5 = X_2+X_4 = X_3+X_6 = 0,$$

with the nine others which have been previously determined; for example, 14, which is the intersection of $U = 0$ and $U' = 0$, is given by

$$14, \quad X_1+X_4 = X_2+X_3 = X_5+X_6 = 0.$$

The interchanges, which correspond to an interchange of the lines $1, 1'$ with $4, 4'$, for example, in the double six, are interchanges of suffixes 1 and 4, 2 and 3, 5 and 6 in the Cremona planes, namely of the pairs of suffixes which occur in the equations of the line 14. This rule gives the permutations in the Cremona representation consequent on any geometrical interchange of pairs of lines in the double six.

7. *The X points of the surface.*

Milne's F points and our X points have been derived by means of different groupings; for example, Milne's F_{123} and F_{456} are our X_2 and X_3. Milne's derivation shows that each X point is associated with three non-intersecting lines of the surface. Thus, if we consider all double sixes, then the notation $X_{1,2,3}$, $X_{1,1',23}$, $X_{56,64,45}$, ... serves for all the X points of the surface. By a consideration of a table of the thirty-six double sixes and use of the results of § 4 it is evident that the points $X_{1,2,3}$, $X_{1',2',3'}$, $X_{23,31,12}$, $X_{14,24,34}$, $X_{15,25,35}$, $X_{16,26,36}$ are all the same point*, namely the point X_2 of § 4.

There are thus 120 X points in all, one Dixon X point for each trihedral pair.

We first consider the twenty X points $X_{p,q,r}(p, q, r = 1, 2, 3, 4, 5, 6)$ associated with a given double six for which 1 and $1'$ are conjugate lines. It is easily seen that the point X_{123}, which is $U = B_1C_2 - B_2C_1$, ..., $U' = -(B_1'C_2' - B_2'C_1')$, ..., is the point

$$X_1 = r_2 - r_3, \quad X_2 = r_3 - r_1, \quad X_3 = r_1 - r_2,$$

$$X_4 = r_5 - r_6, \quad X_5 = r_6 - r_4, \quad X_6 = r_4 - r_5.$$

The interchanges which correspond to an interchange of the lines $3, 3'$ with $4, 4'$ are interchanges of suffixes 1 and 3, 4 and 5, 2 and 6 in the

* *Cf.* (M) 1, 137. The trios of such lines appear as rows and columns in the trihedral pairs III and I respectively.

Cremona representation. Thus the point X_{124} is the point

$$X_1 = r_3 - r_6, \quad X_2 = r_5 - r_4, \quad X_3 = r_6 - r_1,$$

$$X_4 = r_2 - r_5, \quad X_5 = r_4 - r_2, \quad X_6 = r_1 - r_3.$$

The line which joins X_{123}, X_{124} therefore passes through the point

$$X_1 = r_2 - r_6, \quad X_2 = r_3 - r_1 + r_5 - r_4, \quad X_3 = r_6 - r_2,$$

$$X_4 = r_2 - r_6, \quad X_5 = r_6 - r_2, \quad X_6 = r_1 - r_3 + r_4 - r_5,$$

which is the point (12, 34).

Similarly the line which joins X_{456}, X_{356} passes through (56, 34), and the line which joins $X_{1,1',23}$, $X_{1,1',24}$ passes through (12, 56)*.

Twenty-seven such lines pass through each X point of the surface and twelve such lines pass through each point of intersection of lines on the surface. By making obvious interchanges and considering the points in pairs, we find that the four points X_{123}, X_{124}, X_{125}, X_{126} lie in a plane through the line 12. It can be easily verified that the equation of this plane is

$$(r_1 + r_5)(X_1 + X_5) + (r_2 + r_6)(X_2 + X_6) + (r_3 + r_4)(X_3 + X_4) = 0,$$

and so passes through the point of intersection of the planes

$$X_1 = X_5, \quad X_2 = X_6, \quad X_3 = X_4.$$

Also, by considering the points X_{123}, X_{124}, X_{134}, X_{234} in pairs, we find that the points form a quadrangle† whose diagonal points are the points (12, 34), (13, 24), (14, 23). So Milne's plane π_{1234} is the plane of these points of the surface.

Thus the X points of the surface lie by fours on two different types of planes.

First type. Planes through a line in the cubic surface.

There are 540 planes of the first type, twenty through each line of the surface and eighteen through each X point. The eighteen planes through

* As proved otherwise in § 5. $X_{1,1',23}$, $X_{2,2',31}$, $X_{3,3',12}$, $X_{4,4',56}$, $X_{5,5',64}$, $X_{6,6',45}$ are all the same point, namely Dixon's X point associated with I.

† *Cf.* (M) 3, 250, where it is proved that these points are coplanar.

each X point pass through the eighteen lines which are represented by the suffixes in the six-fold representation of the point, *i.e.* the eighteen lines of the three associated double sixes which have that X point as a Dixon X point.

Second type. Planes through three points on the cubic surface.

There are 270 planes of the type π_{1234}. Through each point of intersection of lines on the surface pass six of these planes and through each X point pass nine. For example, the nine planes through $X_{1,1',23}$, whose tangential equation in Dixon's notation is $\xi = 0$, are

$\{(1,4'),(1',4),(23,56)\}$; $\{(1,5'),(1',5),(23,64)\}$; $\{(1,6'),(1',6),(23,45)\}$;

$\{(2,4'),(2',4),(31,56)\}$; $\{(2,5'),(2',5),(31,64)\}$; $\{(2,6'),(2',6),(31,45)\}$;

$\{(3,4'),(3',4),(12,56)\}$; $\{(3,5'),(3',5),(12,64)\}$; $\{(3,6'),(3',6),(12,45)\}$.

Such a set of nine concurrent planes can thus be written down by an obvious process upon the table of two associated trihedral pairs.

The X points can thus be derived directly from the 135 points of intersection of lines on the surface.

The 270 planes π_{1234} are most simply considered as follows. On each line of the surface as base there are five triangles in the surface. The five vertices form a pentad. The 270 faces of these twenty-seven pentads are the π planes. A set of nine concurrent planes contains one face of each of nine pentads associated with the lines of a trihedral pair.

Milne* has proved that the plane π_{1234} touches the envelope Γ_3 and five others of the same type, and touches the envelope Γ and three others of the same type. Since the reciprocal of 12 with respect to T_3 is the line which joins (1, 2′), (1′, 2), it is evident that the twenty-seven lines of Γ_3 are the twelve lines of the double six associated with T_3 together with the fifteen lines which join pairs of points such as (1, 2′), (1′, 2) (fifteen edges, one from each of fifteen pentads of the twenty-seven). Thus, since the line (1, 2′), (1′, 2) for example lies in three faces of such a pentad, it is clear that these faces, which are π planes, not only touch Γ_3, but pass through a line of Γ_3. In fact, Γ_3 touches forty-five such π planes, three through each of the fifteen edges (p, q'), (p', q) $(p, q = 1, 2, 3, 4, 5, 6)$. It is also clear that each

* (M) 3, 250.

of the π planes touches six of the thirty-six Γ_3 envelopes of the surface, for each edge of a pentad is a line of two such envelopes. Further*, the nine π planes through $X_{1,1',23}$ belong to this set of forty-five and so all touch Γ_3. Thus it is evident, from the identity $3\Gamma = 3\Gamma_3 + \xi\phi_3$, that these nine planes also touch Γ, and it is easy to see, by considering the four X points on a π plane, that four Γ envelopes touch each π plane.

8. *The E planes of the surface.*

Associated with each trihedral summit plane there is an E plane. Thus there are forty E planes in all. Considering the ten sets of three associated trihedral pairs all of which have T_3 in common, we can denote the corresponding ten E planes by $E_{1,p,q}$ $(p, q = 2, 3, 4, 5, 6)$, so that $E_{1,p,q}$ is the polar plane of $X_{1,p,q}$.

Now the polar plane of (12, 56), with respect to T_3, is the plane† $E_{123} - E_{124}$. Similarly the polar planes of (12, 64) and (12, 45), with respect to T_3, are $E_{123} - E_{125}$ and $E_{123} - E_{126}$. Thus these three planes all pass through the polar line of 12, namely the line which joins (1, 2′), (1′, 2). It follows that the four planes E_{123}, E_{124}, E_{125}, E_{126} are concurrent at a point on this line.

Thus the forty E planes meet in fours at 270 points, one on each of the edges of the twenty-seven pentads. The E planes which meet on the edges of a given pentad are all different, and so all forty planes occur in this manner. The twenty-seven points of concurrence which lie in any one E plane are the points of intersection of this E plane with the twenty-seven lines of type (1, 2′), (1′, 2) in the three associated trihedral pairs which correspond to the E plane. It is easily shown that these twenty-seven lines lie by threes on twenty-seven planes, three planes through a line. For example, by reciprocation with respect to three suitable Schur quadrics, we find that the three lines (1, 2′), (1′, 2); (5, 6′), (5′, 6); (25, 16), (26, 15) meet in pairs. Thus the twenty-seven points of concurrence in an E plane lie by threes on twenty-seven lines, three lines through a point.

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* The forty-five planes divide into ten sets of nine concurrent planes concurrent in the ten Dixon X points of the ten trihedral pairs which contain the double six associated with Γ_3. Each plane occurs in two sets.

† *Vide* the last result of § 5.

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ON INEQUALITIES FOR CERTAIN INTEGRALS IN THE THEORY OF PICARD FUNCTIONS

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I.

1.1. *Introduction.* If P is a Picard function, that is, a function regular in $|s|<1$, and one not assuming the values 0, 1, and $\bar{P}$ is a superordinate of P, then it is known that

(i) $$\frac{1}{2\pi}\int_0^{2\pi}\{\log^+|\bar{P}(\rho e^{i\theta})|\}^\mu d\theta < B(\mu) \quad (0<\mu<1),$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi}\log^+|P(\rho e^{i\theta})|\,d\theta < A\log\frac{1}{1-\rho}+B,$$

(iii) $$\frac{1}{2\pi}\int_0^{2\pi}|\log P(\rho e^{i\theta})|\,d\theta < B\left(\log\frac{1}{1-\rho}\right)^2+B,$$

(iv) $$\frac{1}{2\pi}\int_0^{2\pi}|\log P(\rho e^{i\theta})|^2 d\theta < \frac{B}{1-\rho}.$$

A is an absolute constant, B is a constant depending on P, and $B(\mu)$ a constant depending on P and μ.

1.2. The results (i), (iii), and (iv) were proved by J. E. Littlewood* with the help of modular functions. The function $\bar{P}(s)$ is related to the modular function $k^2(\tau)$ by the equation $\bar{P}(s)=k^2(\tau)$, $s=(T-T_0)/(T-T_0')$, T_0 is a value of τ in the fundamental region of $k^2(\tau)$, and T_0' is the conjugate of T_0. By the theory of superordinate and subordinate functions† it is enough to prove (iii) and (iv) for the functions $\bar{P}$.

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* J. E. Littlewood, "On inequalities in the theory of functions", *Proc. London Math. Soc.* (2), 23 (1924), 481–519.

† See J. E. Littlewood, *loc cit.*

1.3. The result (ii) was proved by R. Nevanlinna*. He proved that (ii) is a best possible inequality† and determined the constant A‡. In the determination of A, Nevanlinna makes use of a differential inequality.

1.4. If $s = s(Z)$ conformally represents the regular Riemann surface $R(a_1, a_2, \ldots, a_{q-1}, \infty)$, $q \geqslant 3$, with winding points of infinite order at $a_1, a_2, \ldots, a_{q-1}, \infty$ on the interior of the circle $|s| < 1$, then, when $a_1 = 0$, $a_2 = 1$, $q = 3$, $Z = Z(s)$ is a $\bar{P}$. The functions Z include the $\bar{P}$ functions. In the present paper the following two theorems are established for the functions Z.

THEOREM A.

(i) $$\frac{1}{2\pi}\int_0^{2\pi} \{\log^+ |Z(\rho e^{i\theta})|\}^\mu \, d\theta < \frac{A}{1-\mu}$$

$$(0 < \mu < 1, \quad 0 < B \leqslant \rho < 1),$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |Z(\rho e^{i\theta})| \, d\theta < A \log \frac{1}{1-\rho} + B,$$

(iii) $$\left\{\frac{1}{2\pi}\int_0^{2\pi} [\log |Z(\rho e^{i\theta})|]^\mu \, d\theta\right\}^{1/\mu} < B\left(\frac{1}{1-\rho}\right)^{1-(1/\mu)} \qquad (\mu > 1).$$

THEOREM B.

(i) $$\frac{1}{2\pi}\int_0^{2\pi} |\log P(\rho e^{i\theta})|^\mu \, d\theta < A \frac{\mu^2}{1-\mu} \log \frac{1}{1-\rho} + B \quad (0 < \mu < 1),$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi} |\log P(\rho e^{i\theta})| \, d\theta < A\left(\log \frac{1}{1-\rho}\right)^2 + B,$$

(iii) $$\frac{1}{2\pi}\int_0^{2\pi} |\log P(\rho e^{i\theta})|^\mu \, d\theta < \frac{B}{\mu-1}\left(\frac{1}{1-\rho}\right)^{\mu-1} \qquad (1 < \mu \leqslant 2),$$

(iv) $$\left\{\frac{1}{2\pi}\int_0^{2\pi} |\log P(\rho e^{i\theta})|^\mu \, d\theta\right\}^{1/\mu} < B\left(\frac{1}{1-\rho}\right)^{1-(1/\mu)} \qquad (\mu > 2).$$

A is a constant depending on the Riemann surface R, and B a constant depending on the special function Z or $\bar{P}$.

1.5. Theorem A (i) is an extension of (i) to a somewhat larger class of functions than the $\bar{P}$ functions, and in addition gives a form of $B(\mu)$.

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* R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions meromorphes* (1929), 154.

† *Ibid.*, § 72.

‡ R. Nevanlinna, "Sur les valeurs exceptionelles . . .", *Bull. de la Sc. Math.*, 55 (1927), 92–101.

Theorem A (ii) is a special case of (ii).

Theorem B (ii) is similar to (iii), and (iv) is included in Theorem B (iii).

1.6. The paper is divided into sections. In the first section Theorem A is established. With the help of Green's theorem a differential inequality is obtained for a function $\lambda = \lambda(\rho) = \lambda(\rho, \mu)$, which is substantially equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi} \{\log^+|Z(\rho e^{i\theta})|\}^\mu \, d\theta.$$

To prove Theorem B, Green's theorem is again used to furnish a differential inequality. The solution of this inequality is made to depend on Theorem A.

1.7. This investigation was undertaken on the advice of Prof. J. E. Littlewood. I am also much indebted to Prof. Littlewood for many valuable suggestions and criticisms.

II.

2.1. In this section Theorem A and a corollary are proved.

THEOREM A. *Let $s = s(Z)$ be a function which conformally represents the regular Riemann surface $R(a_1, a_2, \ldots, a_{q-1}, \infty)$, where $q \geqslant 3$, on the interior of the circle $|s| < 1$. Then*

(i) $$\frac{1}{2\pi}\int_0^{2\pi} \{\log^+|Z(\rho e^{i\theta})|\}^\mu \, d\theta < \frac{A}{1-\mu} \qquad (0 < \mu < 1, \ 0 < B \leqslant \rho < 1),$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi} \log^+|Z(\rho e^{i\theta})| \, d\theta < A \log\frac{1}{1-\rho} + B,$$

(iii) $$\left\{\frac{1}{2\pi}\int_0^{2\pi} [\log^+|Z(\rho e^{i\theta})|]^\mu \, d\theta\right\}^{1/\mu} < \left(\frac{B}{1-\rho}\right)^{1-1/\mu} \qquad (\mu > 1),$$

A is a constant depending on R, and B a constant depending on the special function Z.

COROLLARY. *If $a_1 = 0$, Theorem A is true with $\big|\log|Z|\big|$ in place of $\log^+|Z|$.*

2.2. We suppose first that $a_1 = 0$.

Let $$V = 1 + \log(1 + |Z|).$$

Consider Green's theorem in the form

$$(2.21) \qquad -\int_L \phi \frac{\partial \psi}{\partial n}\, ds = \iint_D \left\{ \frac{\partial \phi}{\partial \xi}\frac{\partial \psi}{\partial \xi} + \frac{\partial \phi}{\partial \eta}\frac{\partial \psi}{\partial \eta} + \phi\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right)\right\} d\xi\, d\eta,$$

where D is the interior of the circle $|s| = \rho < 1$, L is the circumference of this circle, $\partial/\partial n$ means differentiation along the inward drawn normal.

If we put

$$\phi = V^{\mu-1}, \quad \psi = V,$$

$$\lambda = \lambda(\rho) = \lambda(\rho, \mu) = \frac{1}{2\pi}\int_0^{2\pi} V^{\mu}(\rho e^{i\theta})\, d\theta, \quad \lambda' = \frac{d\lambda}{d\rho},$$

we obtain

$$\frac{2\pi}{\mu}(\rho\lambda') = \iint_D \left[(\mu-1)\, V^{\mu-2}\left\{\left(\frac{\partial V}{\partial \xi}\right)^2 + \left(\frac{\partial V}{\partial \eta}\right)^2\right\} + V^{\mu-1}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right)\right] d\xi\, d\eta.$$

Differentiating with respect to ρ and dividing by $2\pi\rho$, we obtain

$$(2.22) \quad \frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda') = \frac{1}{2\pi}\int_0^{2\pi}\left[(\mu-1)\, V^{\mu-2}\left\{\left(\frac{\partial V}{\partial \xi}\right)^2 + \left(\frac{\partial V}{\partial \eta}\right)^2\right\} + V^{\mu-1}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right)\right] d\theta.$$

2.3. We now evaluate the right-hand side of (2.22). Straightforward calculation gives

$$(2.31) \qquad \left(\frac{\partial V}{\partial \xi}\right)^2 + \left(\frac{\partial V}{\partial \eta}\right)^2 = \frac{1}{(1+|Z|)^2}\left|\frac{dZ}{ds}\right|^2$$

and

$$(2.32) \qquad \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = \frac{1}{|Z|(1+|Z|)^2}\left|\frac{dZ}{ds}\right|^2.$$

We have* to evaluate $|dZ/ds|$.

The function $s(Z)$ conformably represents the regular Riemann surface $R(a_1, a_2, \ldots, a_{q-1}, \infty)$ on $|s| < 1$. Hence

$$(2.33) \qquad W = \frac{2|ds/dZ|}{1-|s|^2}$$

is a uniform† function of Z.

W is continuous at all points in the Z-plane except the winding points. As Z describes a closed contour round one of the winding points $Z = a_\nu$

* Compare R. Nevanlinna's *Fonctions meromorphes* (1929).

† F. Nevanlinna, *Acta Math.*, 50 (1927), 159–188.

s undergoes a linear parabolic transformation, which transforms $|s| \leqslant 1$ into itself. This transformation is necessarily of the form

$$\frac{s+e^{i\alpha}}{s-e^{i\alpha}} = \frac{s'+e^{i\alpha}}{s'-e^{i\alpha}} + i\beta,$$

where α and β are real. If we put

$$\Sigma = \Sigma(s) = \frac{2\pi}{|\beta|} \frac{s+e^{i\alpha}}{s-e^{i\alpha}}, \tag{2.34}$$

the function $\phi(Z) = e^{\Sigma\{S(z)\}}$ is uniform in the neighbourhood of $Z = a$. Since the real part of Σ is negative for $|s| < 1$, $\phi(Z)$ remains finite. It is thus regular in the neighbourhood of $Z = a$; and since its argument increases by $\pm 2\pi$ as a closed contour round a is described, $\phi(Z)$ has a simple zero at $Z = a$; so that in the neighbourhood of $Z = a$

$$\phi(Z) = \gamma_1(Z-a) + \ldots \quad (\gamma_1 \neq 0),$$

and

$$\Sigma(s) = \log(Z-a) + C_0 + C_1(Z-a) + \ldots \tag{2.35}$$

In the neighbourhood of ∞, this gives

$$\Sigma(s) = \log \frac{1}{Z} + C_0 + \frac{C_1}{Z} + \ldots. \tag{2.351}$$

From (2.34) and (2.35) we have

$$\frac{d\Sigma}{dZ} = \frac{4\pi}{|\beta|} \frac{|ds/dZ|}{|s-e^{i\alpha}|^2} = \frac{1}{|Z-a|} |1 + C_1(Z-a) + \ldots|$$

and

$$R(\Sigma) = -\frac{2\pi}{|\beta|} \frac{1-|s|^2}{|s-e^{i\alpha}|^2} = \log|Z-a| + R\{C_0 + C_1(z-a) + \ldots\}$$

and so, from (2.33),

$$W = \frac{1}{|Z-a|} \frac{|1 + C_1(Z-a) + \ldots|}{\log(1/|Z-a|) - R\{C_0 + C_1(Z-a) + \ldots\}}$$

$$= \frac{1}{|Z-a|} \frac{1}{\log(1/|Z-a|)} \left\{1 + O\left(\frac{1}{\log(1/|Z-a|)}\right)\right\}, \tag{2.36}$$

giving, near $Z = a$, by (2.33),

$$\left|\frac{dZ}{ds}\right| = \frac{2}{1-|s|^2} |Z-a| \log \frac{1}{|Z-a|} \left\{1 + O\left(\frac{1}{\log(1/|Z-a|)}\right)\right\}. \tag{2.37}$$

In the neighbourhood of ∞ this becomes

$$\left|\frac{dZ}{ds}\right| = \frac{2}{1-|s|^2}|Z|\log|Z|\left\{1+O\left(\frac{1}{\log|Z|}\right)\right\}. \tag{2.371}$$

For all other points in the Z plane,

$$\left|\frac{dZ}{ds}\right| = O\left(\frac{2}{1-|s|^2}\right). \tag{2.372}$$

The calculation for $|d\bar{P}/ds|$ may be done somewhat more easily; we have

$$\frac{2|ds/d\bar{P}|}{1-|s|^2} = \frac{|d\tau/dk^2|}{v}, \quad \bar{P}(s) = k^2(\tau), \quad \tau = u+iv.$$

To evaluate the expression on the right-hand side, since the expression is a uniform function of k^2, it is sufficient to consider a value of τ in a fundamental region. The evaluation may be done with the help of the ϑ functions. The results are identical with (2.37).

2.4. Suppose that circles of radius $\delta < e^{-1}$ are drawn* in the Z plane with centres at each of the winding points $a_1, a_2, \ldots, a_{q-1}$, and a circle of radius $1/\delta$ with centre at the origin. Further choose δ so small that none of the circles meet.

When we use (2.31) and (2.32) the equation (2.22) becomes

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda') = \frac{1}{2\pi}\left(\sum_{s=1}^{q-1}\int_{|Z-a_s|\leqslant\delta} + \int_{|Z|>1/\delta} + \int_L\right)(J_1+J_2)\,d\theta, \tag{2.41}$$

where L represents the circle $|s| = \rho$, exclusive of the points at which $|Z-a_s| \leqslant \delta$, and $|Z| \geqslant 1/\delta$,

$$J_1 = \frac{(\mu-1)V^{\mu-2}}{(1+|Z|)^2}\left|\frac{dZ}{ds}\right|^2, \qquad J_2 = \frac{V^{\mu-1}}{|Z|(1+|Z|)^2}\left|\frac{dZ}{ds}\right|^2,$$

and $|dZ/ds|^2$ is given by the equations (2.37).

2.5. The letters A and B will be used for positive numbers (not necessarily always the same) independent of ρ and μ. A may depend on the Riemann surface R $(a_1, a_2, \ldots, a_{q-1}, \infty)$. B may depend on Z. When Z is a $\bar{P}$, A will be an absolute constant.

2.6. Consider the neighbourhood of a finite winding point $|Z-a| \leqslant \delta$. From (2.37) we have

$$\left|\frac{dZ}{ds}\right|^2 = \left(\frac{2}{1-\rho^2}\right)^2|Z-a|^2\left(\log\frac{1}{|Z-a|}\right)^2\left(1+\frac{1}{O\{\log 1/(|Z-a|)\}}\right),$$

* See F. Nevanlinna, *loc. cit.*

and, since $|Z-a|^2\{\log(1/|Z-a|)\}^2<A$, $|Z-a|\{\log(1/|Z-a|)\}^2<A$, $V\geqslant 1$, and $\mu>0$, we have

$$J_1+J_2<\left(\frac{2}{1-\rho^2}\right)^2\{A(\mu-1)V^{\mu-2}+AV^{\mu-1}\}$$

$$\leqslant\left(\frac{2}{1-\rho^2}\right)^2\{A(\mu+1)V^{\mu-1}+AV^{\mu-1}\}$$

$$\leqslant\left(\frac{2}{1-\rho^2}\right)^2 A(\mu+1)V^{\mu-1}. \tag{2.61}$$

In the neighbourhood of the infinite winding point $|Z|\geqslant 1/\delta$ we have, by (2.371),

$$J_1=\left(\frac{2}{1-\rho^2}\right)^2(\mu-1)V^{\mu-2}\left(\frac{1}{1+|Z|}\right)^2|Z|^2(\log|Z|)^2\left\{1+O\left(\frac{1}{\log|Z|}\right)\right\}$$

$$=\left(\frac{2}{1-\rho^2}\right)^2(\mu-1)V^{\mu-2}\{(\log|Z|)^2+O(\log|Z|)\}.$$

Since $V=1+\log(1+|Z|)=\log|Z|+O(1)$ when $|Z|\geqslant 1/\delta$, we have

$$J_1=\left(\frac{2}{1-\rho^2}\right)^2(\mu-1)V^{\mu-2}\{V^2+O(V)\}$$

$$=\left(\frac{2}{1-\rho^2}\right)^2\{(\mu-1)V^{\mu}+(\mu-1)V^{\mu-2}O(V)\}$$

$$\leqslant\left(\frac{2}{1-\rho^2}\right)^2(\mu-1)V^{\mu}+\left(\frac{2}{1-\rho^2}\right)^2 A(\mu+1)V^{\mu-1}.$$

For J_2 we observe that

$$\frac{1}{|Z|}(\log|Z|)^2<A,$$

so that

$$J_2<\left(\frac{2}{1-\rho^2}\right)^2 AV^{\mu-1}.$$

Hence in the neighbourhood $|Z|>1/\delta$ of ∞, we have

$$J_1+J_2\leqslant\left(\frac{2}{1-\rho^2}\right)^2(\mu-1)V^{\mu}+\left(\frac{2}{1-\rho^2}\right)^2 A(\mu+1)V^{\mu-1}. \tag{2.611}$$

For all other values of Z, we have, by (2.372),

$$J_1+J_2<\left(\frac{2}{1-\rho^2}\right)^2 A(\mu+1)V^{\mu-1}.$$

When we use these inequalities for J_1+J_2, (2.41) reduces to

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda') < \left(\frac{2}{1-\rho^2}\right)^2 A(\mu+1)\frac{1}{2\pi}\int_0^{2\pi} V^{\mu-1}d\theta + \left(\frac{2}{1-\rho^2}\right)^2 \frac{1}{2\pi}\int_{|Z|\geqslant 1/\delta}(\mu-1)V^{\mu}d\theta.$$

Since $V<A$ for $|Z|<1/\delta$, this becomes finally

$$(2.62)\quad \frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda')$$

$$< \left(\frac{2}{1-\rho^2}\right)^2 \frac{1}{2\pi}\int_0^{2\pi}(\mu-1)V^{\mu}d\theta + \left(\frac{2}{1-\rho^2}\right)^2 A(\mu+1)\frac{1}{2\pi}\int_0^{2\pi} V^{\mu-1}d\theta$$

$$= \left(\frac{2}{1-\rho^2}\right)^2 (\mu-1)\lambda + A(\mu+1)\left(\frac{2}{1-\rho^2}\right)^2 \frac{1}{2\pi}\int_0^{2\pi} V^{\mu-1}d\theta.$$

2.7. It is convenient to consider (2.62) separately for the three cases $0<\mu<1$, $\mu=1$, and $\mu>1$, which give respectively

$$(2.71)\qquad \frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda') \leqslant (\mu-1)\left(\frac{2}{1-\rho_2}\right)^2\lambda + A\left(\frac{2}{1-\rho^2}\right)^2 \quad (0<\mu<1);$$

$$(2.72)\qquad \frac{1}{\rho}\frac{d}{d\rho}(\rho\lambda') < A\left(\frac{2}{1-\rho^2}\right)^2 \quad (\mu=1);$$

$$(2.73)\qquad \frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda') \leqslant (\mu-1)\left(\frac{2}{1-\rho^2}\right)^2\lambda + A\mu\left(\frac{2}{1-\rho^2}\right)^2 \frac{1}{2\pi}\int_0^{2\pi} V^{\mu-1}d\theta \quad (\mu>1).$$

2.8. To integrate* (2.71) we write

$$X = \frac{1}{1-\rho^2}, \quad \lambda = (X-1)^{\mu-1}Y.$$

(2.71) reduces to

$$\frac{1}{\mu}\left\{X(X-1)^{\mu}\frac{d^2Y}{dX^2} + (X-1)^{\mu-1}(2\mu X-1)\frac{dY}{dX} + (\mu-1)^2(X-1)^{\mu-2}Y\right\} < A,$$

and gives *à fortiori*

$$\frac{1}{\mu}\left\{X(X-1)^{\mu}\frac{d^2Y}{dX^2} + (X-1)^{\mu-1}(2\mu X-1)\frac{dY}{dX}\right\} < A.$$

Multiplying by $\mu(X-1)^{\mu-1}>0$, we obtain

$$\frac{d}{dX}\left\{X(X-1)^{2\mu-1}\frac{dY}{dX}\right\} < A\mu(X-1)^{\mu-1}.$$

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* For this method of integration I am indebted to Prof. J. E. Littlewood.

Integrating, we have

$$X(X-1)^{2\mu-1}\frac{dY}{dX} < A(X-1)^{\mu}+B$$

$$\leqslant A(X-1)^{\mu} \qquad (X>B).$$

Hence

$$\frac{dY}{dX} < \frac{A(X-1)^{1-\mu}}{X}$$

$$= A(X-1)^{-\mu}\frac{(X-1)}{X}$$

$$\leqslant A(X-1)^{-\mu} \qquad (X>B).$$

Integrating again, we have

$$Y < \frac{A(X-1)^{1-\mu}}{1-\mu}+B$$

$$\leqslant \frac{A(X-1)^{1-\mu}}{1-\mu} \qquad (X>B).$$

Hence

$$\lambda = (X-1)^{\mu-1}\,Y < \frac{A}{1-\mu},$$

that is to say

$$\lambda(\rho,\mu) < \frac{A}{1-\mu} \quad (0<\mu<1,\ 0<B\leqslant\rho<1). \tag{2.81}$$

2.9. The integral of (2.72) is immediate. We have in succession

$$\rho\lambda' < A\left(\frac{2}{1-\rho^2}\right) \quad (0\leqslant\rho<1),$$

$$\lambda' < \frac{A}{1-\rho} \qquad (0<\rho_0\leqslant\rho<1),$$

$$\lambda = \lambda(\rho,1) < A\log\frac{1}{1-\rho}+B. \tag{2.91}$$

2.10. To integrate (2.73) we observe that

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} V^{\mu}\,d\theta\right\}^{1/\mu}$$

is an increasing function of μ, in the wide sense; thus

$$\frac{1}{2\pi}\int_0^{2\pi} V^{\mu-1}\,d\theta \leqslant \left\{\frac{1}{2\pi}\int_0^{2\pi} V^{\mu}\,d\theta\right\}^{(\mu-1)/\mu} = \lambda^{(\mu-1)/\mu}.$$

Hence (2.73) may be written

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda') < (\mu-1)\left(\frac{2}{1-\rho^2}\right)^2\lambda + A\mu\left(\frac{2}{1-\rho^2}\right)^2\lambda^{(\mu-1)/\mu}.$$

The substitutions $X = 1/(1-\rho^2)$, $\lambda = (X-1)^{\mu-1}\,Y$ reduce this to

$$\frac{1}{\mu}\left\{X(X-1)^{\mu}\frac{d^2Y}{dX^2} + (X-1)^{\mu-1}(2\mu X-1)\frac{dY}{dX} + (\mu-1)^2(X-1)^{\mu-2}\,Y\right\}$$
$$< A\mu\,\lambda^{(\mu-1)/\mu},$$

whence, multiplying by μ, and observing that $(\mu-1)^2(X-1)^{\mu-2}\,Y > 0$, we have

$$X(X-1)^{\mu}\frac{d^2Y}{dX^2} + (X-1)^{\mu-1}(2\mu X-1)\frac{dY}{dX} < A\mu^2\lambda^{(\mu-1)/\mu}.$$

We multiply this by $(X-1)^{\mu-1} > 0$, obtaining

$$\frac{d}{dX}\left(X(X-1)^{2\mu-1}\frac{dY}{dX}\right) < A\mu^2(X-1)^{\mu-1}\lambda^{(\mu-1)/\mu}. \tag{2.101}$$

For $\mu > 1$, it follows from (2.41) that $d(\rho\lambda')/d\rho > 0$, hence $\rho\lambda' > 0$ and $\lambda' > 0$. Thus λ is an increasing function of ρ and so of X. Hence, integrating (2.101), we obtain

$$X(X-1)^{2\mu-1}\frac{dY}{dX} < A\mu^2\lambda^{(\mu-1)/\mu}\int_{X_0}^{X}(x-1)^{\mu-1}dx + B^{\mu}$$
$$\leqslant A\mu\,\lambda^{(\mu-1)/\mu}(X-1)^{\mu} + B^{\mu}$$
$$\leqslant A\mu\,\lambda^{(\mu-1)/\mu}(X-1)^{\mu} \qquad (X > B).$$

Substituting for λ and dividing by $X(X-1)^{2\mu-1} > 0$, we get

$$\frac{dY}{dX} < A\mu\,Y^{(\mu-1)/\mu}\frac{(X-1)^{(1-\mu)/\mu}}{X}$$
$$= A\mu\,Y^{(\mu-1)/\mu}(X-1)^{(1-2\mu)/\mu}\frac{X-1}{X}$$
$$\leqslant A\mu\,Y^{(\mu-1/\mu)}(X-1)^{(1-2\mu)/\mu} \qquad (X > B).$$

Thus
$$Y^{-(\mu-1)/\mu}\frac{dY}{dX} < A\mu(X-1)^{(1-2\mu)/\mu},$$

$$\mu\frac{dY^{1/\mu}}{dX} < A\mu(X-1)^{(1/\mu)-2}.$$

Dividing by $\mu > 0$ and integrating, we get

$$\begin{aligned} Y^{1/\mu} &< A\int_{X_0}^{X} \frac{1}{(x-1)^{2-1/\mu}}\,dx + B \\ &\leqslant A\int_{X_0}^{\infty} \frac{1}{(x-1)^{2-1/\mu}}\,dx + B \\ &\leqslant A+B \\ &= B. \end{aligned}$$

This gives

$$\lambda = (X-1)^{\mu-1}\,Y < B^{\mu}(X-1)^{\mu-1} = B^{\mu}\left(\frac{1}{1-r^2}\right)^{\mu-1}.$$

Replacing $1/(1-r^2)$ by $A/(1-r)$, and taking μ-th roots, we obtain the solution of (2.73), namely

$$\lambda^{1/\mu}(\rho,\,\mu) < \frac{B}{(1-r)^{1-1/\mu}} \quad (\mu > 1). \tag{2.102}$$

2.11. Since $V = 1+\log(1+|Z|)$ and $\log^+|Z| \leqslant \log(1+|Z|) < V$, we have

$$\{\log^+|Z|\}^{\mu} \leqslant V^{\mu},$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi}\{\log^+|Z|\}^{\mu}\,d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} V^{\mu}\,d\theta = \lambda(\rho,\,\mu).$$

Theorem A, in the case $a_1 = 0$, therefore, follows from (2.81), (2.91), (2.102).

If $a_1 \neq 0$, let $Z_1 = Z - a_1$; then Z_1 is a Z and has 0 as one winding point. The inequalities proved above apply to Z. Since

$$\begin{aligned} \log^+|Z| &= \log^+|Z_1+a_1| \leqslant \log^+|Z_1| + \log^+|a_1| + \log 2 \\ &< \log^+|Z_1| + A, \end{aligned}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\{\log^+|Z|\}^{\mu}\,d\theta < A^{\mu}\,\frac{1}{2\pi}\int_0^{2\pi}\{\log^+|Z_1|\}^{\mu}\,d\theta + A^{\mu}.$$

This establishes Theorem A in general.

2.12. To prove the corollary, we observe that, if $a_1 = 0$, $\Sigma = 1/Z$ is a Z, and hence Theorem A is applicable, with $\log^+\Sigma = \log^+ 1/Z$ in place of Z.

But $$\left|\log|Z|\right|^{\mu} = \{\log^+|Z|\}^{\mu} + \left\{\log^+\frac{1}{|Z|}\right\}^{\mu},$$

$$\frac{1}{2\pi}\int_0^{2\pi}|\log|Z||^{\mu}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\{\log^+|Z|\}^{\mu}\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\left\{\log^+\frac{1}{|Z|}\right\}^{\mu}d\theta,$$

and the corollary follows.

III.

3.1. In this section we prove Theorem B.

THEOREM B. *If P is a function regular and uniform in $|s|<1$, and if P does not take the values 0 or 1 in $|s|<1$, then*

(i) $$\frac{1}{2\pi}\int_0^{2\pi}|\log P(\rho e^{i\theta})|^{\mu}\,d\theta < \frac{A\mu^2}{1-\mu}\log\frac{1}{1-\rho}+B \quad (0<\mu<1);$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi}|\log P(\rho e^{i\theta})|\,d\theta < A\left(\log\frac{1}{1-\rho}\right)^2+B;$$

(iii) $$\frac{1}{2\pi}\int_0^{2\pi}|\log P(\rho e^{i\theta})|^{\mu}\,d\theta < \frac{B}{\mu-1}\left(\frac{1}{1-\rho}\right)^{\mu-1} \quad (1<\mu\leqslant 2);$$

(iv) $$\frac{1}{2\pi}\int_0^{2\pi}|\log P(\rho e^{i\theta})|^{\mu}\,d\theta\Big\}^{1/\mu} < \left(\frac{B}{1-\rho}\right)^{1-1/\mu} \quad (\mu>2);$$

where A is an absolute constant and B a constant depending on P.

3.2. Let Z be the function discussed in § II.

Suppose that $a_1=0$, $a_2=1$. Let $\phi=|\log Z|^{\mu-1}$, $\psi=|\log Z|=V_1$. If we write

$$\lambda_1=\lambda_1(\rho)=\lambda(\rho,\mu)=\frac{1}{2\pi}\int_0^{2\pi}|\log Z(\rho e^{i\theta})|^{\mu}\,d\theta,$$

Green's Theorem (2.21) gives as before

(3.21) $$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda_1') = \frac{1}{2\pi}\int_0^{2\pi}\Big[(\mu-1)V_1^{\mu-2}\left\{\left(\frac{\partial V_1}{\partial\xi}\right)^2+\left(\frac{\partial V_1}{\partial\eta}\right)^2\right\} + V_1^{\mu-2}\left(\frac{\partial^2 V_1}{\partial\xi^2}+\frac{\partial^2 V_1}{\partial\eta^2}\right)\Big]d\theta.$$

By straightforward calculations,

(3.22) $$\left(\frac{\partial V_1}{d\xi}\right)^2+\left(\frac{\partial V_1}{\partial\eta}\right)^2 = \frac{1}{|Z|^2}\left|\frac{dZ}{\partial s}\right|^2,$$

(3.23) $$\frac{\partial^2 V_1}{\partial\xi^2}+\frac{\partial^2 V_1}{\partial\eta^2} = \frac{1}{|Z|^2|\log Z|}\left|\frac{dZ}{ds}\right|^2.$$

On substituting in (3.21) and replacing V_1 by $|\log Z|$ we obtain

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda_1') = \frac{\mu}{2\pi}\int_0^{2\pi} \frac{|\log Z|^{\mu-2}}{|Z|^2}\left|\frac{dZ}{ds}\right|^2 d\theta. \tag{3.24}$$

3.3. From (2.371), when Z is in the neighbourhood of ∞, we have

$$\left|\frac{dZ}{ds}\right|^2 < A\left(\frac{2}{1-\rho^2}\right)^2 |Z|^2\{\log|Z|\}^2. \tag{3.31}$$

Since $\{\log 1/(|Z|)\}^2 = \{\log|Z|\}^2$, (2.37) shows that this inequality also applies to a value of Z in the neighbourhood of the winding point $Z=0$. From (2.37) and (2.372) it follows that the same inequality applies for all other values of Z. Hence (3.24) may be written

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda_1') < A\left(\frac{2}{1-\rho^2}\right)^2 \frac{\mu}{2\pi}\int_0^{2\pi} |\log Z|^{\mu-2}\,|\log|Z|^2\,d\theta. \tag{3.32}$$

We consider this inequality separately for the cases $0<\mu\leqslant 2$, $\mu>2$.

3.4. If $0<\mu\leqslant 2$, (3.32) gives, since $|\log Z| \geqslant \big|\log|Z|\big|$,

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda_1') \leqslant A\left(\frac{2}{1-\rho^2}\right)^2 \frac{\mu}{2\pi}\int_0^{2\pi} \Big|\log|Z|\Big|^{\mu-2}\Big|\log|Z|\Big|^2 d\theta$$

$$= A\left(\frac{2}{1-\rho^2}\right)^2 \frac{\mu}{2\pi}\int_0^{2\pi} \Big|\log|Z|\Big|^{\mu} d\theta. \tag{3.41}$$

3.5. For $0<\mu<1$, we have from Theorem A (i), corollary, and (3.41),

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda_1') < A\,\frac{\mu}{1-\mu}\left(\frac{2}{1-\rho^2}\right)^2 \quad (0<B\leqslant\rho<1). \tag{3.51}$$

Integrating, we have

$$\rho\lambda_1' < A\,\frac{\mu^2}{1-\mu}\,\frac{2}{1-\rho^2} \quad (0<B<\rho<1),$$

$$\lambda_1' < \frac{A\mu^2}{1-\mu}\,\frac{1}{1-\rho} \qquad (0\leqslant B<\rho<1).$$

Integrating again, we have

$$\lambda_1 < \frac{A\mu^2}{1-\mu}\log\frac{1}{1-\rho} + B, \tag{3.52}$$

[*Added in proof.*—Prof. J. E. Littlewood has pointed out to me that (3.52) may be written in a somewhat different way, giving a better inequality as ρ tends to 1, and showing further that (3.52) is not a best possible inequality.

From (3.52) we have

$$\lambda_1(1-\epsilon) < \frac{A}{\epsilon} \log \frac{1}{1-\rho} \quad (0 < B \leqq \rho < 1).$$

Since $\lambda_1^{1/\mu}$ is an increasing function of μ, we then have, if $\mu < 1-\epsilon$,

$$\lambda_1^{1/\mu}(\mu) < \left(\frac{A}{\epsilon} \log \frac{1}{1-\rho}\right)^{1/(1-\epsilon)},$$

and so

$$\lambda_1 < \frac{A}{\epsilon} \left(\log \frac{1}{1-\rho}\right)^{\mu(1+\epsilon)}.$$

This is the modification of (3.52).]

3.6. For $\mu = 1$, we have from Theorem A (ii), corollary, and (3.41),

$$\frac{1}{\rho} \frac{d}{d\rho} (\rho\lambda_1') < A \left(\frac{2}{1-\rho^2}\right)^2 \log \frac{1}{1-\rho} \quad (0 < \rho_0 \leqq \rho < 1),$$

$$\frac{d}{d\rho} (\rho\lambda_1') \leqq A \left(\frac{2}{1-\rho}\right)^2 \log \frac{1}{1-\rho} \quad (0 < \rho_0 \leqq \rho < 1).$$

Integrating, we have

$$\rho\lambda_1' < \frac{A}{1-\rho} \left(\log \frac{1}{1-\rho} - 1\right) + B,$$

$$\lambda_1' < \frac{A}{1-\rho} \log \frac{1}{1-\rho} \quad (0 < \rho_0 \leqq \rho < 1).$$

Integrating again, we have

$$\lambda_1 < A \left(\log \frac{1}{1-\rho}\right)^2 + B \quad (0 < \rho < 1). \tag{3.61}$$

3.7. For $1 < \mu \leqq 2$, by Theorem A (iii), corollary, (3.41) may be written

$$\frac{1}{\rho} \frac{d}{d\rho} (\rho\lambda_1') < B \left(\frac{2}{1-\rho^2}\right)^2 \frac{1}{1-\rho)^{\mu-1}}$$

giving

$$\frac{d}{d\rho} (\rho\lambda_1') < B \frac{1}{(1-\rho)^{\mu+1}} \quad (0 < \rho_0 \leqq \rho < 1).$$

Integrating, we have

$$\rho\lambda_1' < \frac{B}{(1-\rho)^{\mu-1}} + B,$$

$$\lambda_1' \leqq \frac{A}{(1-\rho)^{\mu}} \quad (0 < \rho_0 \leqq \rho < 1).$$

Integrating again, we have

$$\lambda_1 < \frac{B}{\mu-1} \frac{1}{(1-\rho)^{\mu-1}} + B$$

$$\leqq \frac{B}{(\mu-1)} \frac{1}{(1-\rho)^{\mu-1}}. \tag{3.71}$$

3.8. When $\mu > 2$ we must start from (3.32). We have

$$\frac{1}{2\pi}\int_0^{2\pi} |\log Z|^{\mu-2} \left|\log|Z|\right|^2 d\theta$$

$$\leqslant \left\{\frac{1}{2\pi}\int_0^{2\pi} |\log Z|^{\mu} d\theta\right\}^{(\mu-2)/\mu} \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\log|Z|\right|^{\mu} d\theta\right\}^{2/\mu}$$

$$\leqslant B\lambda_1^{(\mu-2)/\mu}\left(\frac{2}{1-\rho^2}\right)^{2-2/\mu},$$

by Theorem A (iii), corollary, so that (3.32) gives

$$\frac{1}{\mu\rho}\frac{d}{d\rho}(\rho\lambda_1') < B\mu\left(\frac{2}{1-\rho^2}\right)^{2-2/\mu}\lambda_1^{(\mu-2)/\mu}.$$

We make the substitutions $X = 1/(1-\rho^2)$, $\lambda_1 = (X-1)^{\frac{1}{2}\mu-1}\,Y$, and obtain

$$X(X-1)^{\frac{1}{2}\mu}\frac{d^2Y}{dX^2} + (\mu X-1)(X-1)^{\frac{1}{2}\mu-1}\frac{dY}{dX}$$

$$+[(\tfrac{1}{2}\mu-1)^2(X-1)^{\frac{1}{2}\mu-2} + (\tfrac{1}{2}\mu-1)(X-1)^{\frac{1}{2}\mu-1}]\,Y$$

$$< B\mu^2\lambda_1^{(\mu-2)/\mu}X^{2-2/\mu},$$

and hence (since $\mu > 2$)

$$X(X-1)^{\frac{1}{2}\mu}\frac{d^2Y}{dX^2} + (\mu X-1)(X-1)^{\frac{1}{2}\mu-1}\frac{dY}{dX} < B\mu^2\lambda_1^{(\mu-2)/\mu}X^{2-2/\mu}.$$

Multiplying by $(X-1)^{\frac{1}{2}\mu-1}$, we have

$$\frac{d}{dX}\left(X(X-1)^{\mu-1}\frac{dY}{dX}\right) < B\mu^2\lambda_1^{(\mu-2)/\mu}(X-1)^{\frac{1}{2}\mu-1}X^{2-2/\mu}$$

$$\leqslant B\mu^2\lambda_1^{(\mu-2)/\mu}(X-1)^{\frac{1}{2}\mu+1-2/\mu} \qquad (X > X_0).$$

Integrating, we have

$$X(X-1)^{\mu-1}\frac{dY}{dX} < B\mu\lambda_1^{(\mu-2)/\mu}(X-1)^{\frac{1}{2}\mu+2-2/\mu} + B^{\mu}$$

$$\leqslant B\mu\lambda_1^{(\mu-2)/\mu}(X-1)^{\frac{1}{2}\mu+2-2/\mu} = B\mu\,Y^{(\mu-2)/\mu}(X-1)^{\mu}$$

$$(X > X_0).$$

Dividing by $X(X-1)^{\mu-1}\,Y^{(\mu-2)/\mu}$, we get

$$Y^{-(\mu-2)/\mu}\frac{dY}{dX} < B\mu\frac{X-1}{X} < B\mu \quad (X > X_0).$$

Integrating, we have

$$\tfrac{1}{2}\mu Y^{2/\mu} < B\mu X + B\mu \leqslant B\mu(X-1),$$

$$Y < B^{\mu}(X-1)^{\frac{1}{2}\mu},$$

so that

$$\lambda_1 < B^{\mu}(X-1)^{\mu-1} \leqslant B^{\mu}\left(\frac{1}{1-\rho}\right)^{\mu-1}.$$

Thus, if $\mu > 2$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |\log Z|^{\mu}\right\}^{1/\mu} < B\left(\frac{1}{1-\rho}\right)^{1-1/\mu}. \tag{3.81}$$

3.9. Since $a_1 = 0$, and $a_1 = 1$, and it may be supposed that $q = 3$, inequalities (3.52), (3.61), (3.71), (3.81) prove Theorem B for the $\bar{P}$ functions.

The extension to the P functions may be made by the theory of superordinate and subordinate functions.

Let G_1 be a value of $\log P(0)$, then $\log P$ with $\log P(0) = G_1$ is an analytic function, regular and uniform in $|s| < 1$. $\log \bar{P}$ with $\log P(0) = G_1$ is a superordinate of $P(0)$ in $|s| \leqslant \rho < 1$, and hence

$$\frac{1}{2\pi}\int_0^{2\pi} |\log P(\rho e^{i\theta})|^{\mu} d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} |\log \bar{P}(\rho e^{i\theta})|^{\mu} d\theta.$$

This completes the proof of Theorem B.

ON WARING'S PROBLEM FOR ODD POWERS

By R. D. JAMES*.

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1. *Introduction.*

The first part of this paper is concerned with the representation of an integer n in the form

$$n = \sum_{i=1}^{s} u_i^k + \sum_{j=1}^{t} P(v_j), \qquad u_i \geqslant 0, \qquad N \geqslant v_j \geqslant M, \tag{1.1}$$

where

$$P(v) = N^{-1}\left((N+v)^k + (N-v)^k - 2N^k\right), \tag{1.2}$$

and N and M are integers depending on n. If s, s_1, and t are integers satisfying certain conditions, and if k is an odd integer greater than or equal to 5, we show that all sufficiently large numbers are sums of $s+s_1$ non-negative k-th powers and t values of $P(v)$. The proof is similar to that given by Gelbcke (1)† for the usual Waring problem.

In the second part we use this result, and the fact that the number of primes $d \equiv 2 \pmod{k}$, k odd, between x and $x+\epsilon x$ increases indefinitely with x, to prove that all sufficiently large numbers are sums of $s+s_1+2t+1$ non-negative k-th powers. Let $G(k)$ denote the least value of r such that all large numbers are sums of r non-negative k-th powers. If k is an odd integer greater than or equal to 5, we show that

$$G(k) \leqslant 2^{k-2}(k-3)+k+9+\zeta_k,$$

where

$$\zeta_k = \left[\frac{(k-2)\log 2 - \log k + \log(k-2)}{\log k - \log(k-1)}\right].$$

In particular, $G(5) \leqslant 35$, $G(7) \leqslant 164$, $G(9) \leqslant 824$, ... These values are an improvement on those previously known‡.

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† See the list of references at the end of the paper.

‡ Hardy and Littlewood (5) proved that

$$G(k) \leqslant 2^{k-2}(k-2)+k+5+\zeta_k,\ G(5) \leqslant 41,\ G(6) \leqslant 87,\ G(7) \leqslant 192,\ G(8) \leqslant 425,\ G(9) \leqslant 949,\ \ldots.$$

2. *Notation.*

Let k be an odd integer greater than or equal to 5. Let $a = 1/k$; $b = 1/(k-1)$; $K = 2^{k-1}$; $c = 2^{1-k} = 1/K$;

(2.1) $\qquad L = \frac{13}{80}$ when $k = 5$, $\qquad L = (2-2a)c$ when $k \geqslant 7$.

Let s and t be integers satisfying the conditions

(2.21) $\qquad sa + 2tb \geqslant 2^{k-2}(1-2b+L)+4b,$

(2.22) $\qquad s \geqslant 4$ when $k = 5$, $\quad s \geqslant 1$ when $k \geqslant 7$,

(2.23) $\quad 2^9 \geqslant t \geqslant 11$ when $k = 5$, $\quad 2^{2k-1} \geqslant t \geqslant 2^{k-3}$ when $k \geqslant 7$.

Let $B = 2^{-k}(2^k-6)^{\frac{1}{2}}(2^k-2)^{-\frac{1}{2}}$. Then, using (2.23), we have

$$2t\Big((1+B)^k+(1-B)^k-2\Big)+4$$

$$= 2t\left(2\binom{k}{1}B^{k-1}+2\binom{k}{3}B^{k-3}+\ldots+2\binom{k}{2}B^2\right)+4$$

$$< 2^{2k}\left(2\binom{k}{1}+2\binom{k}{3}+\ldots+2\binom{k}{2}\right)B^2+4 = 2^{2k}(2^k-2)\,B^2+4 = 2^k-2,$$

that is,

(2.31) $\qquad 2t\Big((1+B)^k+(1-B)^k-2\Big)+4 < 2^k-2.$

Let N be any integer for which

(2.32) $\quad 2t\Big(\big((1+[BN]N^{-1})^k+(1-[BN]N^{-1})^k-2\big)N^{k-1}+1\Big)$

$$\leqslant n \leqslant (2^k-2)N^{k-1}.$$

Since the function $(1+v)^k+(1-v)^k$ is an increasing function of v, the left side of (2.32) is less than

$$2t\Big((1+B)^k+(1-B)^k-2\Big)N^{k-1}+2t < 2t\Big((1+B)^k+(1-B)^k-2\Big)N^{k-1}+2^{2k}$$

(2.321) $\qquad\qquad < \Big(2t\Big((1+B)^k+(1-B)^k-2\Big)+3\Big)N^{k-1}$

if $N > 5$. It follows from (2.31) that at least one integer N satisfying (2.32) always exists when n is sufficiently large.

Let $M = [BN]$. Then $M < N$, since $B < 1$. Also,

(2.33) $\qquad M = [BN] > BN-1 > \frac{1}{2}BN$

for N sufficiently large. We may write (2.32) in the form

$$2t\Big(P(M)+1\Big) \leqslant n \leqslant P(N). \tag{2.34}$$

From (2.32) and (2.33) we have

$$2t\Big((1+\tfrac{1}{2}B)^k+(1-\tfrac{1}{2}B)^k-2\Big)N^{k-1} < n \leqslant (2^k-2)N^{k-1},$$

$$A_1 N^{k-1} < n \leqslant A_3 N^{k-1}, \tag{2.35}$$

$$P(N) = (2^k-2)N^{k-1} < (2^k-2)A_1^{-1} n = A_2 n, \tag{2.36}$$

where A_1, A_2, A_3 depend only on k and t. Let

$$f(x) = \sum_{u=0}^{n^a} x^{u^k}; \qquad g(x) = \sum_{v=M}^{N} x^{P(v)};$$

$$\rho = \exp(2\pi i l/q), \quad (l, q) = 1;$$

$$S_\rho = \sum_{u=0}^{q-1} \rho^{u^k}; \qquad T_\rho = \sum_{v=0}^{q-1} \rho^{P(v)};$$

$$A_q(j) = q^{-s-t} \sum_{\substack{l=0 \\ (l,q)=1}}^{q-1} S_\rho^{\,s}\, T_\rho^{\,t}\, \rho^{-j};$$

$$\mathfrak{S}(j, h) = \sum_{q=1}^{h} A_q(j).$$

Let $v(j) = v(j, N, M)$ be defined by

$$P\Big(v(j)\Big) = j, \quad M \leqslant v(j) \leqslant N, \quad P(M) \leqslant j \leqslant P(N). \tag{2.4}$$

Let

$$\psi_\rho(x) = \Gamma(1+a)\, q^{-1} S_\rho \left(1 + \sum_{j=1}^{n} \frac{a(a+1)\dots(a+j-1)}{j!} \left(\frac{x}{\rho}\right)^j\right);$$

$$\phi_\rho(x) = q^{-1} T_\rho \sum_{j=P(M)+1}^{P(N)} \Big(v(j)-v(j-1)\Big) \left(\frac{x}{\rho}\right)^j;$$

$$\theta = \begin{cases} \frac{9}{14}, & k = 5, \\ \frac{1}{2}, & k \geqslant 7; \end{cases} \qquad \delta = \begin{cases} \frac{13}{20}, & k = 5, \\ 1, & k \geqslant 7. \end{cases}$$

We define $a(j)$ as the coefficient of $\Gamma^s(1+a)\,\Gamma^t(1+b)\,q^{-s-t} S_\rho^{\,s}\, T_\rho^{\,t}(x/\rho)^j$ in the expansion of $\psi_\rho^{\,s}(x)\,\phi_\rho^{\,t}(x)$; that is,

$$\psi_\rho^{\,s}(x)\,\phi_\rho^{\,t}(x) = \Gamma^s(1+a)\,\Gamma^t(1+b)\,q^{-s-t} S_\rho^{\,s}\, T_\rho^{\,t} \sum_{j=tP(M)+t}^{sn+tP(N)} a(j) \left(\frac{x}{\rho}\right)^j.$$

Let $r(n) = r(n, N, M)$ be the number of solutions of (1.1); for

$$2t\Big(P(M)+1\Big) \leqslant j \leqslant n \leqslant P(N)$$

let

$$\sigma(j) = \sigma(j, n, N, M) = r(j, N, M) - \Gamma^s(1+a)\,\Gamma^t(1+b)\,a(j)\,\mathfrak{S}(j, n^{\delta a}).$$

We shall use ϵ to denote an arbitrarily small positive number and $C_1, C_2, \ldots$ to denote positive numbers depending at most on k, s, t, and ϵ.

On the circle $|x| = 1$ we take the points $\rho = \exp(2\pi i l/q)$ which correspond to the Farey fractions l/q with denominators less than or equal to n^{1-b} and divide the circumference into subarcs by means of the mediants. We write

$$x = \rho e^{2\pi i y}.$$

Then, on the arc associated with ρ, we have

$$-y_1 \leqslant y \leqslant y_2,$$

where
$$\frac{1}{2qn^{1-b}} \leqslant y_1 < \frac{1}{qn^{1-b}}, \qquad \frac{1}{2qn^{1-b}} \leqslant y_2 < \frac{1}{qn^{1-b}}.$$

We consider minor arcs and major arcs given by

Minor arcs $\mathfrak{m}$: $\quad n^b < q \leqslant n^{1-b}$,

Major arcs $\mathfrak{M}$: $\quad 1 \leqslant q \leqslant n^a$,

$\mathfrak{M}_3$: $\quad n^a < q \leqslant n^b$.

Each arc $\mathfrak{M}$ is divided into two parts. That part of $\mathfrak{M}$ for which $|y| \leqslant 1/(2q^\theta n^{1-\theta a})$ is called an arc of type $\mathfrak{M}_1$, and the remaining part of $\mathfrak{M}$ in which $|y| > 1/(2q^\theta n^{1-\theta a})$ is called an arc of type $\mathfrak{M}_2$. There are always arcs of type $\mathfrak{M}_2$, since

$$\frac{1}{2q^\theta n^{1-\theta a}} \leqslant \frac{1}{2qn^{1-a}} < \frac{1}{2qn^{1-b}}$$

when $q \leqslant n^a$.

3. *Preliminary lemmas for $f(x)$ and $\psi_\rho(x)$.*

LEMMA 1. *If j is an integer greater than or equal to 1 and $\beta > 0$, then*

$$\left|\frac{\Gamma(\beta+1+j)}{j!} - j^\beta\right| < \gamma(\beta)\, j^{\beta-1}, \tag{3.11}$$

$$\gamma_3(\beta)\, j^\beta < \frac{\Gamma(\beta+1+j)}{j!} < \gamma_4(\beta)\, j^\beta, \tag{3.12}$$

where $\gamma(\beta)$, $\gamma_3(\beta)$, and $\gamma_4(\beta)$ depend only on β.

Proof. The first inequality is **3**, Theorem 277. Also, when $j \geqslant 1$, (3.11) implies

$$\frac{\Gamma(\beta+1+j)}{j!} < j^{\beta}+\gamma(\beta)\, j^{\beta-1} < \Big(1+\gamma(\beta)\Big)\, j^{\beta}.$$

When $j > 2\gamma(\beta)$, (3.11) implies

$$\frac{\Gamma(\beta+1+j)}{j!} > j^{\beta}-\gamma(\beta)\, j^{\beta-1} > \tfrac{1}{2} j^{\beta}.$$

When $j \leqslant 2\gamma(\beta)$, we have

$$\frac{\Gamma(\beta+1+j)}{j!} > \Gamma(\beta+1) = \frac{\Gamma(\beta+1)}{\Big(2\gamma(\beta)\Big)^{\beta}} \Big(2\gamma(\beta)\Big)^{\beta} \geqslant \frac{\Gamma(\beta+1)}{\Big(2\gamma(\beta)\Big)^{\beta}}\, j^{\beta}.$$

This proves (3.12).

LEMMA 2. (**1**, Theorem 8.) *When* $|y| \leqslant \frac{1}{2}$,

$$|\psi_{\rho}(x)| < C_2 q^{-c+\epsilon}\, \mathrm{Min}\,(n^{a}, |y|^{-a}).$$

LEMMA 3. *On* $\mathfrak{m}$,

$$|f(x)| < C_3 n^{a-ac+\epsilon}.$$

Proof. Let

$$\tau(j) = \sum_{h=0}^{j^a} \rho^{h^k} \quad (j \geqslant 0); \qquad \tau(-1) = 0.$$

Then

$$(3.31) \quad f(x) = \sum_{j=0}^{n} \Big(\tau(j)-\tau(j-1)\Big)\Big(\frac{x}{\rho}\Big)$$

$$= \Big(1-\frac{x}{\rho}\Big) \sum_{j=0}^{n-1} \tau(j)\Big(\frac{x}{\rho}\Big) + \tau(n)\Big(\frac{x}{\rho}\Big)^{n}.$$

By **3**, Theorem 267, with $m = [j^a]+1 \leqslant 2j^a$, $r = -1$, we have

$$|\tau(j)|^{K} < C_1 (2j^a)^{\epsilon} q^{\epsilon} \Big((2j^a)^{K-1}+(2j^a)^{K} q^{-1}+(2j^a)^{K-k} q\Big)$$

$$\leqslant C_1 (2n^a)^{\epsilon} q^{\epsilon} \Big((2n^a)^{K-1}+(2n^a)^{K} q^{-1}+(2n^a)^{K-k} q\Big)$$

$$< C_4 n^{aK-a+\epsilon}$$

(since $n^a < n^b < q \leqslant n^{1-b} < n^{1-a}$ for $\mathfrak{m}$);

$$(3.32) \qquad |\tau(j)| < C_5 n^{a-ac+\epsilon}.$$

Also,

$$\left|1-\frac{x}{\rho}\right| = |1-e^{2\pi iy}| = 2|\sin \pi y| \leqslant 2\pi|y| < 2\pi q^{-1} n^{-1+b}. \tag{3.33}$$

Then, from (3.31), (3.32), and (3.33), we obtain

$$\begin{aligned} |f(x)| &\leqslant \left|1-\frac{x}{\rho}\right| \sum_{j=0}^{n-1} |\tau(j)| + |\tau(n)| \\ &< (2\pi q^{-1} n^{-1+b} n + 1)\, C_5 n^{a-ac+\epsilon} \\ &< (2\pi+1)\, C_5 n^{a-ac+\epsilon}, \end{aligned}$$

since $q^{-1} n^b < 1$ on $\mathfrak{m}$.

LEMMA 4. *On* $\mathfrak{M}_3$,

$$|f(x)| < C_6 n^{a-ac+\epsilon}.$$

Proof. (1) Let $|y| \leqslant 1/(qn^{1-a})$; then

$$\left|1-\frac{x}{\rho}\right| \leqslant 2\pi|y| \leqslant 2\pi q^{-1} n^{-1+a}. \tag{3.41}$$

Since $n^a < q < n^{1-a}$ on $\mathfrak{M}_3$, the inequality (3.32) holds also on $\mathfrak{M}_3$. Hence, from (3.31), (3.41), and (3.32), we obtain

$$\begin{aligned} |f(x)| &< (2\pi|y|n+1) C_5 n^{a-ac+\epsilon} \\ &\leqslant (2\pi q^{-1} n^{-1+a} n + 1) C_5 n^{a-ac+\epsilon} \\ &< (2\pi+1) C_5 n^{a-ac+\epsilon}, \end{aligned}$$

since $q^{-1} n^a < 1$ on $\mathfrak{M}_3$.

(2) Let $|y| > 1/(qn^{1-a})$; then we have

$$x = e^{2\pi i(l/q+y)}, \quad n^a < q \leqslant n^b,$$

$$1/(qn^{1-a}) < |y| < 1/(qn^{1-b}), \quad 2q^{-1}|y|^{-1} > 1. \tag{3.42}$$

From the theory of the Farey series it is known that there exist integers l_1, q_1, and a number y_1 such that

$$\frac{l_1}{q_1} + y_1 = \frac{l}{q} + y, \tag{3.43}$$

$$0 \leqslant l_1 \leqslant q_1 \leqslant 2q^{-1}|y|^{-1} < n \quad [(l_1, q_1) = 1], \tag{3.44}$$

$$|y_1| < \tfrac{1}{2} q |y| q_1^{-1} \leqslant q_1^{-2}. \tag{3.45}$$

It follows from (3.45) and (3.42) that

$$|y_1| < \frac{1}{2q_1 n^{1-b}},$$

and so we cannot have $n^b < q_1 \leqslant n^{1-b}$. For if this was the case x would belong to a minor arc $\mathfrak{m}$. Again, $1 \leqslant q_1 \leqslant n^b$ is impossible when $n > 2^{k-1}$, since otherwise

$$\begin{aligned} |lq_1 - l_1 q| &= qq_1|y - y_1| \leqslant qq_1(|y| + |y_1|) \\ &< qq_1|y| + q^2|y| \leqslant 2n^b q|y| < 2n^b qq^{-1}n^{-1+b} \\ &= 2n^{2b-1} < 2n^{-b} < 1, \end{aligned}$$

and therefore $lq_1 - l_1 q = 0$. Since $(l, q) = 1$, $(l_1, q_1) = 1$, we should then have $q = q_1$, $l = l_1$, $y = y_1$,

$$|y_1| = q|y|q_1^{-1} > \tfrac{1}{2}q|y|q_1^{-1},$$

which contradicts (3.45). Hence $q_1 > n^{1-b} > n^a$. Write

$$\rho_1 = e^{2\pi i l_1/q_1}; \qquad \tau_1(j) = \sum_{h=0}^{j^a} \rho_1^{h^k} \quad (j \geqslant 0); \qquad \tau_1(-1) = 0.$$

Then

$$f(x) = \left(1 - \frac{x}{\rho_1}\right) \sum_{j=0}^{n-1} \tau_1(j) \left(\frac{x}{\rho_1}\right)^j + \tau_1(n) \left(\frac{x}{\rho_1}\right)^n. \tag{3.46}$$

First, suppose that $q_1 > n^{1-a}$. Then, from **3**, Theorem 267, with

$$q = q_1, \quad m = [j^a] + 1 \leqslant 2n^a, \quad r = -1,$$

we obtain

$$\begin{aligned} |\tau_1(j)|^K &< C_7 n^{aK-1+\epsilon} q_1^{1+\epsilon}, \\ |\tau_1(j)| &< C_8 n^{a-c+\epsilon} q_1^{c+\epsilon} \\ &< C_9 n^{a-c+\epsilon} q^{-c-\epsilon} |y|^{-c-\epsilon} \quad \text{[by (3.44)]} \\ &< C_9 n^{a-c+\epsilon} q^{-c-\epsilon} q^{c+\epsilon} n^{(1-a)(c+\epsilon)} \quad \text{[since } |y| > 1/(qn^{1-a})\text{]}, \end{aligned}$$

$$|\tau_1(j)| < C_9 n^{a-ac+\epsilon}. \tag{3.47}$$

Second, suppose that $q_1 \leqslant n^{1-a}$. From **3**, Theorem 267, with

$$q = q_1, \quad m = [j^a] + 1 \leqslant 2n^a, \quad r = -1,$$

we get

$$|\tau_1(j)| < C_{10} n^{a-ac+\epsilon}. \tag{3.48}$$

Then, from (3.46) and (3.47) or (3.48),

$$\begin{aligned}|f(x)| &\leqslant \left|1-\frac{x}{\rho_1}\right| \sum_{j=0}^{n-1} |\tau_1(j)| + |\tau_1(n)| \\ &\leqslant (2\pi|y_1|n+1)\, C_{11}\, n^{a-ac+\epsilon} \\ &< (2\pi+1)\, C_{11}\, n^{a-ac+\epsilon},\end{aligned}$$

since (3.45) and $q_1 > n^{1-b}$ imply that $|y_1|n < q_1^{-2}n < n^{-2+2b}n < 1$. This completes the proof.

LEMMA 5. (1, Theorem 12.)

$$|f(x)-\psi_\rho(x)| < C_{12}\, q^{1-c+\epsilon}(n|y|+1).$$

LEMMA 6. *On* $\mathfrak{M}_1$,

$$|f(x)| < C_{13}\, n^a\, q^{-c+\epsilon}.$$

Proof. From Lemmas 5 and 2 we obtain

$$\begin{aligned}|f(x)| &< |\psi_\rho(x)| + C_{12}\, q^{1-c+\epsilon}(n|y|+1) \\ &< C_2\, n^a\, q^{-c+\epsilon} + C_{12}\, q^{1-c+\epsilon}\, n|y| + C_{12}\, q^{1-c+\epsilon}.\end{aligned}$$

Since $q \leqslant n^a$ on $\mathfrak{M}_1$ and

$$qn|y| \leqslant qn(2q^\theta n^{1-\theta a})^{-1} \leqslant qn(2qn^{1-a})^{-1} = \tfrac{1}{2}n^a,$$

this gives

$$|f(x)| < (C_2 + \tfrac{1}{2}C_{12} + C_{12})\, n^a\, q^{-c+\epsilon}.$$

LEMMA 7. *On* $\mathfrak{M}_2$,

$$\begin{aligned}|f(x)| &< C_{14}\, n^{a-c+\epsilon}\, q^{-c-\epsilon}\, |y|^{-c-\epsilon} && \textit{when} \quad |y| \leqslant 1/(qn^{1-a}), \\ |f(x)| &< C_{15}\, n^{a-ac+\epsilon} && \textit{when} \quad |y| \geqslant 1/(qn^{1-a}).\end{aligned}$$

Proof. On $\mathfrak{M}_2$,

$$x = e^{2\pi i(l/q+y)}, \quad 1 \leqslant q \leqslant n^a,$$

$$1/(2q^\theta n^{1-\theta a}) \leqslant |y| < 1/(qn^{1-b}), \quad 2q^{-1}|y|^{-1} > 1.$$

We determine l_1, q_1, and y_1 as in Lemma 4. As before, we have $q_1 > n^{1-b}$ and

$$\begin{aligned}|\tau_1(j)| &< C_9\, n^{a-c+\epsilon}\, q^{-c-\epsilon}\, |y|^{-c-\epsilon} && \text{when} \quad q_1 > n^{1-a}, \\ |\tau_1(j)| &< C_{10}\, n^{a-ac+\epsilon} && \text{when} \quad q_1 \leqslant n^{1-a}.\end{aligned}$$

Then $$|f(x)| \leqslant \left|1-\frac{x}{\rho_1}\right| \sum_{j=0}^{n-1} |\tau_1(j)| + |\tau_1(n)|,$$

$$(3.7)\quad |f(x)| < \max\left((2\pi+1)\,C_9\,n^{a-c+\epsilon}q^{-c-\epsilon}|y|^{-c-\epsilon},\ (2\pi+1)\,C_{10}\,n^{a-ac+\epsilon}\right).$$

Since $$n^{a-c+\epsilon}q^{-c-\epsilon}|y|^{-c-\epsilon} \gtreqless n^{a-ac+\epsilon},$$

according as $|y| \gtreqless 1/(qn^{1-a})$, it follows from (3.7) that

$$|f(x)| < (2\pi+1)\,(C_9+C_{10})\,n^{a-c+\epsilon}q^{-c-\epsilon}|y|^{-c-\epsilon} \quad \text{when} \quad |y| \leqslant 1/(qn^{1-a}),$$

$$|f(x)| < (2\pi+1)\,(C_9+C_{10})\,n^{a-ac+\epsilon} \quad \text{when} \quad |y| \geqslant 1/(qn^{1-a}).$$

4. *Preliminary lemmas for $g(x)$ and $\phi_\rho(x)$.*

LEMMA 8. *Let ρ be a primitive q-th root of unity, $m>0$, let r be an integer, and let*

$$T = \sum_{h=r+1}^{r+m} \rho^{P(h)}.$$

Then, for every $\epsilon > 0$,

$$|T|^{\frac{1}{2}K} < C_{16}\,m^{\epsilon}q^{\epsilon}(m^{\frac{1}{2}(K-2)}+m^{\frac{1}{2}K}q^{-1}+m^{\frac{1}{2}(K-2k+2)}q).$$

Proof. From **3**, Theorem 265, with $\Psi(z)=P(z)$, $\lambda = 2kl/q$, and k replaced by $k-1$, it follows that

$$(4.1)\quad |T|^{\frac{1}{2}K} < 4^{\frac{1}{2}K}\Bigg(m^{\frac{1}{2}(K-2)}+m^{\frac{1}{2}(K-2k+2)} \times \sum_{h_1,\ldots,h_{k-2}=1}^{m} \min\left(m, \frac{1}{\{2lk!\,h_1\ldots h_{k-2}/q\}}\right)\Bigg).$$

The coefficient of the highest power of v, namely v^{k-1}, in $P(v)$ is independent of n and N, and it is the only coefficient involved in the inequality (4.1). Hence the lemma follows from (4.1) exactly as Theorem 267 follows from Theorem 265 and 266 in Landau (**3**, pp. 253-256).

LEMMA 9. *If $P(M) \leqslant j \leqslant P(N)$, then*

$$C_{17}j^b \leqslant v(j) \leqslant C_{18}j^b;$$

if $P(M)+1 \leqslant j \leqslant P(N)$, then

$$C_{19}j^{b-1} < v(j)-v(j-1) < C_{20}j^{b-1}.$$

Proof. We note that both $P(v)$ and $P'(v)$ are polynomials in v with positive coefficients. They are increasing functions of v and, from the

definition (2.4), we have $v(j) \geqslant v(j-1)$. Then, from (2.4) and (2.33), we obtain

$$\begin{aligned} j^{-b} v(j) &= \Big(P\big(v(j)\big)\Big)^{-b} v(j) \\ &= \left(2\binom{k}{1} + 2\binom{k}{3}\left(\frac{N}{v(j)}\right)^2 + \ldots + 2\binom{k}{2}\left(\frac{N}{v(j)}\right)^{k-3}\right)^{-b} \\ &\geqslant \left(2\binom{k}{1} + 2\binom{k}{3}\left(\frac{N}{M}\right)^2 + \ldots + 2\binom{k}{2}\left(\frac{N}{M}\right)^{k-3}\right)^{-b} \\ &> \left(2\binom{k}{1} + 2\binom{k}{3}\left(\frac{2}{B}\right)^2 + \ldots + 2\binom{k}{2}\left(\frac{2}{B}\right)^{k-3}\right)^{-b} = C_{17}. \end{aligned}$$

Also,

$$\begin{aligned} j^{-b} v(j) &= \Big(P\big(v(j)\big)\Big)^{-b} v(j) \\ &\leqslant \left(2\binom{k}{1} + 2\binom{k}{3} + \ldots + 2\binom{k}{2}\right)^{-b} = (2^k - 2)^{-b} = C_{18}. \end{aligned}$$

This proves the first result of the lemma. Next, by the mean value theorem,

$$1 = P\big(v(j)\big) - P\big(v(j-1)\big) = \big(v(j) - v(j-1)\big)\, P'\big(u(j)\big),$$

where $v(j-1) \leqslant u(j) \leqslant v(j)$. Hence

$$\begin{aligned} v(j) - v(j-1) &= \Big(P'\big(u(j)\big)\Big)^{-1} \leqslant \Big(P'\big(v(j-1)\big)\Big)^{-1} \\ &= \left(2(k-1)\binom{k}{1}\big(v(j-1)\big)^{k-2} + \ldots + 2 \cdot 2\binom{k}{2} N^{k-3} v(j-1)\right)^{-1} \\ &< v(j-1)\Big(2P\big(v(j-1)\big)\Big)^{-1} = \tfrac{1}{2} v(j-1) \cdot (j-1)^{-1} \\ &\leqslant \tfrac{1}{2} C_{18} (j-1)^{b-1} \leqslant C_{20} j^{b-1}. \end{aligned}$$

Finally,

$$\begin{aligned} v(j) - v(j-1) &\geqslant \Big(P'\big(v(j)\big)\Big)^{-1} \\ &> v(j)\Big((k-1)\, P\big(v(j)\big)\Big)^{-1} \\ &\geqslant C_{17}(k-1)^{-1} j^{b-1} = C_{19} j^{b-1}, \end{aligned}$$

which completes the proof.

LEMMA 10. *If* $2t\big(P(M)+1\big) \leqslant j \leqslant n$, *then*

$$a(j) > C_{21} j^{sa+tb-1}.$$

Proof. Since

$$\psi_\rho(x) = \Gamma(1+a)\, q^{-1} S_\rho\left(1+\sum_{j=1}^{n} \frac{a(a+1)\dots(a+j-1)}{j!}\left(\frac{x}{\rho}\right)^j\right)$$

$$= \text{the first } n+1 \text{ terms of the expansion of } \Gamma(1+a) q^{-1} S_\rho (1-x/\rho)^{-a},$$

we have

$$\psi_\rho^{\,s}(x) = \text{the first } n+1 \text{ terms of the expansion of } \Gamma^s(1+a)\, q^{-s} S_\rho^{\,s} (1-x/\rho)^{-sa}$$

$$+\text{terms of higher degree.}$$

That is,

$$\psi_\rho^{\,s}(x) = \Gamma^s(1+a)\, q^{-s} S_\rho^{\,s}\left(1+\sum_{j=1}^{n} \frac{sa(sa+1)\dots(sa+j-1)}{j!}\left(\frac{x}{\rho}\right)^j\right)$$

$$+\text{higher powers of } x/\rho$$

$$= \Gamma^s(1+a)\, q^{-s} S_\rho^{\,s} \sum_{j=0}^{n} \frac{\Gamma(sa+j)}{\Gamma(sa)\, j!}\left(\frac{x}{\rho}\right)^j + \text{higher powers.}$$

Let $$A(j) = \frac{\Gamma(sa+j)}{\Gamma(sa)\, j!}; \quad D(j) = \frac{v(j)-v(j-1)}{\Gamma(1+b)}.$$

Then, by (3.12) with $\beta = sa-1$, and Lemma 9,

$$A(j) > \big(\Gamma(sa)\big)^{-1} \gamma_3(sa-1)\, j^{sa-1} = C_{22} j^{sa-1}, \tag{4.31}$$

$$D(j) > C_{19}\big(\Gamma(1+b)\big)^{-1} j^{b-1} = C_{23} j^{b-1}. \tag{4.32}$$

Since $$v(j)-v(j-1) \geqslant \Big(P'\big(v(j)\big)\Big)^{-1} \geqslant v(j+1)-v(j),$$

we have

$$D(j) \geqslant D(j+1). \tag{4.33}$$

Then

$$\psi_\rho^{\,s}(x)\, \phi_\rho^{\,t}(x) = \Gamma^s(1+a)\, \Gamma^t(1+b)\, q^{-s-t} S_\rho^{\,s}\, T_\rho^{\,t} \sum_{i=0}^{n} A(i)\left(\frac{x}{\rho}\right)^i \left(\sum_{j=P(M)+1}^{n} D(j)\left(\frac{x}{\rho}\right)^j\right)^t$$

$$+\text{terms of degree greater than } n.$$

Hence $$a(j) = a_t(j) = \sum_{i=0}^{n} \sum_{\substack{j_1=P(M)+1 \\ i+j_1+\ldots+j_t=j}}^{n} \cdots \sum_{j_t=P(M)+1}^{n} A(i)\,D(j_1)\ldots D(j_t)$$

$$(4.34) \qquad = \sum_{i=0}^{j-tP(M)-t} \sum_{\substack{j_1=P(M)+1 \\ i+j_1+\ldots+j_t=j}}^{j-(t-1)(P(M)+1)} \cdots \sum_{j_t=P(M)+1}^{j-(t-1)(P(M)+1)} A(i)\,D(j_1)\ldots D(j_t).$$

(1) Let $t=1$. Then $2P(M)+2 \leqslant j \leqslant n$ and

$$\begin{aligned} a_1(j) &= \sum_{i=0}^{j-P(M)-1} \sum_{\substack{j_1=P(M)+1 \\ i+j_1=j}}^{j} A(i)\,D(j_1) \\ &= \sum_{i=0}^{j-P(M)-1} A(i)\,D(j-i) \\ &\geqslant D(j) \sum_{i=0}^{j-P(M)-1} A(i) \quad \text{[by (4.33)]} \\ &> C_{22}\,C_{23}\,j^{b-1} \sum_{i=0}^{j-P(M)-1} i^{sa-1} \quad \text{[by (4.31) and (4.32)]} \\ &\geqslant C_{22}\,C_{23}\,j^{b-1} \int_0^{j-P(M)-1} u^{sa-1}\,du \\ &= C_{22}\,C_{23}(sa)^{-1} j^{b-1}\Big(j-P(M)-1\Big)^{sa} \\ &= C_{22}\,C_{23}(sa)^{-1}\Big(j^{-1}\Big(j-P(M)-1\Big)\Big)^{sa} j^{sa+b-1} \\ &\geqslant C_{22}\,C_{23}\,(sa)^{-1}\Big(\Big(2P(M)+2\Big)^{-1}\Big(P(M)+1\Big)\Big)^{sa} j^{sa+b-1} \\ &= C_{22}\,C_{23}(sa)^{-1}\,2^{-sa}\,j^{sa+b-1} = C_{21}(1)\,j^{sa+b-1}. \end{aligned}$$

(2) We complete the proof by induction. We assume that

$$a_t(j) > C_{21}(t)\,j^{sa+tb-1}$$

for $2t\Big(P(M)+1\Big) \leqslant j \leqslant n$. Then, for $(2t+2)\Big(P(M)+1\Big) \leqslant j \leqslant n$, we have

$$\begin{aligned} a_{t+1}(j) &= \sum_{i=0}^{j-(t+1)(P(M)+1)} \sum_{\substack{j_1=P(M)+1 \\ i+j_1+\ldots+j_{t+1}=j}}^{j-tP(M)-t} \cdots \sum_{j_{t+1}=P(M)+1}^{j-tP(M)-t} A(i)\,D(j_1)\ldots D(j_{t+1}) \\ &> \sum_{j_{t+1}=P(M)+1}^{j-2t(P(M)+1)} D(j_{t+1}) \sum_{\substack{i=0 \\ i+j_1+\ldots+j_t=j-j_{t+1}}}^{j-j_{t+1}-tP(M)-t} \cdots \sum_{j_t=P(M)+1}^{j-j_{t+1}-(t-1)(P(M)+1)} A(i)\ldots D(j_t) \\ &> \sum_{j_{t+1}=P(M)+1}^{j-2t(P(M)+1)} D(j_{t+1})\,C_{21}(t)\,(j-j_{t+1})^{sa+tb-1}, \end{aligned}$$

since $j-j_{t+1} \geqslant j-\Big(j-2t\big(P(M)+1\big)\Big) = 2t\big(P(M)+1\big)$. Hence, by (4.32),

$$\begin{aligned}
\alpha_{t+1}(j) &> C_{23}\, j^{b-1} \sum_{j_{t+1}=P(M)+1}^{j-2t(P(M)+1)} C_{21}(t)(j-j_{t+1})^{sa+tb-1} \\
&\geqslant C_{23}\, j^{b-1}\, C_{21}(t) \int_{P(M)+1}^{j-2tP(M)-2t} (j-u)^{sa+tb-1}\, du \\
&= C_{23}\, C_{21}(t)\,(sa+tb)^{-1} j^{b-1}\Big(\big(j-P(M)-1\big)^{sa+tb} - \big(2tP(M)+2t\big)^{sa+tb}\Big) \\
&= C_{23}\, C_{21}(t)\,(sa+tb)^{-1}\Big(\Big(j^{-1}\big(j-P(M)-1\big)\Big)^{sa+tb} \\
&\qquad\qquad - \Big(j^{-1}\big(2tP(M)+2t\big)\Big)^{sa+tb}\Big) j^{sa+(t+1)b-1} \\
&\geqslant C_{23}\, C_{21}(t)\,(sa+tb)^{-1}\,(2t+2)^{-sa-tb}\big((2t+1)^{sa+tb} - (2t)^{sa+tb}\big)\, j^{sa+(t+1)b-1} \\
&= C_{21}(t+1)\, j^{sa+(t+1)b-1}.
\end{aligned}$$

LEMMA 11. *When* $|y| \leqslant \frac{1}{2}$,

$$|\phi_\rho(x)| < C_{24}\, q^{-2c+\epsilon} \min\,(n^b, |y|^{-b}).$$

Proof. We have

$$\begin{aligned}
|\phi_\rho(x)| &= |q^{-1}T_\rho| \Big| \sum_{j=P(M)+1}^{P(N)} \big(v(j)-v(j-1)\big)\, e^{2\pi i y j} \Big| \\
&= |q^{-1}T_\rho| \Big| \sum_{j=0}^{P(N)-P(M)-1} \Big(v\big(j+P(M)+1\big) - v\big(j+P(M)\big)\Big)\, e^{2\pi i y (j+P(M)+1)} \Big| \\
&= |q^{-1}T_\rho| \Big| \sum_{j=0}^{P(N)-P(M)-1} \Big(v\big(j+P(M)+1\big) - v\big(j+P(M)\big)\Big)\, e^{2\pi i y j} \Big| \\
&< C_{25}\, q^{-2c+\epsilon} \Big(v\big(P(M)+1\big) - v\big(P(M)\big)\Big)\, (\sin \pi |y|)^{-1}
\end{aligned}$$

by Lemma 8 with $m = q$, $r = -1$, and **1**, Theorem 1, with

$$a_j = v\big(j+P(M)+1\big) - v\big(j+P(M)\big).$$

From Lemma 9, (2.33), and (2.35), we obtain

$$\begin{aligned}
v\big(P(M)+1\big) - v\big(P(M)\big) &< C_{20}\big(P(M)\big)^{b-1} \\
&< C_{20}\big(P(\tfrac{1}{2}BN)\big)^{b-1} = C_{20}\Big(\big((1+\tfrac{1}{2}B)^k + (1-\tfrac{1}{2}B)^k - 2\big)\, N^{k-1}\Big)^{b-1} \\
&< C_{20}\big((1+\tfrac{1}{2}B)^k + (1-\tfrac{1}{2}B)^k - 2\big)\, A_3^{1-b}\, n^{b-1} = C_{26}\, n^{b-1}.
\end{aligned}$$

Hence $$|\phi_\rho(x)| < C_{25}\, q^{-2c+\epsilon}\, C_{26}\, n^{b-1}\, (\sin \pi |y|)^{-1},$$
$$< C_{27}\, n^{b-1} |y|^{-1} q^{-2c+\epsilon}.$$

Then, if $|y|^{-1} \leqslant n$,
$$|\phi_\rho(x)| < C_{27}\, q^{-2c+\epsilon} |y|^{-b}$$
$$= C_{27}\, q^{-2c+\epsilon}\, \text{Min}\, (n^b,\ |y|^{-b}).$$

Again, by Lemmas 8 and 9 and (2.36),
$$|\phi_\rho(x)| < C_{25}\, q^{-2c+\epsilon} \sum_{j=P(M)+1}^{P(N)} C_{20}\, j^{b-1},$$
$$< C_{28}\, A_2{}^b\, n^b\, q^{-2c+\epsilon}.$$

If $|y|^{-1} \geqslant n$, then
$$|\phi_\rho(x)| < C_{28}\, A_2{}^b\, n^b\, q^{-2c+\epsilon}$$
$$= C_{28}\, A_2{}^b\, q^{-2c+\epsilon}\, \text{Min}\, (n^b,\ |y|^{-b}).$$

This completes the proof.

LEMMA 12. *Let $R(j)$ denote the number of solutions of*
$$j = P(v_1) + P(v_2) \quad (v_1 \geqslant 0,\ \ v_2 \geqslant 0).$$

Then
$$\sum_{j=0}^{w} R^2(j) < C_{31}\, w^{2b+\epsilon}.$$

Proof. $\Sigma\, R^2(j)$ is equal to the number of solutions of

(4.51) $$0 \leqslant P(v_1) + P(v_2) = P(v_3) + P(v_4) \leqslant w.$$

Since $P(v_i) \leqslant w$ and the coefficients of $P(v)$ are positive, we have
$$2k v_i^{k-1} \leqslant w, \quad v_i \leqslant C_{32}\, w^b.$$

For fixed values of v_2, v_4, such that
$$w_1 = P(v_4) - P(v_2) \neq 0,$$
the equation

(4.52) $$P(v_1) - P(v_3) = w_1$$

has at most $2(k-2)\, T(|w_1|)$ solutions, where $T(m)$ denotes the number of positive divisors of m. For let $v_1 = v_3 + u$. Then
$$w_1 = P(v_1) - P(v_3) = P(v_3+u) - P(v_3)$$
$$= 2\binom{k}{1}\left((v_3+u)^k - v_3{}^k\right) + \ldots + 2\binom{k}{2} N^{k-3}\left((v_3+u)^2 - v_3{}^2\right),$$

(4.53) $$w_1 = 2u\left(k(k-1)\, v_3^{k-2} + \ldots\right)$$

implies that u divides w_1. Hence u takes at most $2T(|w_1|)$ values and to each of these there correspond at most $k-2$ values of v_3, since (4.53) is of degree $k-2$ in v_3. Since v_1 is then determined by $v_1 = v_3+u$, there are at most $2(k-2)\,T(|w_1|)$ solutions of (4.52). Also $v_2 \leqslant C_{32} w^b$, $v_4 \leqslant C_{32} w^b$, so that $w_1 \neq 0$ for at most $C_{32}^2 w^{2b}$ pairs v_2, v_4. Hence the number of solutions of (4.51) in which $P(v_4)-P(v_2) \neq 0$ is at most

$$2C_{32}^2 w^{2b}(k-2)\,T(|w_1|) < C_{33} w^{2b} |w_1|^{\epsilon}$$

$$\leqslant C_{33} w^{2b} \Big(\max P(v)\Big)^{\epsilon} \leqslant C_{33} w^{2b} w^{\epsilon}$$

by (4.52) and (4.51).

If $w_1 = P(v_4)-P(v_2) = 0$, we have $P(v_1)-P(v_3) = 0$. Since

$$P(v_1)-P(v_3) = (v_1-v_3)\,P'(u) \quad (v_3 \leqslant u \leqslant v_1),$$

and $P'(u) \neq 0$ for $u \neq 0$, by the remark in Lemma 9, it follows that $v_1 = v_3$ and similarly that $v_2 = v_4$. Thus (4.51) has at most $C_{32}^2 w^{2b}$ solutions in which $P(v_4)-P(v_2) = 0$. The total number of solutions of (4.51) is, therefore, at most

$$C_{33} w^{2b+\epsilon} + C_{32}^2 w^{2b} < C_{31} w^{2b+\epsilon}.$$

Then
$$\sum_{j=0}^{w} R^2(j) < C_{31} w^{2b+\epsilon}.$$

LEMMA 13. *On* $\mathfrak{m}$,

$$|g(x)| < C_{34} n^{b-2bc+\epsilon}.$$

Proof. Let

$$\beta(j) = \sum_{h=M}^{v(j)} \rho^{P(h)} \quad [j \geqslant P(M)], \qquad \beta\Big(P(M)-1\Big) = 0.$$

Then
$$g(x) = \sum_{v=M}^{N} x^{P(v)} = \sum_{j=P(M)}^{P(N)} \Big(\beta(j)-\beta(j-1)\Big)\Big(\frac{x}{\rho}\Big)^j,$$

$$g(x) = \Big(1-\frac{x}{\rho}\Big) \sum_{j=P(M)}^{P(N)-1} \beta(j)\Big(\frac{x}{\rho}\Big)^j + \beta\Big(P(N)\Big)\Big(\frac{x}{\rho}\Big)^{P(N)}. \tag{4.6}$$

By Lemma 8, with $m = [v(j)]-M+1 < v(j) \leqslant N$, $r = M-1$, we have

$$|\beta(j)|^{\frac{1}{2}K} < C_{16} N^{\epsilon} q^{\epsilon} (N^{\frac{1}{2}(K-2)} + N^{\frac{1}{2}K} q^{-1} + N^{\frac{1}{2}(K-2k+2)} q)$$

$$< C_{35} n^{\epsilon} q^{\epsilon} (n^{\frac{1}{2}(b(K-2))} + n^{\frac{1}{2}(bK)} q^{-1} + n^{\frac{1}{2}(b(K-2k+2))} q)$$

$$< C_{36} n^{\frac{1}{2}(b(K-2))} n^{\epsilon},$$

since $n^b < q \leqslant n^{1-b}$ on $\mathfrak{m}$. Hence

$$|\beta(j)| < C_{37} n^{b-2bc+\epsilon}.$$

From (4.6) it follows that

$$\begin{aligned}|g(x)| &< \left|1-\frac{x}{\rho}\right| \sum_{j=P(M)}^{P(N)-1} C_{37}\, n^{b-2bc+\epsilon} + C_{37}\, n^{b-2bc+\epsilon} \\ &< \left(2\pi|y|P(N)+1\right) C_{37}\, n^{b-2bc+\epsilon} \\ &< (2\pi q^{-1} n^{-1+b} A_2\, n+1)\, C_{37}\, n^{b-2bc+\epsilon} \\ &< C_{38}\, n^{b-2bc+\epsilon},\end{aligned}$$

since $q^{-1} n^b < 1$ on m.

LEMMA 14.

$$|g(x)-\phi_\rho(x)| < C_{39}\, q^{1-2c+\epsilon}(n|y|+1).$$

Proof. We may write

$$\begin{aligned}\phi_\rho(x) &= q^{-1} T_\rho \sum_{j=P(M)+1}^{P(N)} \left(v(j)-M-v(j-1)+M\right)\left(\frac{x}{\rho}\right) \\ &= \left(1-\frac{x}{\rho}\right) \sum_{j=P(M)}^{P(N)-1} q^{-1} T_\rho \left(v(j)-M\right)\left(\frac{x}{\rho}\right)^j \\ &\qquad + q^{-1} T_\rho \left(v\left(P(N)\right)-M\right)\left(\frac{x}{\rho}\right)^{P(N)},\end{aligned}$$

since $v\left(P(M)\right)-M = 0$.

With (4.6) this gives

$$\begin{aligned}(4.71)\quad g(x)-\phi_\rho(x) &= \left(1-\frac{x}{\rho}\right) \sum_{j=P(M)}^{P(N)-1} \left(\beta(j)-q^{-1} T_\rho\left(v(j)-M\right)\right)\left(\frac{x}{\rho}\right)^j \\ &\qquad + \left(\beta\left(P(N)\right)-q^{-1} T_\rho\left(v\left(P(N)\right)-M\right)\right)\left(\frac{x}{\rho}\right)^{P(N)}.\end{aligned}$$

(1) Let $v(j)-M < 2q$. Then, by Lemma 8 with

$$m = [v(j)]-M+1 \leqslant 2q, \quad r = M-1,$$

we have $$|\beta(j)| < C_{40}\, q^{1-2c+\epsilon}.$$

Also, by Lemma 8, with $m = q$, $r = -1$, we have

$$\left|q^{-1} T_\rho\left(v(j)-M\right)\right| < C_{25}\, q^{-2c+\epsilon}\, 2q.$$

Therefore

$$(4.72)\qquad \left|\beta(j)-q^{-1} T_\rho\left(v(j)-M\right)\right| < (C_{40}+2C_{25}) q^{1-2c+\epsilon}.$$

(2) Let $v(j)-M \geqslant 2q$. Write

$$v(j) = v_1 q + v_2, \quad M = M_1 q + M_2,$$

where v_1 and M_1 are integers and $0 \leqslant v_2 < q$, $0 \leqslant M_2 < q$. Then

$$\beta(j) = \sum_{h=M_1 q+M_2}^{(M_1+1)q-1} \rho^{P(h)} + \sum_{h=(M_1+1)q}^{(M_1+2)q-1} \rho^{P(h)} + \ldots + \sum_{h=v_1 q}^{v_1 q+v_2} \rho^{P(h)}.$$

That is, $\beta(j)$ consists of $v_1-M_1-1 = [v(j)/q]-[M/q]-1$ sums T_ρ with at most q additional terms at the beginning and at most q at the end. Hence Lemma 8, with $m \leqslant q$, $r+1 = M_1 q+M_2$ or $r+1 = v_1 q$, gives

$$\left|\beta(j)-\left([v(j)/q]-[M/q]-1\right) T_\rho\right| < 2C_{42} q^{1-2c+\epsilon},$$

$$\left|\beta(j)-\left(v(j)-M\right) q^{-1} T_\rho\right| < 2T_\rho + 2C_{42} q^{1-2c+\epsilon} < 2(C_{25}+C_{42}) q^{1-2c+\epsilon}. \tag{4.73}$$

From (4.71), (4.72), and (4.73) it follows that

$$\begin{aligned} |g(x)-\phi_\rho(x)| &\leqslant \left|1-\frac{x}{\rho}\right| \sum_{j=P(M)}^{P(N)-1} C_{43} q^{1-2c+\epsilon} + C_{43} q^{1-2c+\epsilon} \\ &< \left(2\pi|y|P(N)+1\right) C_{43} q^{1-2c+\epsilon} \\ &< (2\pi|y|A_2 n+1) C_{43} q^{1-2c+\epsilon} \\ &< C_{39} q^{1-2c+\epsilon}(n|y|+1). \end{aligned}$$

LEMMA 15. *On $\mathfrak{M}_1$ and $\mathfrak{M}_3$,*

$$|g(x)| < C_{49} n^b q^{-2c+\epsilon}.$$

Proof. From Lemmas 14 and 11 we obtain

$$\begin{aligned} |g(x)| &< |\phi_\rho(x)| + C_{39} q^{1-2c+\epsilon}(n|y|+1) \\ &< C_{24} n^b q^{-2c+\epsilon} + C_{39} q^{1-2c+\epsilon} n|y| + C_{39} q^{1-2c+\epsilon} \\ &< (C_{24}+C_{39}+C_{39}) n^b q^{-2c+\epsilon}, \end{aligned}$$

since $|y| < q^{-1} n^{-1+b}$ and $q \leqslant n^b$ on $\mathfrak{M}_1$ and $\mathfrak{M}_3$.

LEMMA 16. *On $\mathfrak{M}_2$,*

$$|g(x)| < C_{50} n^{b-2c+\epsilon} q^{-2c-\epsilon} |y|^{-2c-\epsilon}.$$

Proof. We determine l_1, q_1, and y_1 as in Lemma 4. Let

$$\rho_1 = e^{2\pi i l_1/q_1}; \quad \beta_1(j) = \sum_{h=M}^{v(j)} \rho_1^{P(h)}, \quad j \geqslant P(M); \quad \beta_1\left(P(M)-1\right) = 0.$$

Then

$$g(x)=\left(1-\frac{x}{\rho_1}\right)\sum_{j=P(M)}^{P(N)-1}\beta_1(j)\left(\frac{x}{\rho_1}\right)^j+\beta_1\big(P(N)\big)\left(\frac{x}{\rho_1}\right)^{P(N)}.$$

From Lemma 8, with $m=[v(j)]-M+1<N$, $r=M-1$, and $q=q_1$, we have

$$\begin{aligned}|\beta_1(j)|^{\frac{1}{2}K}&<C_{16}N^{\epsilon}q_1{}^{\epsilon}(N^{\frac{1}{2}(K-2)}+N^{\frac{1}{2}K}q_1^{-1}+N^{\frac{1}{2}(K-2k+2)}q_1)\\&<C_{51}n^{\epsilon}q_1{}^{\epsilon}(n^{\frac{1}{2}(b(K-2))}+n^{\frac{1}{2}bK}q_1^{-1}+n^{\frac{1}{2}(b(K-2k+2))}q_1)\\&<C_{52}n^{\epsilon}n^{\frac{1}{2}(b(K-2k+2))}q_1^{1+\epsilon},\end{aligned}$$

since $q_1>n^{1-b}$. Hence, by (3.44),

$$|\beta_1(j)|<C_{53}n^{b-2c+\epsilon}q_1^{2c+\epsilon}\leqslant 2^{2c+\epsilon}C_{53}n^{b-2c+\epsilon}q^{-2c-\epsilon}|y|^{-2c-\epsilon};$$

$$\begin{aligned}|g(x)|<\left|1-\frac{x}{\rho_1}\right|&\sum_{j=P(M)}^{P(N)-1}2^{2c+\epsilon}C_{53}n^{b-2c+\epsilon}q^{-2c-\epsilon}|y|^{-2c-\epsilon}\\&+2^{2c+\epsilon}C_{53}n^{b-2c+\epsilon}q^{-2c-\epsilon}|y|^{-2c-\epsilon}\\<\big(2\pi|y|P(N)+1\big)&2^{2c+\epsilon}C_{53}n^{b-2c+\epsilon}q^{-2c-\epsilon}|y|^{-2c-\epsilon}\\<(2\pi A_2+1)&\,2^{2c+\epsilon}C_{53}n^{b-2c+\epsilon}q^{-2c-\epsilon}|y|^{-2c-\epsilon},\end{aligned}$$

since (3.45) and $q_1>n^{1-b}$ make $|y|P(N)<q_1^{-2}A_2n<A_2n^{-1+2b}<A_2$.

5. *The proof of a main lemma.*

Our object in this section is to prove that

$$\sum_{j=2t(P(M)+1)}^{n}|\sigma(j)|^2<C_{87}n^{2sa+2lb-1-L+\epsilon}.$$

LEMMA 17. *If $k\geqslant 7$,*

$$\sum_{\mathfrak{M}_1}\int_{C-\mathfrak{M}_1}|\psi_\rho{}^s(x)|^2|\phi_\rho{}^t(x)|^2|dx|<C_{54}n^{2sa+2lb-1-L-2a+\epsilon},$$

where the integral is taken over the entire circle $|x|=1$ with the exception of the arc $\mathfrak{M}_1$ itself. The summation extends over all arcs $\mathfrak{M}_1$.

Proof. By Lemmas 2 and 11,

$$\begin{aligned}\int_{C-\mathfrak{M}_1}|\psi_\rho{}^s(x)|^2|\phi_\rho{}^t(x)|^2|dx|&<C_2^{2s}C_{24}^{2t}q^{-2sc-4lc+\epsilon}\int_{1/(2q^0n^{1-\theta a})}^{\infty}y^{-2sa-2lb}\,dy\\&<C_{55}n^{2sa+2lb-1-(2sa+2lb-1)\theta a+\epsilon}q^{-2sc-4lc+(2sa+2lb-1)\theta};\end{aligned}$$

$$\begin{aligned}\sum_{\mathfrak{M}_1}\int_{C-\mathfrak{M}_1}|\psi_\rho{}^s(x)|^2|\phi_\rho{}^t(x)|^2|dx|&\\<C_{55}n^{2sa+2lb-1-(2sa+2lb-1)\theta a+\epsilon}&\sum_{q=1}^{\eta^a}2q\,q^{-2sc-4lc+(2sa+2lb-1)\theta}.\end{aligned}$$

In the exponent of q,

$$1-2sc-4tc+(2sa+2tb-1)\theta = \tfrac{1}{2}+s(a-2c)+t(b-4c) > \tfrac{1}{2}.$$

Hence

$$(5.1) \qquad \sum_{\mathfrak{M}_1} \int_{C-\mathfrak{M}_1} |\psi_\rho^s(x)|^2 |\phi_\rho^t(x)|^2 |dx| < C_{54} n^{2sa+2tb-1-(2sc+4tc-2)a+\epsilon}.$$

In the exponent of n,

$$(2sc+4tc-2)a > (2sac+4tbc-2b)a(k-1) \geqslant (1-4b+L+8bc)(1-a)$$

[by (2.21)]

$$= 1-4b+L+8bc-a+4ab-La-8abc$$
$$= L-La+1-a-4a+8ac$$
$$= L+2a+1-7a+8ac-La > L+2a,$$

since $L < 8c$ and $1 \geqslant 7a$ when $k \geqslant 7$. The result now follows from (5.1).

When $k = 5$, let $\mathfrak{M}_1'$ denote the arcs $\mathfrak{M}_1$ for which $1 \leqslant q \leqslant n^{\delta a}$, and let $\mathfrak{M}_1''$ denote those for which $n^{\delta a} < q \leqslant n^a$.

LEMMA 18.

$$\sum_{\mathfrak{M}_1'} \int_{C-\mathfrak{M}_1'} |\psi_\rho^s(x)|^2 |\phi_\rho^t(x)|^2 |dx| < C_{56} n^{2sa+2tb-1-L-2\delta a+\epsilon}.$$

Proof. As in Lemma 17, we have

$$\sum_{\mathfrak{M}_1} \int_{C-\mathfrak{M}_1'} |\psi_\rho^s(x)|^2 |\phi_\rho^t(x)|^2 |dx|$$
$$< C_{55} n^{2sa+2tb-1-(2sa+2tb-1)\theta a+\epsilon} \sum_{q=1}^{n^{\delta a}} 2q\, q^{-2sc-4tc+(2sa+2tb-1)\theta}.$$

In the exponent of q,

$$1-2sc-4tc+(2sa+2tb-1)\theta = \tfrac{5}{14}+s\left(\frac{9}{7k}-\frac{1}{2^{k-2}}\right)+t\left(\frac{9}{7(k-1)}-\frac{1}{2^{k-3}}\right) > \tfrac{5}{14}.$$

Hence

$$(5.2) \qquad \sum_{\mathfrak{M}_1'} \int_{C-\mathfrak{M}_1'} |\psi_\rho^s(x)|^2 |\phi_\rho^t(x)|^2 |dx|$$
$$< C_{56} n^{2sa+2tb-1-(2sa+2tb-1)(1-\delta)\theta a-(2sc+4tc-2)\delta a+\epsilon}.$$

In the exponent of n,

$$(2sa+2tb-1)(1-\delta)\theta a+(2sc+4tc-2)\delta a$$
$$\geqslant (\tfrac{8}{5}+\tfrac{11}{2}-1)\tfrac{7}{20}\cdot\tfrac{9}{14}\cdot\tfrac{1}{5}+(\tfrac{1}{2}+\tfrac{11}{4}-2)\tfrac{13}{20}\cdot\tfrac{1}{5}$$
$$= \tfrac{5491}{20000}+\tfrac{13}{80} > \tfrac{13}{50}+\tfrac{13}{80} = 2\delta a+L.$$

The result then follows from (5.2).

LEMMA 19.

$$\sum_{\mathfrak{M}_1''} \int_{\mathfrak{M}_1''} |\psi_\rho^{\,s}(x)|^2 |\phi_\rho^{\,t}(x)|^2 |dx| < C_{57}\, n^{2sa+2tb-1-L+\epsilon}.$$

Proof. From Lemmas 2 and 11,

$$\begin{aligned}\int_{\mathfrak{M}_1''} |\psi_\rho^{\,s}(x)|^2 |\phi_\rho^{\,t}(x)|^2 |dx| &< C_2^{2s}\, C_{24}^{2t}\, q^{-2sc-4tc+\epsilon}\left(n^{2sa+2tb}\int_0^{1/n} dy + \int_{1/n}^{\infty} y^{-2sa-2tb}\, dy\right)\\ &< C_{58}\, n^{2sa+2tb-1+\epsilon} q^{-2sc-4tc};\end{aligned}$$

$$\begin{aligned}\sum_{\mathfrak{M}_1''} \int_{\mathfrak{M}_1''} |\psi_\rho^{\,s}(x)|^2 |\phi_\rho^{\,t}(x)|^2 |dx| &< C_{58}\, n^{2sa+2tb-1+\epsilon} \sum_{q=n^{\delta a}}^{n^a} 2q\, q^{-2sc-4tc}\\ &< C_{57}\, n^{2sa+2tb-1+\epsilon-(2sc+4tc-2)\delta a}.\end{aligned}$$

We have

$$(2sc+4tc-2)\delta a \geqslant (\tfrac{1}{2}+\tfrac{1}{4}-2)\,\tfrac{1}{2}\tfrac{3}{0}\cdot\tfrac{1}{5} = \tfrac{1}{8}\tfrac{3}{0} = L,$$

which completes the proof.

LEMMA 20.

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_{59}\, n^{2sa+2tb-1-L+\epsilon}.$$

Proof. By Lemmas 3 and 13,

$$\begin{aligned}\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f^s(x)|^2 |g^t(x)|^2 |dx| &\leqslant \max|f(x)|^{2s} \cdot \max|g(x)|^{2t-4} \sum_{\mathfrak{m}} \int_{\mathfrak{m}} |g^2(x)|^2 |dx|\\ &< C_{60}\, n^{2sa+2tb-4b-(2sac+4tbc-8bc)+\epsilon} \int_{-\frac{1}{2}}^{\frac{1}{2}} |g^2(e^{2\pi i y})|^2\, dy.\end{aligned}$$

Now

$$g^2(x) = \sum_{j=2P(M)}^{2P(N)} R_1(j)\, x^j,$$

where, in the notation of Lemma 12, $0 \leqslant R_1(j) \leqslant R(j)$.

Then **3**, Theorem 223, gives

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} |g^2(e^{2\pi i y})|^2\, dy &= \sum_{j=2P(M)}^{2P(N)} |R_1(j)|^2\\ &< \sum_{j=0}^{2P(N)} R^2(j) < C_{31}\big(2P(N)\big)^{2b+\epsilon}\\ &< C_{31}(2A_2)^{2b+\epsilon}\, n^{2b+\epsilon}\end{aligned}$$

by Lemma 12 and (2.36). Therefore

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_{60} C_{31} (2A_2)^{2b+\epsilon} n^{2sa+2tb-1-(2sa+4tb-8b)c+1-2b+\epsilon}.$$

By (2.21), $$(2sa+4tb-8b)c-(1-2b) \geqslant L,$$

and the result follows.

LEMMA 21.

$$\sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f^s(x) g^t(x) - \psi_\rho^s(x) \phi_\rho^t(x)|^2 |dx| < C_{61} n^{2sa+2tb-1-L+\epsilon}.$$

Proof. Since

$$\begin{aligned} |f^s g^t - \psi_\rho^s \phi_\rho^t|^2 &\leqslant 2|f|^{2s}|g^t-\phi_\rho^t|^2 + 2|\phi_\rho|^{2t}|f^s-\psi_\rho^s|^2 \\ &\leqslant 2^{2t+1}|f|^{2s}(|g-\phi_\rho|^{2t}+|g-\phi_\rho|^2|\phi_\rho|^{2t-2}) \\ &\qquad + 2^{2s+1}|\phi_\rho|^{2t}(|f-\psi_\rho|^{2s}+|f-\psi_\rho|^2|\psi_\rho|^{2s-2}), \end{aligned}$$

we have

$$\begin{aligned} &\sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f^s g^t - \psi_\rho^s \phi_\rho^t|^2 |dx| \\ &\quad \leqslant C_{62}\Big(\sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f|^{2s}|g-\phi_\rho|^{2t}|dx| + \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f|^{2s}|g-\phi_\rho|^2|\phi_\rho|^{2t-2}|dx| \\ &\qquad + \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |\phi_\rho|^{2t}|f-\psi_\rho|^{2s}|dx| + \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |\phi_\rho|^{2t}|f-\psi_\rho|^2|\psi_\rho|^{2s-2}|dx|\Big) \\ &\quad = C_{62} \sum_{\mathfrak{M}_1} (I_1+I_2+I_3+I_4). \end{aligned} \tag{5.51}$$

(1) From Lemmas 6 and 14,

$$\begin{aligned} I_1 &< C_{13}^{2s} C_{39}^{2t} n^{2sa+\epsilon} q^{2t-2sc-4tc+\epsilon} \left(\int_0^{1/n} dy + n^{2t} \int_{1/n}^{1/(2q^\nu n^{1-\theta a})} y^{2t} dy \right) \\ &< C_{63} n^{2sa+\epsilon} q^{2t-2sc-4tc+\epsilon} (n^{-1} + n^{-1} n^{(2t+1)\theta a} q^{-(2t+1)\theta}) \\ &\leqslant 2C_{63} n^{2sa-1+(2t+1)\theta a+\epsilon} q^{2t-(2t+1)\theta-2sc-4tc+\epsilon}, \end{aligned}$$

since $n^{(2t+1)\theta a} q^{-(2t+1)\theta} \geqslant 1$ on $\mathfrak{M}_1$. If the exponent of q is less than or equal to -2, we have

$$\sum_{\mathfrak{M}_1} I_1 < C_{64} n^{2sa+2tb-1+\epsilon} \cdot n^{-(2tb-(2t+1)\theta a)}.$$

Since, by (2.23),

$$(5.511)\quad 2tb-(2t+1)\theta a \geqslant 22b-23\theta a > \begin{cases} \frac{21}{2}a > (2-2a)c = L, & k \geqslant 7, \\ \frac{101}{14}a > \frac{13}{80} = L, & k = 5, \end{cases}$$

this gives

$$(5.512)\qquad \sum_{\mathfrak{N}_1} I_1 < C_{65}\, n^{2sa+2tb-1-L+\epsilon}.$$

If the exponent of q is greater than -2, we have

$$\sum_{\mathfrak{N}_1} I_1 < C_{66}\, n^{2sa+2tb-1+\epsilon} \cdot n^{-(2sc+4tc-2)a}.$$

As in the proof of Lemma 17, we have

$$(5.513)\qquad (2sc+4tc-2)a > L-La+1-5a+8ac > L,$$

since $L < 8c$ and $1 \geqslant 5a$. Hence (5.512) holds in either case.

(2) From Lemmas 6, 14, and 11,

$$I_2 < C_{13}^{2\epsilon} C_{39}^2 C_{24}^{2t-2}\, n^{2sa+\epsilon} q^{2-2sc-4tc+\epsilon}\left(n^{2tb-2b}\int_0^{1/n} dy + n^2 \int_{1/n}^{1/(2q^6 n^{1-6a})} y^{2-(2t-2)b}\, dy\right).$$

By (2.23) the exponent of y in the second integral is

$$\leqslant \begin{cases} 2-(2^{k-2}-2)b < -1 & \text{when} \quad k \geqslant 7, \\ 2-(22-2)b = -3 & \text{when} \quad k = 5. \end{cases}$$

Then $$I_2 < C_{67}\, n^{2sa+2tb-1-2b+\epsilon} q^{2-2sc-4tc}.$$

If the exponent of q is less than or equal to -2, we have

$$\sum_{\mathfrak{N}_1} I_2 < C_{68}\, n^{2sa+2tb-1+\epsilon} n^{-2b}$$

and

$$2b > \begin{cases} (2-2a)c = L & \text{when} \quad k \geqslant 7, \\ \frac{13}{80} = L & \text{when} \quad k = 5. \end{cases}$$

If the exponent of q is greater than -2, we have

$$\sum_{\mathfrak{N}_1} I_2 < C_{69}\, n^{2sa+2tb-1+\epsilon} n^{-(2sc+4tc-2)a-2(b-a)}$$

and, by (5.513),

$$(2sc+4tc-2)a+2(b-a) > (2sc+4tc-2)a > L.$$

Hence, in both cases,

$$\sum_{\mathfrak{M}_1} I_2 < C_{70} n^{2sa+2lb-1-L+\epsilon}. \tag{5.52}$$

(3) From Lemmas 11 and 5,

$$\begin{aligned} I_3 &< C_{12}^{2s} C_{24}^{2t} q^{2s-2sc-4lc+\epsilon} \left(n^{2lb} \int_0^{1/n} dy + n^{2s} \int_{1/n}^{1/(2q^3 n^{1-\theta a})} y^{2s-2lb}\, dy \right) \\ &< C_{12}^{2s} C_{24}^{2t} q^{2s-2sc-4lc+\epsilon} \left(n^{2lb} \int_0^{1/n} dy + n^{2s+2lb} \int_{1/n}^{1/(2q^3 n^{1-\theta a})} y^{2s}\, dy \right) \\ &< C_{71} n^{2lb} q^{2s-2sc-4lc+\epsilon} (n^{-1} + n^{-1+(2s+1)\theta a} q^{-(2s+1)\theta}) \\ &\leqslant 2C_{71} n^{2lb-1+(2s+1)\theta a} q^{2s-(2s+1)\vartheta-2sc-4lc+\epsilon}, \end{aligned}$$

since $n^{(2s+1)\theta a} q^{-(2s+1)\theta} \geqslant 1$ on $\mathfrak{M}_1$. If the exponent of q is less than or equal to -2, we have

$$\sum_{\mathfrak{M}_1} I_3 < C_{72} n^{2sa+2lb-1+\epsilon} n^{-(2s-(2s+1)\theta)a},$$

and, by (2.22),

$$\left(2s-(2s+1)\theta\right)a \geqslant \begin{cases} (2-3\theta)a = \frac{1}{2}a > (2-2a)c = L, & k \geqslant 7, \\ (8-9\theta)a = \frac{3}{70} > \frac{13}{80} = L, & k = 5. \end{cases}$$

If the exponent of q is greater than -2,

$$\sum_{\mathfrak{M}_1} I_3 < C_{73} n^{2sa+2lb-1+\epsilon} n^{-(2sc+4lc-2)a} < C_{73} n^{2sa+2lb-1-L+\epsilon}$$

by (5.513). Hence, in either case,

$$\sum_{\mathfrak{M}_1} I_3 < C_{74} n^{2sa+2lb-1-L+\epsilon}. \tag{5.53}$$

(4) From Lemmas 11, 5, and 2

$$\begin{aligned} I_4 &< C_2^{2s} C_{12}^2 C_{24}^{2l} q^{2-2sc-4lc+\epsilon} \left(n^{(2s-2)a+2lb} \int_0^{1/n} dy + n^2 \int_{1/n}^{1/(2q^3 n^{1-\theta a})} y^{2-(2s-2)a-2lb}\, dy \right) \\ &< C_{75} q^{2-2sc-4lc+\epsilon} n^{2sa+2lb-1-2a}. \end{aligned}$$

If the exponent of q is less than or equal to -2,

$$\sum_{\mathfrak{M}_1} I_4 < C_{76} n^{2sa+2lb-1-2a+\epsilon}$$

and

$$2a > \begin{cases} (2-2a)c = L, & k \geqslant 7, \\ \frac{13}{80} = L, & k = 5. \end{cases}$$

If the exponent of q is greater than -2,

$$\sum_{\mathfrak{M}_1} I_4 < C_{77} n^{2sa+2tb-1-(2sc+4tc-2)a+\epsilon}$$

and, by (5.513), $(2sc+4tc-2)a > L$. Hence, in both cases,

$$\sum_{\mathfrak{M}_1} I_4 < C_{78} n^{2sa+2tb-1-L+\epsilon}. \tag{5.54}$$

The required result now follows from (5.51), (5.512), (5.52), (5.53), and (5.54).

LEMMA 22.

$$\sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_{79} n^{2sa+2tb-1-L+\epsilon}.$$

Proof. By Lemmas 7 and 16,

$$\int_{\mathfrak{M}_2} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_{14}^{2s} C_{50}^{2t} n^{2sa+2tb-2sc-4tc+\epsilon} q^{-2sc-4tc-\epsilon} \int_{1/(2q^\theta n^{1-\theta a})}^{1/(qn^{1-a})} y^{-2sc-4tc-\epsilon} dy$$

$$+ C_{15}^{2s} C_{50}^{2t} n^{2sa+2tb-2sac-4tc+\epsilon} q^{-4tc-\epsilon} \int_{1/(qn^{1-a})}^{\infty} y^{-4tc-\epsilon} dy$$

$$< C_{80} n^{2sa+2tb-1-(2sc+4tc-1)\theta a+\epsilon} q^{-2sc-4tc+(2sc+4tc-1)\theta}$$

$$+ C_{81} n^{2sa+2tb-1-(2sc+4tc-1)a+\epsilon} q^{-1};$$

$$\sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} |f^s(x)|^2 |g^t(x)|^2 |dx|$$

$$< C_{82} n^{2sa+2tb-1+\epsilon} \begin{cases} n^{-(2sc+4tc-1)\theta a} & \text{if the exponent of } q \leqslant -2, \\ n^{-(2sc+4tc-2)a} & \text{if the exponent of } q > -2, \end{cases}$$

$$+ C_{83} n^{2sa+2tb-1-(2sc+4tc-2)a+\epsilon}.$$

By (5.513), $(2sc+4tc-2)a > L$ and, by the proof of Lemma 17,

$(2sc+4tc-1)\theta a > (L+2a)\frac{1}{2} > a > (2-2a)c = L$ when $k \geqslant 7$,

$(2sc+4tc-1)\theta a \geqslant (\frac{1}{2}+\frac{1}{4}-1)\frac{9}{14} \cdot \frac{1}{5} = \frac{81}{280} > \frac{13}{80} = L$ when $k = 5$,

whence the result follows.

LEMMA 23.

$$\sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_{84} n^{2sa+2tb-1-L+\epsilon}.$$

Proof. By Lemmas 4 and 15,

$$\int_{\mathfrak{M}_3} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_6^{2s} C_{49}^{2t} n^{2sa+2tb-2sac+\epsilon} q^{-4tc+\epsilon} \int_0^{1/(qn^{1-b})} dy$$
$$< C_{85} n^{2sa+2tb-1+b-2sac+\epsilon} q^{-1-4tc};$$

$$\sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} |f^s(x)|^2 |g^t(x)|^2 |dx| < C_{85} n^{2sa+2tb-1+b-2sac+\epsilon} \sum_{q=n^a}^{n^b} 2q q^{-1-4tc}$$
$$< C_{86} n^{2sa+2tb-(2sc+4tc-2)a+b-a+\epsilon} \quad \text{[by (2.23)]},$$

and, by the proof of Lemma 17,

$$(2sc+4tc-2)a-(b-a) > L+2a-(b-a) > L \qquad \text{when} \quad k \geqslant 7,$$

$$(2sc+4tc-2)a-(b-a) \geqslant (\tfrac{1}{2}+\tfrac{1}{4}-2)\tfrac{1}{5}-\tfrac{1}{20} = \tfrac{1}{5} > \tfrac{13}{80} = L \quad \text{when} \quad k = 5.$$

LEMMA 24.

$$\int_C \left| f^s(x)\, g^t(x) - \sum_{1 \leqslant q \leqslant n^{\delta a}} \sum_{\rho} \psi_\rho^{\,s}(x)\, \phi_\rho^{\,t}(x) \right|^2 |dx| < C_{87} n^{2sa+2tb-1-L+\epsilon}.$$

The integral is taken over the circle $|x| = 1$ *and the summation extends over all* ρ *for which* $1 \leqslant q \leqslant n^{\delta a}$. *There are* $\sum_{q=1}^{n^{\delta a}} \phi(q) < n^{2\delta a}$ *terms in the sum.*

Proof. (1) Let $k \geqslant 7$ so that $\delta = 1$. Then (1, Theorem 16)

$$\int_C \left| f^s g^t - \sum_\rho \psi_\rho^{\,s} \phi_\rho^{\,t} \right|^2 |dx| < 2 \sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f^s|^2 |g^t|^2 |dx| + 2 \sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} |f^s|^2 |g^t|^2 |dx|$$
$$+ 2 \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} |f^s|^2 |g^t|^2 |dx| + 2n^{2a} \sum_{\mathfrak{M}_1} \int_{C-\mathfrak{M}_1} |\psi_\rho^{\,s}|^2 |\phi_\rho^{\,t}|^2 |dx|$$
$$+ 2 \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f^s g^t - \psi_\rho^{\,s} \phi_\rho^{\,t}|^2 |dx|.$$

To the terms on the right we apply Lemmas 20, 22, 23, 17, and 21, respectively. This proves the lemma when $k \geqslant 7$.

(2) Let $k = 5$. Then (1, Theorem 17)

$$\int_C \left| f^s g^t - \sum_\rho \psi_\rho^{\,s} \phi_\rho^{\,t} \right|^2 |dx| < 2 \sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f^s|^2 |g^t|^2 |dx| + 2 \sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} |f^s|^2 |g^t|^2 |dx|$$
$$+ 2 \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} |f^s|^2 |g^t|^2 |dx| + 4n^{2\delta a} \sum_{\mathfrak{M}_1'} \int_{C-\mathfrak{M}_1'} |\psi_\rho^{\,s}|^2 |\phi_\rho^{\,t}|^2 |dx|$$
$$+ 4 \sum_{\mathfrak{M}_1''} \int_{\mathfrak{M}_1''} |\psi_\rho^{\,s}|^2 |\phi_\rho^{\,t}|^2 |dx| + 4 \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f^s g^t - \psi_\rho^{\,s} \phi_\rho^{\,t}|^2 |dx|.$$

The result for $k = 5$ now follows from Lemmas 20, 22, 23, 18, 19, and 21, respectively.

THEOREM 1.

$$\sum_{j=2t(P(M)+1)}^{n} |\sigma(j)|^2 < C_{87}\, n^{2s\alpha+2t\beta-1-L+\epsilon}.$$

Proof. We have

$$\psi_\rho^{\,s}(x)\,\phi_\rho^{\,t}(x) = \Gamma^s(1+a)\,\Gamma^t(1+b)\,q^{-s-t}\,S_\rho^{\,s}\,T_\rho^{\,t} \sum_{j=tP(M)+t}^{sn+tP(N)} \alpha(j)\left(\frac{x}{\rho}\right)^j,$$

$$\sum_{1\leqslant q\leqslant n^{\delta a}}^{\rho} \psi_\rho^{\,s}(x)\,\phi_\rho^{\,t}(x) = \Gamma^s(1+a)\,\Gamma^t(1+b) \sum_{j=tP(M)+t}^{sn+tP(N)} \alpha(j) \sum_{1\leqslant q\leqslant n^{\delta a}}^{\rho} q^{-s-t}\,S_\rho^{\,s}\,T_\rho^{\,t}\,\rho^{-j}\,x^j$$

$$= \Gamma^s(1+a)\,\Gamma^t(1+b) \sum_{j=tP(M)+t}^{sn+tP(N)} \alpha(j)\,\mathfrak{S}(j,\, n^{\delta a})\,x^j.$$

Also,

$$f^s(x)\,g^t(x) = \sum_{j=tP(M)}^{n} r(j)\,x^j + \sum_{j=n+1}^{sn+tP(N)} r_1(j)\,x^j,$$

where $0 \leqslant r_1(j) \leqslant r(j)$. Hence

$$f^s(x)\,g^t(x) - \sum_{\rho} \psi_\rho^{\,s}(x)\,\phi_\rho^{\,t}(x) = \sum_{j=2t(P(M)+1)}^{n} \sigma(j)\,x^j + \ldots.$$

From this it follows (**3**, Theorem 223) that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| f^s(e^{2\pi i y})\,g^t(e^{2\pi i y}) - \sum_{\rho} \psi_\rho^{\,s}(e^{2\pi i y})\,\phi_\rho^{\,t}(e^{2\pi i y}) \right|^2 dy$$

$$= \sum_{j=2t(P(M)+1)}^{n} |\sigma(j)|^2 + \text{a positive number.}$$

But, by Lemma 24, this integral is less than

$$C_{87}\, n^{2s\alpha+2t\beta-1-L+\epsilon},$$

and thus

$$\sum_{j=2t(P(M)+1)}^{n} |\sigma(j)|^2 < C_{87}\, n^{2s\alpha+2t\beta-1-L+\epsilon}.$$

7. *The singular series.*

Let p be any prime and let Θ denote the highest power of p which divides k. Let

$$\gamma = \begin{cases} \Theta+1, & p > 2. \\ \Theta+2, & p = 2. \end{cases}$$

Let $U(m) = U(s, t, j, m)$ denote the number of solutions of

$$\sum_{i=1}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \equiv j \pmod{m} \qquad (1 \leqslant u_i \leqslant m, \quad 1 \leqslant v_r \leqslant m); \tag{7.11}$$

and let $V(m) = V(s, t, j, m)$ denote the number of solutions of the same congruence in which at least one u_i is prime to m. Let s_0 and t_0 be integers such that*

$$V(s_0, t_0, j, p^\gamma) > 0 \tag{7.12}$$

for every j and every prime p.

We now make the further assumptions that s and t satisfy the conditions

$$s \geqslant s_0, \quad t \geqslant t_0. \tag{7.13}$$

LEMMA 25. *We have*

$$|A_q(j)| < C_{88} q^{-\frac{27}{40}+\epsilon},$$

and hence

$$\mathfrak{S}(j, \infty) = \sum_{q=1}^{\infty} A_q(j)$$

converges absolutely.

Proof. By Lemma 8, with $m = q$, $r = -1$, and **3**, Theorem 315,

$$|A_q(j)| \leqslant q^{-s-t} q |S_\rho|^s |T_\rho|^t < C_{88} q^{1-sa-2tc+\epsilon}.$$

The exponent of q is equal to

$$1-(sa+2tb)(k-1)c-sa\big(1-(k-1)c\big)+\epsilon$$

$$\leqslant 1-\tfrac{1}{2}\big((k-3)+(k-1)L\big)-4c-sa\big(1-(k-1)c\big)+\epsilon$$

$$\begin{cases} < 1-\frac{1}{2}(k-3)+\epsilon \leqslant -2+\epsilon & \text{when } k \geqslant 9, \\ \leqslant 1-\frac{1}{2}(4+6\cdot\frac{2}{7}\cdot\frac{6}{64})-\frac{1}{16}-\frac{1}{7}(1-\frac{6}{64})+\epsilon = -\frac{285}{224}+\epsilon & \text{when } k = 7, \\ \leqslant 1-\frac{1}{2}(2+4\cdot\frac{13}{80})-\frac{1}{4}-\frac{4}{5}(1-\frac{1}{4})+\epsilon = -\frac{47}{40}+\epsilon & \text{when } k = 5. \end{cases}$$

Since $-2 < -\frac{285}{224} < -\frac{47}{40}$, the result follows.

LEMMA 26. *If $s \geqslant s_0 = 2k-1$, $t \geqslant t_0 = 0$, then*

$$V(p^\gamma) > 0$$

for every j and every prime p.

* In Lemma 26 it is shown that such integers exist.

Proof. It is known (**3**, Theorem 301) that the congruence

$$\sum_{i=1}^{s} u_i^k \equiv j \pmod{p^\gamma} \quad (1 \leqslant u_i \leqslant p^\gamma)$$

has a solution with at least one u_i prime to p for every j and p whenever

$$s \geqslant (p-1)^{-1}(p^\gamma-1)(k, p-1)+1.$$

Also,

$$(p-1)^{-1}(p^\gamma-1)(k, p-1)+1$$

$$= \begin{cases} (2-1)^{-1}(2^2-1)+1 < 2k-1 & \text{when } p=2, \\ (p-1)^{-1}(p^{\Theta+1}-1)(kp^{-\Theta}, p-1) \leqslant (p-1)^{-1}pk+1 < 2k-1 & \text{when } p>2, \end{cases}$$

which proves the lemma.

LEMMA 27. *If $k=5$ and $s \geqslant s_0=4$, $t \geqslant t_0=3$, then*

$$V(p^\gamma) > 0$$

for every j and every prime p.

Proof. (1) Let $p \equiv 2$, 3, or 4 (mod 5), $p \neq 2$. Then $\gamma = 1$ and

$$(p-1)^{-1}(p^\gamma-1)(k, p-1)+1 = (5, p-1)+1 = 2,$$

so that, by **3**, Theorem 301, $s \geqslant 2$, $t \geqslant 0$ is sufficient in this case.

(2) Let $p=2$. Then $\gamma = 2$ and

$$(p-1)^{-1}(p^\gamma-1)(k, p-1)+1 = (2-1)^{-1}(2^2-1)+1 = 4.$$

Hence $s \geqslant 4$, $t \geqslant 0$ is sufficient when $p=2$.

(3) Let $p \equiv 1 \pmod 5$, $j \equiv 0 \pmod p$. Then $\gamma = 1$ and, by **7**, Theorem 9, with $m=5$ and $a=0$, the congruence $u_1^5+u_2^5 \equiv 0 \pmod p$ has a primitive solution*, so that $s \geqslant 2$, $t \geqslant 0$ is sufficient.

(4) Let $p \equiv 1 \pmod 5$, $p > 73$, $j \not\equiv 0 \pmod p$. Then $\gamma = 1$ and, by **3**, Theorems 293 and 320, when $s \geqslant 4$, $p > 73$, we have

$$p^{1-s}V(s, 0, j, p) = p^{1-s}N(p) = \chi_p > 1-k^s p^{\frac{1}{2}(1-s)}$$

$$\geqslant 1-5^4 p^{-\frac{3}{2}} > 0.$$

(5) Let $p = 11$, 31, 41, 61, or 71; $j \not\equiv 0 \pmod p$, $j \not\equiv \pm 5 \pmod{11}$. These are all the primes not exceeding 73 which are congruent to 1 (mod 5). By trial we verify that $s \geqslant 4$, $t \geqslant 0$ is sufficient in these cases.

* In fact, $u_1 = 1$, $u_2 = p-1$ is a solution.

(6) Let $p = 11$, $j \equiv \pm 5 \pmod{11}$. When $k = 5$, we have

$$P(v) = 10v^4 + 20N^2 v^2$$

and, if $v_1 \equiv 2$, $v_2 \equiv v_3 \equiv 3 \pmod{11}$, then

$$\sum_{r=1}^{3} (10v_r^4 + 20N^2 v_r^2) \equiv 9 \pmod{11}.$$

Hence the congruence

$$\sum_{i=1}^{4} u_i^5 + \sum_{r=1}^{3} (10v_r^4 + 20N^2 v_r^2) \equiv 5 \pmod{11}$$

has the solution $u_1 \equiv u_2 \equiv u_3 \equiv u_4 \equiv 2$, $v_1 \equiv 2$, $v_2 \equiv v_3 \equiv 3 \pmod{11}$; and the congruence

$$\sum_{i=1}^{4} u_i^5 + \sum_{r=1}^{3} (10v_r^4 + 20N^2 v_r^2) \equiv -5 \pmod{11}$$

has the solution $u_1 \equiv u_2 \equiv u_3 \equiv 2$, $u_4 \equiv 0$, $v_1 \equiv 2, v_2 \equiv v_3 \equiv 3 \pmod{11}$.

(7) Let $p = 5$. Then $\gamma = 2$ and it is easily verified that

$$\sum_{i=1}^{3} u_i^5 \equiv j \pmod{25}$$

has a solution with u_1 prime to 5 for every j. Since (1)–(7) include all possible cases, the proof is complete.

LEMMA 28. $$\sum_{q} \sum_{m} A_q(j) = m^{-(s+t-1)} U(m).$$

Proof. We follow closely the proof of **4**, Theorem 1. If ρ ranges over all m-th roots of unity and if $q = q(\rho)$ denotes the degree of ρ, then

$$\begin{aligned} m^{-(s+t-1)} U(m) &= m^{-s} \sum_{u_1, \ldots, u_s = 1}^{m} \sum_{v_1, \ldots, v_t = 1}^{m} \sum_{\rho} \rho^{\sum u_i^k + \sum P(v_r) - j} \\ &= \sum_{\rho} q^{-s-t} \rho^{-j} \left(q m^{-1} \sum_{u=1}^{m} \rho^{u^k} \right)^s \left(q m^{-1} \sum_{v=1}^{m} \rho^{P(v)} \right) \\ &= \sum_{\rho} q^{-s-t} \rho^{-j} \left(\sum_{u=1}^{q} \rho^{u^k} \right)^s \left(\sum_{v=1}^{q} \rho^{P(v)} \right)^t \\ &= \sum_{\rho} q^{-s-t} S_\rho^{\,s} T_\rho^{\,t} \rho^{-j} = \sum_{q} \sum_{m} A_q(j). \end{aligned}$$

LEMMA* 29. (**4**, Theorem 3.) *If* $a \equiv y^k \pmod{p^\gamma}$, $p \nmid y$, $l \geqq \gamma$, *then the congruence*

$$a \equiv h^k \pmod{p^l}$$

has a solution.

* In this lemma a is any integer and not $1/k$.

LEMMA 30. *If $s \geqslant s_0$, $t \geqslant t_0$, then, for $l \geqslant \gamma$,*

$$U(p^l) \geqslant p^{(l-\gamma)(s+t-1)}.$$

Proof. By Lemmas 26 or 27 there is at least one solution of the congruence

$$y^k + \sum_{i=2}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \equiv j \pmod{p^\gamma} \tag{7.7}$$

in which $p \nmid y$, $1 \leqslant u_i \leqslant p^\gamma$, $1 \leqslant v_r \leqslant p^\gamma$. Hence there are at least $p^{(l-\gamma)(s+t-1)}$ solutions of (7.7) in which $p \nmid y$, $1 \leqslant u_i \leqslant p^l$, $1 \leqslant v_r \leqslant p^l$. By Lemma 29, with $a = j - \sum_{i=2}^{s} u_i^k - \sum_{r=1}^{t} P(v_r)$, there is a solution of (7.11), with $m = p^l$, corresponding to each solution of (7.7). Thus

$$U(p^l) \geqslant p^{(l-\gamma)(s+t-1)}.$$

THEOREM 2. *If s and t satisfy the conditions* (2.21), (2.22), (2.23), *and* (7.13), *then*

$$\mathfrak{S}(j, \infty) > C_{89};$$

$$\mathfrak{S}(j, n^{\delta a}) > C_{90} \quad \textit{when} \quad n > C_{91}.$$

Proof. Let $(m_1, m_2) = 1$ and consider the numbers of solutions of the congruences

$$\sum_{i=1}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \equiv j \pmod{m_1 m_2} \tag{7.811}$$

$$(1 \leqslant u_i \leqslant m_1 m_2, \quad 1 \leqslant v_r \leqslant m_1 m_2);$$

$$\sum_{i=1}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \equiv j \pmod{m_1} \tag{7.812}$$

$$(1 \leqslant u_i \leqslant m_1, \quad 1 \leqslant v_r \leqslant m_1);$$

$$\sum_{i=1}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \equiv j \pmod{m_2} \tag{7.813}$$

$$(1 \leqslant u_i \leqslant m_2, \quad 1 \leqslant v_r \leqslant m_2).$$

Let $u_{11}, \ldots, u_{s1}, v_{11}, \ldots, v_{t1}$ and $u_{12}, \ldots, u_{s2}, v_{12}, \ldots, v_{t2}$ be any solutions of (7.812) and (7.813) respectively. We may determine $u_1, \ldots, u_s, v_1, \ldots, v_t$ so that

$$u_i \equiv u_{i1}, \quad v_r \equiv v_{r1} \pmod{m_1},$$

$$u_i \equiv u_{i2}, \quad v_r \equiv v_{r2} \pmod{m_2},$$

$$(1 \leqslant u_i \leqslant m_1 m_2, \quad 1 \leqslant v_r \leqslant m_1 m_2).$$

Then $u_1, \ldots, u_s, v_1, \ldots, v_t$ is a solution of (7.811). Conversely, every solution of (7.811) determines a solution of each of (7.812) and (7.813).

Hence $U(m_1 m_2) = U(m_1)\,U(m_2)$. Then we have

$$(m_1 m_2)^{-(s+t-1)}\,U(m_1 m_2) = m_1^{-(s+t-1)}\,U(m_1)\,m_2^{-(s+t-1)}\,U(m_2),$$

and thus, by Lemma 28,

$$\sum_{q \mid m_1 m_2} A_q(j) = \sum_{q \mid m_1} A_q(j) \sum_{q \mid m_2} A_q(j).$$

If p_m denotes the m-th prime, then

$$\sum_{q\ p_1^{l} \ldots p_l^{l}} A_q(j) = \prod_{p \leqq p^{l}} \sum_{q \mid p^{l}} A_q(j). \tag{7.82}$$

By Lemma 25 we have

$$\sum_{q\ p^{l}} A_q(j) = 1 + \sum_{\lambda=1}^{l} A_{p^{\lambda}}(j) > 1 - C_{88} \sum_{\lambda=1}^{\infty} p^{-(\frac{47}{48}-\epsilon)\lambda},$$

$$\sum_{q\ p^{l}} A_q(j) > 1 - C_{88}(p^{\frac{47}{48}-\epsilon} - 1)^{-1}. \tag{7.83}$$

By Lemmas 28 and 30,

$$\sum_{q\ p^{l}} A_q(j) \geqslant p^{-\gamma(s+t-1)}. \tag{7.84}$$

Finally, by Lemma 25, (7.82), (7.83), and (7.84),

$$\begin{aligned}\mathfrak{S}(j, \infty) &= \lim_{l \to \infty} \prod_{p \leqq p_l} \sum_{q\ p^{l}} A_q(j) \\ &\geqslant \prod_{p} \max\left(p^{-\gamma(s+t-1)},\ 1 - C_{88}(p^{\frac{47}{48}-\epsilon} - 1)^{-1}\right) \geqslant C_{89}.\end{aligned}$$

This proves the first result. The second result now follows, since, by Lemma 25,

$$\begin{aligned}\mathfrak{S}(j, n^{\delta a}) &= \mathfrak{S}(j, \infty) - \sum_{q > n^{\delta a}} A_q(j) \\ &> C_{89} - C_{88} \sum_{q > n^{\delta a}} A_q(j) > C_{89} - C_{92}\, n^{-(7\delta a/40)+\epsilon\delta a} > C_{90}\end{aligned}$$

when $n > C_{91}$.

8. *The proof of Theorem* 3.

In this section we prove

THEOREM 3. *Let s and t be integers satisfying the conditions* (2.21), (2.22), (2.23), *and* (7.13). *Let s_1 be the least integer greater than or equal to* 2 *for which*

$$L > (1-2a)(1-a)^{s_1-2}. \tag{8.1}$$

Then, if k is an odd integer greater than or equal to 5, all sufficiently large numbers are sums of $s+s_1$ non-negative k-th powers and t values of $P(v)$.

We first prove two lemmas.

LEMMA 31. (**3**, Theorem 345; **5**, Lemma 22.) *Let $Q_{s_1}(\xi)$ denote the number of positive integers $m \leqslant \xi$ for which the equation*

$$m = \sum_{\nu=1}^{s_1} z_\nu^k \quad (z_\nu \geqslant 0)$$

has a solution. Then

$$Q_{s_1}(\xi) > C_{93}\, \xi^{1-(1-2a)(1-a)^{s_1-2}-\epsilon}.$$

LEMMA 32. *Let $H(n)$ denote the number of integers j which satisfy the inequalities*

$$2t\big(P(M)+1\big) \leqslant j \leqslant n,$$

and for which the equation

$$j = \sum_{i=1}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \quad (0 \leqslant u_i,\ M \leqslant v_r \leqslant N) \tag{8.2}$$

has no solution. Then

$$H(n) < C_{94}\, n^{1-L+\epsilon}.$$

Proof. In the $H(n)$ terms of the sum

$$\sum_{j=2t(P(M)+1)}^{n} |\sigma(j)|^2,$$

for which equation (8.2) has no solution, we have $r(j) = 0$ and hence, by Lemma 10, Theorem 1, and (2.32)–(2.35),

$$|\sigma(j)|^2 = |\Gamma^s(1+a)\,\Gamma^t(1+b)\,\alpha(j)\mathfrak{S}(j,\ n^{\delta a})|^2 > C_{95} j^{2sa+2tb-2} > C_{96} n^{2sa+2tb-2}.$$

In the remaining terms $|\sigma(j)|^2 \geqslant 0$. Thus

$$H(n)\,.\,C_{96}\, n^{2sa+2tb-2} < \sum_{j=2t(P(M)+1)}^{n} |\sigma(j)|^2 < C_{87}\, n^{2sa+2tb-1-L+\epsilon}.$$

It then follows that

$$H(n) < C_{94}\, n^{1-L+\epsilon}.$$

Proof of Theorem 3. Let W be any integer and N any integer* for which

$$2t\big(P(M)+1\big) + N^{k-1} \leqslant W \leqslant (2^k-2)\,N^{k-1}. \tag{8.32}$$

* From (2.321) and (2.31) it is obvious that such an integer exists when W (and hence N) is sufficiently large.

Suppose that W is not representable in the form

$$W = \sum_{i=1}^{s} u_i^k + \sum_{\nu=1}^{s_1} z_\nu^k + \sum_{r=1}^{t} P(v_r) \quad (0 \leqslant u_i,\ 0 \leqslant z_\nu,\ M \leqslant v_r \leqslant N). \tag{8.33}$$

Since there are $Q_{s_1}(N^{k-1})$ positive integers $W_1 \leqslant N^{k-1}$ which are representable in the form

$$W_1 = \sum_{\nu=1}^{s_1} z_\nu^k \quad (0 \leqslant z_\nu),$$

there must be at least $Q_{s_1}(N^{k-1})$ integers $j = W - W_1$ which are *not* representable in the form

$$j = \sum_{i=1}^{s} u_i^k + \sum_{r=1}^{t} P(v_r) \quad (0 \leqslant u_i,\ M \leqslant v_r \leqslant N).$$

The integers j satisfy the inequalities

$$2t\Big(P(M)+1\Big) \leqslant j \leqslant W. \tag{8.34}$$

Hence, by Lemma 32, with $n = W \leqslant (2^k-2)N^{k-1}$, the number of non-representable integers is at most

$$H\Big((2^k-2)N^{k-1}\Big) < C_{94}(2^k-2)^{1-L+\epsilon} N^{(k-1)(1-L+\epsilon)}. \tag{8.35}$$

On the other hand, Lemma 31, with $\xi = N^{k-1}$, shows that the number of non-representable integers is at least

$$C_{93} N^{(k-1)(1-(1-2\alpha)(1-\alpha)^{s_1-2}-\epsilon)}. \tag{8.36}$$

From the inequalities (8.1), (8.32)–(8.36), we obtain

$$C_{93} N^{(k-1)(1-(1-2\alpha)(1-\alpha)^{s_1-2}-\epsilon)} < C_{94} N^{(k-1)(1-L+\epsilon)},$$

$$N < C_{98}, \quad W < C_{99}.$$

That is, every integer $W \geqslant C_{99}$ is a sum of $s+s_1$ non-negative k-th powers and t values of $P(v)$.

9. *The proof of Theorems* 4 *and* 5.

THEOREM 4. *Let s, s_1, and t be integers satisfying the conditions* (2.21), (2.22), (2.23), (7.13), *and* (8.1). *Then, if k is an odd integer greater than or equal to* 5, *we have*

$$G(k) \leqslant s+s_1+2t+1.$$

THEOREM 5. *If k is an odd integer, we have*

$$G(5) \leqslant 35,$$

$$G(k) \leqslant 2^{k-2}(k-3)+k+9+\zeta_k \quad (k \geqslant 7).$$

We require a lemma from prime number theory.

LEMMA 33. *For $w > C_{100}$ there is at least one prime d which does not divide w and which satisfies the conditions*

$$(w/F)^{a^2} < d \leqslant (w/E)^{a^2}; \tag{9.1}$$

$$d \equiv 2 \pmod{k},$$

where

$$E = 2t\Big((1+B)^k+(1-B)^k-2\Big)+4+2t, \quad F = 2^k-2+2t.$$

Proof. By §121 of **6** the number of primes $d \equiv 2 \pmod{k}$ between x and $x+\epsilon x$ increases indefinitely with x. Hence [with $x = (w/F)^{a^2}$, $\epsilon = (F/E)^{a^2}-1$] there are at least k^2+1 primes satisfying (9.1) when w is sufficiently large. Since their product is greater than $(w/F)^{1+a^2} > w$ (for w sufficiently large) at least one does not divide w.

Proof of Theorem 4. We determine d as in Lemma 33. Then, by **3**, Theorems 292 and 301, with $l = k$, $\gamma = 1$, $p = P = d$, the congruence $w \equiv w_1{}^k \pmod{d^k}$ has a solution with $0 < w_1 < d^k$. Let

$$w - w_1{}^k = d^k w_2, \quad w_2 = 2td^{k^2-k}+W. \tag{9.21}$$

From (9.1), (9.21), and (2.321), with $N = d^k$, we have

$$Ed^{k^2} \leqslant w < Fd^{k^2}, \tag{9.22}$$

$$(E-1)d^{k^2-k} < w_2 < Fd^{k^2-k}, \tag{9.23}$$

$$\begin{aligned} 2t\Big(P([Bd^k])+1\Big)+d^{k^2-k} &\leqslant (E-1-2t)d^{k^2-k} < W \\ &< (F-2t)d^{k^2-k} = (2^k-2)d^{k^2-k}. \end{aligned} \tag{9.24}$$

Then, by Theorem 3, with $N = d^k$, there is a solution of (8.33) when $W > C_{99}$, that is when $w > C_{101}$. From (9.21) it follows that the equation

$$\begin{aligned} w &= w_1{}^k+2td^{k^2}+d^kW \\ &= w_1{}^k+2td^{k^2}+d^k\Big(\sum_{i=1}^{s} u_i{}^k+\sum_{\nu=1}^{s_1} z_\nu{}^k+d^{-k}\sum_{r=1}^{t}\Big((d^k+v_r)^k+(d^k-v_r)^k-2d^{k^2}\Big)\Big) \\ &= w_1{}^k+\sum_{i=1}^{s}(du_i)^k+\sum_{\nu=1}^{s_1}(dz_\nu)^k+\sum_{r=1}^{t}\Big((d^k+v_r)^k+(d^k-v_r)^k\Big) \end{aligned}$$

has a solution in which $0 < w_1$, $0 \leqslant u_i$, $0 \leqslant z_r$, $[Bd^k] \leqslant v_r \leqslant d^k$ when $w > C_{101}$. That is,

$$G(k) \leqslant s+s_1+2t+1.$$

Proof of Theorem 5. By **3**, Theorem 350, we have $s_1 = 4+\zeta_k$ when $k \geqslant 7$ and by trial $s_1 = 8$ when $k = 5$. It is readily verified that

$$s = 2k-1, \quad s_1 = 4+\zeta_k, \quad t = 2^{k-3}(k-3)-\tfrac{1}{2}(k-5)$$

satisfy all the conditions of Theorem 4 when $k \geqslant 7$. Hence

$$G(k) \leqslant s+s_1+2t+1 = 2^{k-2}(k-3)+k+9+\zeta_k.$$

When $k = 5$, Lemma 27 shows that it is sufficient to take $s = 4$, $s_1 = 8$, $t = 11$, and then

$$G(5) \leqslant s+s_1+2t+1 = 35.$$

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A GENERALIZATION OF THE KUMMER 16_6 CONFIGURATION (I)

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The two figures of which a generalization is to be discussed are:

(i) the Segre figure* of ten points, fifteen planes, and fifteen solids in [4], and

(ii) the Kummer 16_6 figure† of nodes and tropes of a Kummer quartic surface in [3].

The most general figures of these types are those in which the incidences are all expressible in terms of

(i) a set of $\binom{2r+1}{r}$ points in $\left[\binom{2r}{r}-\binom{2r}{r-2}-1\right]$, and

(ii) a set of 2^{2r} points in $[2^r-1]$,

the Segre and Kummer figures being the case $r=2$. We shall call the first the Segre-type figure, and the second either the Kummer-type figure, or, since the points lie by sets of $2r+2$ in 2^{2r} $[r]$'s, the $(2^{2r})_{2r+2}$ figure.

In this paper the case $r=3$, namely the figure consisting of thirty-five points in [13], and the 64_8 figure in [7], is discussed in detail, consideration of the more general figure being deferred to a later paper. The method adopted is to find a notation for the ten points of the Segre figure which expresses symmetrically the incidence relations‡, and then to consider the consequences of extending the range of the suffixes of the symbols. A summary of the essential features of the Segre figure from this aspect is given in § 1.

The 16_6 figure is the projection of the Segre figure from a point of the Segre manifold which consists of the lines meeting sets of five from among

* *Cf.* Baker (2), 113, for a discussion of the dual figure, and 151 for the figure itself.

† *Cf.* Hudson (1), 5.

‡ The notation is that used by Baker in (2), 115, for the dual figure.

the fifteen planes*. In the same way the 64_8 may be obtained from the Segre-type figure in [13] by projecting from a tangent [5] to a certain fivefold in the [13], and the more general cases are obtainable by similar projections.

Considerable use is made of the representation on a [5] of the Segre-type manifold and its projection; corresponding to the representation of the Segre manifold in [4] by quadric surfaces in [3] through five points†, there is a representation of the manifold in [13] by cubic primals in [5] with seven fixed double points, and just as the representation of the projection of the Segre manifold is obtained by restricting the quadrics to pass through another fixed point, so in the generalization the cubics are made to have another fixed double point.

There is one property of the Segre-Kummer figure which generalizes in a rather striking manner: since the Segre locus is a cubic threefold in [4] it projects from an ordinary point of itself into a *repeated* solid, *i.e.* a solid of which every point is the projection of two points of the manifold, and there is thus a (1, 2) correspondence between the points of the solid and the manifold. The coincident point-pairs of the involution on the Segre manifold project into the Kummer surface. The same holds good in the more general case: the Segre-type fivefold in [13] projects from a tangent [5] into a fivefold in [7] in such a way that every point of this latter manifold is the projection of a pair of points of an involution on the former, and the points derived from the coincident point-pairs form a Kummer-type threefold. Further, there is a group of sixty-four collineations and sixty-four correlations of the 64_8 and the Kummer-type threefold into themselves precisely analogous to the group of self-transformations of the 16_6 and the Kummer surface‡.

A summary of the principal properties of the Segre-type figure will be found in § 11, and of the 64_8 in § 13, and in the table in § 16. The group of self-transformations is set out in § 20.

A generalization of the Kummer figure from the function theory point of view was evolved some forty years ago by Wirtinger§, and about twenty years later Barrau‖ obtained a similar figure as the set of points whose coordinates were permutations of the same set of eight numbers, with suitable arrangements of signs¶. Although the figure discussed here

* Baker (**2**), 139. A résumé of the properties is given here in § 12.

† Baker (**2**), 152.

‡ *Cf.* Hudson (**1**), 40, for the transformations of 16_6.

§ Wirtinger (**3**), for the case corresponding to $r = 3$, and (**4**) for the general case.

‖ Barrau (**9**).

¶ The corresponding description of the 16_6 is to be found in Hudson (**1**), 28. Barrau's scheme of sixty-four points is reproduced in this paper in § 22.

bears some striking resemblances to the Wirtinger figure, the differences between the two are sufficiently marked to make it difficult to explain one figure as a special case of the other. These differences are discussed in § 22. On the other hand, that the figure discussed here is very closely related to Wirtinger's figure is clearly brought out in the alternative notation recently proposed by Bath and Richmond* for period-characteristics and period-marks of ϑ functions.

At the same time as I was first working on this configuration, Mr. Richmond had a similar piece of work in hand, and had made some notes on the Segre-type figure treated in this paper from a point of view not very different from that adopted here. Mr. Richmond very kindly put these notes at my disposal, and I wish to acknowledge here my debt to Mr. Richmond, and to express my thanks.

I. The analogue in [13] of Segre's figure in [4].

1. *The notation on which the analogy is based.*

Consider the set of ten points represented by the symbols $A_{\alpha\beta\gamma}$ for which $\alpha, \beta, \gamma = 1, 2, \ldots, 6$, no two of the suffixes are equal, and their order is immaterial, and which are such that $A_{\alpha\beta\gamma}$ and $A_{\alpha'\beta'\gamma'}$ represent the same point when $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are all different, the points being connected by the fifteen syzygies

$$\sum_{\gamma} A_{ab\gamma} = 0. \qquad (1.1)$$

A typical syzygy of this set is

$$A_{123}+A_{124}+A_{125}+A_{126} = 0,$$

which may also be written

$$A_{456}+A_{563}+A_{634}+A_{345} = 0.$$

If there are no syzygies other than these, five of the points are independent, *i.e.* the figure lies in [4].

The four points connected by the relation (1.1) lie together in a plane which we call ϖ_{ab}. There are fifteen of these planes, two of which meet in a point if their symbols have a common suffix, $(\varpi_{ab}, \varpi_{bc}) = A_{abc}$, and in a line if they have no common suffix,

$$(\varpi_{ab}, \varpi_{cd}) = [A_{abe}, A_{abf}] = [A_{cdf}, A_{cde}].$$

The solid defined by the pair of planes ϖ_{ab}, ϖ_{cd} contains also the plane ϖ_{ef}, and there are thus altogether fifteen of these solids, with three

* Bath and Richmond (10).

planes in each solid, and three solids through each plane. Each set of five planes having a common suffix is an "associated" set, any four of them being general planes of the [4] and defining uniquely the fifth. The lines which meet any four of a set of five associated planes also meet the fifth plane of the set, and the six such systems of ∞^2 lines generate the same threefold, Segre's cubic locus, the symbol for a general point of which may be written in any of the fifteen forms of which the pair

$$\sum_{\alpha\beta} y_\alpha y_\beta A_{\alpha\beta 6} \equiv \sum_{\alpha\beta} y_\alpha y_\beta A_{\alpha'\beta' 5} \quad (\alpha, \beta, \alpha', \beta' = 1, 2, 3, 4).$$

is typical, the y_α being parameters.

2. *The figure defined by a certain set of thirty-five points in* [13].

To generalize the Segre figure we increase by one the number of suffixes in the symbols of the set of points, and investigate the consequences of the analogous set of syzygies. That is, we take the thirty-five points represented by the symbols $A_{\alpha\beta\gamma\delta}$ for which $\alpha, \beta, \gamma, \delta = 1, 2, \ldots, 8$, no two of the suffixes being equal, and their order not being significant, and which are such that $A_{\alpha\beta\gamma\delta}$ and $A_{\alpha'\beta'\gamma'\delta'}$ represent the same point when $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ are all different. We suppose that these points are connected by the fifty-six syzygies

$$\sum_{\delta} A_{abc\delta} = 0. \tag{2.1}$$

A typical syzygy is

$$A_{1234}+A_{1235}+A_{1236}+A_{1237}+A_{1238} = 0,$$

and this may also be written

$$A_{5678}+A_{6784}+A_{7845}+A_{8456}+A_{4567} = 0.$$

We find later that only twenty-one of these syzygies are independent, so that the figure lies in [13].

The five points connected by the relation (2.1) lie together in a [3], which we call Π_{abc}. There are fifty-six of these solids, meeting in the following manner:

Π_{abc} meets Π_{abd} in the point A_{abcd},

Π_{abc} does not meet Π_{ade},

Π_{abc} meets Π_{def} in the line $A_{abcg}A_{abch} = A_{defh}A_{defg}$.

Through the point A_{1234} there pass eight of these solids, viz. $\Pi_{\alpha\beta\gamma}$ $(\alpha, \beta, \gamma = 1, 2, 3, 4)$ and $\Pi_{\alpha'\beta'\gamma'}$ $(\alpha', \beta', \gamma' = 5, 6, 7, 8)$. Two members of

the same set of four have only A_{1234} common, but two members of different sets have altogether a line common. Thus the eight solids lie in a space of dimension 9, at most.

Now consider the set of points whose symbols have two fixed suffixes, say 7, 8. There are fifteen of these points, and they are connected by the six syzygies

$$\sum_{\beta} A_{\alpha\beta 78} = 0 \quad (\alpha, \beta = 1, 2, \ldots, 6),$$

all of which are independent. Thus the points lie in a [8], which we call Δ_{78}. There are altogether twenty-eight of these [8]'s Δ_{ab}. The ten points $A_{\alpha\beta 78}$, where $\alpha, \beta = 1, \ldots, 5$, all lie in Δ_{78}, and therefore are connected by at least one syzygy. The only such relation that can be deduced from the six relations above is

$$\sum_{\alpha, \beta} A_{\alpha\beta 78} = 0 \quad (\alpha, \beta = 1, 2, \ldots, 5). \qquad (2.2)$$

There are twenty-eight sets of six such syzygies, each involving ten points. The general syzygy of the type can be written in either of the two forms

$$\sum_{\alpha, \beta} A_{\alpha\beta cd} = 0 \quad (\alpha, \beta \neq c, d, e), \qquad (2.2)$$

or

$$\sum_{\alpha'\beta'\gamma'} A_{\alpha'\beta'\gamma' e} = 0 \quad (\alpha', \beta', \gamma' \neq c, d, e). \qquad (2.2')$$

In Δ_{78} we have six solids $\Pi_{\alpha 78}$, of which every pair has a common point, these points being the fifteen $A_{\alpha\beta 78}$. This figure in the [8] is one that is known*; it is completely defined by any five solids in [8] every pair of which has a common point. Any plane which meets five of the solids and does not pass through a point common to two of them meets also the sixth, and on this account the solids are said to be "associated". Further, the six points of incidence in any of these planes lie on a conic. Any four of the solids are met in planes by a single [5], which contains the point common to the remaining two solids. *E.g.*, the [5] meeting Π_{178}, Π_{278}, Π_{378}, Π_{478} in planes contains the seven points which are connected by the syzygy

$$A_{5678} = \sum_{\alpha, \beta} A_{\alpha\beta 78} \quad (\alpha, \beta = 1, 2, 3, 4), \qquad (2.3)$$

which may also be written

$$A_{1234} = \sum_{\alpha', \beta'} A_{\alpha'\beta' 56} \quad (\alpha', \beta' = 1, 2, 3, 4). \qquad (2.3')$$

* (7), 473. The points $A_{\alpha\beta 78}$ in Δ_{78} correspond exactly to the points $A_{\alpha\beta}$ in that paper.

There are fifteen of these [5]'s in each [8] Δ_{ab}, and, from the equivalent forms of the syzygy (2.3) and (2.3′), we see that Δ_{ab} and Δ_{cd} have one of these [5]'s common. We may thus denote a [5] of this family by Λ^{ab}_{cd}, where this is the same as Λ^{cd}_{ab} and Λ^{ba}_{cd}, the seven points in it being connected by the relation

$$A_{abcd} \equiv A_{efgh} = \sum_{\gamma,\delta} A_{\gamma\delta ab} \equiv \sum_{\alpha,\beta} A_{\alpha\beta cd}$$

$$(\alpha, \beta, \gamma, \delta = e, f, g, h). \quad (2.3)$$

There are 210 of these [5]'s. We have seen that the intersection of two [8]'s Δ_{ab} whose symbols have no common suffix is a [5]; a pair of [8]'s of the other possible type, say Δ_{ab} and Δ_{bc}, have common a solid Π_{abc}, through which a third [8], Δ_{ac}, passes.

To find the dimension of the space in which the figure defined by the whole set of thirty-five points lies, we select the set of twenty points $A_{\alpha\beta\gamma 8}$ in which one suffix is always 8, and none is ever 7 (or, what is the same thing, one suffix is always 7 and none is ever 8). The remaining fifteen points are easily expressible in terms of these by means of the syzygies (2.1). The twenty points are connected in sets of ten by the six relations

$$\sum_{\alpha,\beta} A_{\alpha\beta c 8} = 0 \quad (\alpha, \beta, c = 1, \ldots, 6). \quad (2.2)$$

These are independent, and there are no other independent relations, so that this set of twenty points, and consequently the whole figure, lies in [13]*. It is often more convenient to discuss the figure in terms of these twenty points rather than in terms of the whole set of thirty-five. To avoid confusion we write

$$\mathbf{8}_{\lambda\mu\nu} \equiv A_{\lambda\mu\nu 8} \equiv A_{\lambda'\mu'\nu' 7} \equiv \mathbf{7}_{\lambda'\mu'\nu'},$$

using, as far as possible, α, β, γ for suffixes when the range is 1, ..., 8; λ, μ, ν when the range is 1, ..., 6; and later ρ, σ, τ when the range is 1, ..., 7. The syzygies then take the forms

$$\mathbf{8}_n \equiv \sum_{\lambda\mu} \mathbf{8}_{\lambda\mu n} \equiv \sum_{\lambda'\mu'\nu'} \mathbf{7}_{\lambda'\mu'\nu'} = 0 \quad (2.4)$$

and

$$\mathbf{7}_n \equiv \sum_{\lambda\mu} \mathbf{7}_{\lambda\mu n} \equiv \sum_{\lambda'\mu'\nu'} \mathbf{8}_{\lambda'\mu'\nu'} = 0 \quad (2.4')$$

$$(\lambda, \mu, \lambda', \mu', \nu' \neq n).$$

* We have also the six syzygies

$$\sum_{\alpha\beta} A_{\alpha\beta c 7} = 0 \quad (\alpha, \beta, c = 1, \ldots, 6),$$

but these are easily deducible from the others.

The four [8]'s Δ_{12}, Δ_{34}, Δ_{56}, Δ_{78} all pass through the plane

$$A_{1278}A_{3478}A_{5678},$$

and any pair of them have a common [5]. Thus they lie in a space of at most twelve dimensions, and from consideration of the syzygies it follows that the dimension is exactly twelve. There are 105 of these primes. $[\Delta_{12}, \Delta_{34}, \Delta_{56}, \Delta_{78}]$ contains twenty-seven of the points $A_{\alpha\beta\gamma\delta}$, namely all thirty-five with the exception of the eight whose suffixes are $\alpha = 1$ or 2, $\beta = 3$ or 4, $\gamma = 5$ or 6, $\delta = 7$ or 8.

3. *The eight times ruled fivefold* Ω.

We use as reference points the set of twenty $\mathbf{8}_{\lambda\mu\nu} \equiv \mathbf{7}_{\lambda'\mu'\nu'}$ which are connected by the two sets of six syzygies (2.4) and (2.4′). Consider the fivefold the symbol for whose general point is

$$P \equiv \Sigma y_\lambda y_\mu y_\nu \mathbf{8}_{\lambda\mu\nu} = \prod_{\kappa=1}^{6} y_\kappa \, \Sigma y_{\lambda'}^{-1} y_{\mu'}^{-1} y_{\nu'}^{-1} \mathbf{7}_{\lambda'\mu'\nu'}, \tag{3.1}$$

the y_λ being parameters. It can be proved that P may also be expressed as

$$\sum_{\alpha\beta\gamma} z_\alpha z_\beta z_\gamma A_{\alpha\beta\gamma k},$$

where α, β, γ range over any *six* of the values 1, ..., 8 which do not include k, and the parameters z are simple functions of the parameters y. Thus the manifold stands in a symmetrical relation to all the points $A_{\alpha\beta\gamma\delta}$, as in fact appears later, without requiring algebraic verification, when the representation is considered.

The points P which lie in $\Delta_{68} = [A_{\lambda\mu 68}]$ $(\lambda, \mu = 1, ..., 5)$ are clearly those for which y_6 is constant. They form a fourfold ω_{68} whose general point is*

$$P_{6,8} \equiv \sum_{\lambda\mu} y_\lambda y_\mu \mathbf{8}_{\lambda\mu 6} \quad (\lambda, \mu = 1, ..., 5).$$

This manifold is known†; it is of order 10. It may be defined thus: meeting the six associated solids $\Pi_{\alpha 68}$ in Δ_{68} there are ∞^3 planes, and the six points of incidence in each of the planes lie on a conic, say ϑ_{68}. The

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* The point $P_{6,8}$ is not the same as $P_{8,6}$, the symbol for which is

$$\sum_{\lambda\mu} z_\lambda z_\mu A_{\lambda\mu 68} \quad (\lambda, \mu = 1, ..., 5),$$

but as P varies both generate the same manifold $\omega_{68} = (\Omega, \Delta_{68})$. We emphasize the distinction by inserting the comma between the suffixes.

† (7), 478. The symbol for the point $P_{6,8}$ is there written $\Sigma \mu_\alpha \mu_\beta A_{\alpha\beta}$. $A_{\alpha\beta}$ is the same point as $\mathbf{8}_{\alpha\beta 6}$ above.

locus of these conics is ω_{68}. The conic which passes through $P_{6,8}$ meets Π_{568} in the point*

$$P^5_{68} = \sum_\lambda y_\lambda (y_\lambda - y_5)^{-1} A_{\lambda 568} \quad (\lambda = 1, 2, 3, 4).$$

From the form $\Sigma y_\lambda y_\mu 8_{\lambda\mu 6}$ of its general point, it follows, taking account of the single syzygy connecting the base points, viz. $\sum_{\lambda\mu} 8_{\lambda\mu 6} = 0$, that ω_{68} is represented on a fourfold space by the system of quadrics that pass through six fixed points. Thus there are on ω_{68}:

Six solids, corresponding to the base points, viz. $\Pi_{\lambda 68}$;

Six ∞^3 families of lines, corresponding to the lines through the various base points of the representation;

Fifteen multiple points, corresponding to the joins of pairs of the base points of the representation, viz. $A_{\lambda\mu 68}$†;

One ∞^6 family of conics, corresponding to the lines of the [4];

One ∞^3 family of conics, corresponding to the normal rational curves through the six base points, namely the conics ϑ_{68}. One of these conics passes through any point of ω_{68} and in particular through any point of a solid $\Pi_{\lambda 68}$.

The theorems have been stated for the particular manifold ω_{68}, but they are all, of course, true of any of the twenty-eight manifolds $\omega_{ab} = (\Omega, \Delta_{ab})$.

4. *The eight families of lines on* Ω.

Since Ω is a fivefold, the points in the immediate neighbourhood of P lie in a [5], say $\mathfrak{P}$, which is defined by the six points obtained by differentiating the symbol for the point P with respect to each parameter in turn.

$$\partial P / \partial y_6 \equiv \sum_{\lambda\mu} y_\lambda y_\mu 8_{\lambda\mu 6}.$$

This is the point that we have previously called $P_{6,8}$, which, as P varies, generates ω_{68}. Any line on Ω which passes through P lies in $\mathfrak{P}$.

From the form

$$P \equiv \Pi y_\kappa \Sigma y_\lambda^{-1} y_\mu^{-1} y_\nu^{-1} 7_{\lambda\mu\nu},$$

we find, by differentiating with respect to the parameters y_λ^{-1}, other

* We see later than $P^5_{68} = P^5_{85}$, and do not therefore put a comma between the suffixes.

† The points are triple, since four quadrics, which have common a line and four base points, meet further in seven points.

points in $\mathcal{P}$. In particular we have, omitting the multiple,

$$\partial P/\partial y_6^{-1} \equiv \Sigma y_\lambda^{-1} y_\mu^{-1} \mathbf{7}_{\lambda\mu 6},$$

which is the same point as $P_{6,7}$ lying in Δ_{67}. We can write $P_{6,7}$ in the form

$$y_6 P_{6,7} \equiv \Sigma y_\lambda y_\mu y_\nu \mathbf{8}_{\lambda\mu\nu} \quad (\lambda, \mu, \nu = 1, \ldots, 5),$$

which shows that there is a syzygy

$$y_6(P_{6,7} + P_{6,8}) = P.$$

Thus the join of $P_{6,7}$ and $P_{6,8}$ passes through P, and by symmetry we must have all seven points $P_{6,\alpha}$ collinear with P. This establishes the existence on Ω of one ∞^4 family of lines, whose members pass one through any point of Ω and meet the seven [8]'s $\Delta_{\alpha 6}$ one of whose suffixes is 6. By symmetry there are eight families of lines, $l_1, \ldots, l_8$. One line of each family passes through any point P, and the eight lines through P all lie in $\mathcal{P}$. The line l_a through P meets Δ_{ab} in $P_{a,b}$, which, as P varies on Ω, varies over the manifold ω_{ab}. The six systems of lines that lie on ω_{ab} belong to the six systems l_α other than l_a and l_b.

Now consider the two points in which $\mathcal{P}$ meets any of the manifolds ω_{ab}, say in particular $P_{6,8}$ and $P_{8,6}$, which lie on ω_{68}. We have seen that

$$P_{6,8} \equiv \Sigma y_\lambda y_\mu \mathbf{8}_{\lambda\mu 6},$$

and it is not difficult to show that

$$P_{8,6} \equiv \Sigma y_\lambda y_\mu (y_\lambda - y_6)^{-1}(y_\mu - y_6)^{-1} \mathbf{8}_{\lambda\mu 6}.$$

Further, the conic ϑ_{68} through $P_{6,8}$ meets Π_{568} in the point

$$P_{68}^5 \equiv \Sigma y_\lambda (y_\lambda - y_5)^{-1} \mathbf{8}_{\lambda 56} \quad (\lambda = 1, 2, 3, 4),$$

and in the same way the conic ϑ_{86} through $P_{8,6}$ meets Π_{568} in the point

$$\Sigma z_\lambda (z_\lambda - z_5)^{-1} \mathbf{8}_{\lambda 56},$$

where

$$z_\lambda = y_\lambda (y_\lambda - y_6).$$

This is the same point as P_{68}^5, and thus the two conics ϑ_{68} must be the same. Thus $\mathcal{P}$ meets Δ_{68} in a line which is a chord of ω_{68} and lies in one of the planes that meet all six solids $\Pi_{\alpha 68}$.

Next let P vary on l_a. Since $P_{a,b}$ is fixed, the conic ϑ_{ab} on ω_{ab} is fixed, and thus, as P varies, $P_{b,a}$ describes the conic ϑ_{ab}. Further, from their symbols, the ranges described by P on l_a, and by $P_{b,a}$ on ϑ_{ab}, are related, so that the locus of lines l_b which meet a given line l_a is a quadric surface in [3], whose other system of generators consists of the lines l_a meeting these lines l_b.

Before leaving this figure, we prove a result that is of importance later, viz: the eight points $A_{abc\delta}$, P^a_{bc}, P^b_{ca}, P^c_{ab} which lie in Π_{abc} lie on a twisted cubic curve. In particular, take the points in Π_{678}. We have

$$P^6_{78} \equiv \Sigma (y_\alpha - y_6)^{-1} A_{\alpha 678},$$

$$P^7_{86} = \Sigma y_\alpha^{-1} A_{\alpha 678},$$

and

$$P^8_{67} \equiv \Sigma y_\alpha A_{\alpha 678}.$$

These three points and the five points $A_{\alpha 678}$ all lie on the twisted cubic γ_{678} which is the locus as θ varies of the point $\Sigma(\theta - y_\alpha)^{-1} A_{\alpha 678}$, being given respectively by $\theta = y_6, 0, \infty, y_\alpha$.

5. *The tangent* [9], $\nabla_{\alpha\beta\gamma\delta}$, *at a singular point* $A_{\alpha\beta\gamma\delta}$ *of* Ω.

The point $A_{1238} = \mathbf{8}_{123}$ on Ω is given by $y_4 = y_5 = y_6 = 0$, so that the points of the manifold in the neighbourhood of this point are given by $\Sigma \delta_\lambda \partial P / \partial y_\lambda$ wherein y_4, y_5, y_6 are put equal to zero after differentiation and $\delta_4, \delta_5, \delta_6$ are small. Written at full length this point is

$$(\delta_1 y_2 y_3 + \delta_2 y_3 y_1 + \delta_3 y_1 y_2)\mathbf{8}_{123} + \sum_{\kappa = 4, 5, 6} \delta_\kappa (y_2 y_3 \mathbf{8}_{23\kappa} + y_3 y_1 \mathbf{8}_{31\kappa} + y_1 y_2 \mathbf{8}_{12\kappa}).$$

To obtain the tangent cone to Ω at 8_{123} we remove the restriction that $\delta_4, \delta_5, \delta_6$ should be small, and allow all the parameters to vary arbitrarily. By an obvious change of parameters we can rewrite this general point of the tangent cone as

$$\xi \mathbf{8}_{123} + \Sigma \eta_\kappa (\zeta_1 \mathbf{8}_{23\kappa} + \zeta_2 \mathbf{8}_{31\kappa} + \zeta_3 \mathbf{8}_{12\kappa}).$$

Thus all the points lie in a [9], namely the [9] that contains the eight solids $\Pi_{\alpha\beta\gamma}$ through $\mathbf{8}_{123}$, and form a cone which is the locus of solids which join to $\mathbf{8}_{123}$ the planes which meet a certain three planes in projectively related systems. The cone consists of all the solids that can be drawn through $\mathbf{8}_{123}$ to meet in lines the solids of either of the sets of four Π_{123}, Π_{238}, Π_{318}, Π_{128}, or Π_{456}, Π_{567}, Π_{647}, Π_{457}. It can be represented by the vanishing of two row minors of a determinant of order 3 whose elements are linear functions of the coordinates in [9], so that its curve sections are normal elliptic sextic curves. Thus the cone is of order 6, and therefore

$A_{\alpha\beta\gamma\delta}$ is a sixfold point on Ω.

We denote the tangent [9] at the point by $\nabla_{\alpha\beta\gamma\delta}$. It has the remarkable property that it meets every tangent [5] to Ω in a plane, *i.e.* lies with it in a prime. This is easily verified in the case of A_{1238} by

considering the prime which is the locus of points $\Sigma x_{\lambda\mu\nu}\mathbf{8}_{\lambda\mu\nu}$ for which*

$$(a_2-a_3)\{a_4(a_5-a_6)x_{156}+a_5(a_6-a_4)x_{164}+a_6(a_4-a_5)x_{145}\}$$
$$+\text{two similar terms}=0. \quad (5.1)$$

This clearly contains the ten points $\mathbf{8}_{123}$, $\mathbf{8}_{23\kappa}$, $\mathbf{8}_{31\kappa}$, $\mathbf{8}_{12\kappa}$ ($\kappa=4, 5, 6$) (as well as $\mathbf{8}_{456}$) which define the tangent [9], and the six points $\Sigma a_\lambda a_\mu \mathbf{8}_{\lambda\mu n}$ which define the tangent [5] at the point $\Sigma a_\lambda a_\mu a_\nu \mathbf{8}_{\lambda\mu\nu}$ of Ω. This verification applies to the twenty points $A_{\lambda\mu\nu 8}$, and the theorem follows for the remaining fifteen points $A_{\lambda\mu 78}$ by symmetry.

It is not conversely true that every prime through $\nabla_{\alpha\beta\gamma\delta}$ contains a tangent [5]; on the other hand, if the prime contains one such [5], it contains ∞^3†.

6. *The osculating primes of* Ω.

At any point of a fivefold which lies in a space of sufficiently high dimension, there is an osculating space defined by the points $\partial^2 P/\partial y_\alpha \partial y_\beta$, the y_α being the parameters in terms of which the point P of the manifold is expressed. In the case of Ω there are no points $\partial^2 P/\partial {y_\alpha}^2$, and of the fifteen points

$$\partial^2 P/\partial y_m \partial y_n = P^8_{mn} = \sum_\lambda y_\lambda \mathbf{8}_{\lambda mn} \quad (\lambda, m, n = 1, \ldots, 6)$$

only thirteen are independent. For we have the obvious syzygy [using (2.4)]

$$\sum_{\mu\nu}\sum_\lambda y_\lambda \mathbf{8}_{\lambda\mu\nu} \equiv \sum_\kappa y_\kappa \mathbf{8}_\kappa = 0.$$

Then we can also express all the points in terms of $\mathbf{7}_{\lambda\mu\nu}$, and we get (omitting a multiple)

$$\partial^2 P/\partial y_m^{-1} \partial y_n^{-1} = \Sigma y_\lambda^{-1} \mathbf{7}_{\lambda mn},$$

---

* Not every linear equation in the twenty coordinates represents a prime, but this equation satisfies the necessary conditions, which are set out at the beginning of § 7.

† The equation of a general prime through ∇_{1238} is

$$\Sigma k_{\rho\kappa\kappa'} x_{\rho\kappa\kappa'} = 0 \quad (\rho = 1, 2, 3; \ \kappa, \kappa' = 4, 5, 6), \quad (5.2)$$

where $k_{1\kappa\kappa'}+k_{2\kappa\kappa'}+k_{3\kappa\kappa'}=0$ and $k_{\rho 56}+k_{\rho 64}+k_{\rho 45}=0$.

The condition that it shall contain one tangent [5] is that all two-row minors of

$$\begin{vmatrix} k_{156} & k_{256} & k_{356} \\ k_{164} & k_{264} & k_{364} \\ k_{145} & k_{245} & k_{345} \end{vmatrix}$$

vanish.

as fifteen further points lying in the same space. These are connected by a relation derived from (2.4′), viz.

$$\sum_{\mu\nu}\sum_{\lambda} y_\lambda^{-1}\mathbf{7}_{\lambda\mu\nu} \equiv \sum_{\kappa} y_\kappa^{-1}\mathbf{7}_\kappa = 0,$$

which is quite distinct from the first. Thus all the points in the second neighbourhood of P lie in a prime say $\mathfrak{P}^*$.

$\mathfrak{P}$ contains the points P^s_{bc}, and therefore by symmetry must contain all the points P^a_{bc}, and thus the plane $(\mathfrak{P}, \Pi_{abc})$ is that defined by the points $P^a_{bc}\ P^b_{ca}\ P^c_{ab}$.

7. *The representation of* Ω *on* [5].

We can establish a point to point correspondence between Ω and a [5] R' by considering the parameters y, in terms of which the points of Ω are expressed, as coordinates in R'. Take a general point in [13] to have the symbol $\Sigma x_{\lambda\mu\nu}\mathbf{8}_{\lambda\mu\nu}$; then, on account of the syzygies

$$\mathbf{8}_n \equiv \Sigma\,\mathbf{8}_{\lambda\mu n} = 0, \tag{2.4}$$

an equation

$$\Sigma k_{\lambda\mu\nu} x_{\lambda\mu\nu} = 0 \tag{7.1}$$

represents a prime only if the six relations

$$\sum_{\lambda\mu} k_{\lambda\mu n} \equiv 0 \tag{7.2}$$

are satisfied.

To the point P of Ω whose "coordinates" are

$$\rho x_{\lambda\mu\nu} = y_\lambda y_\mu y_\nu,$$

there corresponds the point P' of R' whose coordinates are y_λ. Thus to the section of Ω by the prime

$$\left.\begin{aligned}\Sigma k_{\lambda\mu\nu} x_{\lambda\mu\nu} &= 0 \\ \Sigma k_{\lambda\mu n} &= 0\end{aligned}\right\} \quad \begin{aligned}&(7.1)\\ &(7.2)\end{aligned}$$

there corresponds the cubic fourfold Σ', primal in [5], whose equation is

$$\Sigma k_{\lambda\mu\nu} y_\lambda y_\mu y_\nu = 0. \tag{7.3}$$

There are no terms in this equation involving powers of any of the y_α higher than the first, so that Σ' has double points at the vertices of the simplex of reference, say $W_1', \ldots, W_6'$. Further, the relations (7.2)

* That these syzygies are independent, and that there are no others independent of them is confirmed later when the representation of Ω is considered.

among the coefficients are just sufficient to make Σ' have a double point at the unit point, say W_7'. Any cubic primal with these seven points double has an equation of the form (7.3) with the restriction (7.2), so that these seven points completely specify the representation*.

From this we confirm two facts about Ω: (i) Cubic primals in [5] with seven fixed double points have freedom 13, so that Ω lies in [13] and is normal. (ii) The representation has no invariants, and a finite number of collineations into itself, so that the same is true of Ω. *I.e.*, it is possible to define Ω from fifteen arbitrarily chosen points, which follows from the possibility of arranging the original thirty-five base points as a system connected by syzygies which involve only numerical coefficients (in the actual scheme the coefficients are all unity), while it is not possible to arrange it as a system lying in space of n, less than 13 dimensions, together with $13-n$ points outside that space. The freedom of Ω is 195.

From the representation we can calculate the order of Ω. Every primal Σ' contains all twenty-one lines $W_\rho' W_\sigma'$†, and thus four of the primals meet in a curve of order 81, of which part is these twenty-one lines, and the rest is of order 60 with tenfold points at W_ρ'. This residual curve is met by a fifth primal Σ' in 180 points of which 140 are absorbed by the base points. Thus an arbitrary [8] meets Ω in forty points, *i.e.* Ω is of order 40.

The point $y_1 = y_2 = y_3 = y_4 = y_5 = 0$, namely W_6', corresponds to the whole manifold $\Sigma y_\mu y_\nu \mathbf{8}_{\mu\nu 6}$ on Ω, *i.e.* ω_{68}, while the unit point, whose equations we may write as $y_\lambda - y_\mu = 0$, corresponds to the whole manifold $\Sigma(y_\lambda - y_6)(y_\mu - y_6) A_{\lambda\mu 78}$, namely ω_{78}. In the representation the point W_7' plays no different part from the other six points W_ρ', *i.e.* the elements are symmetrical in the suffixes 1, ..., 7 at least, while on Ω itself, in the form in which the symbol for P was written in terms of the parameters y_α, the suffixes 1, ..., 6 were a symmetrical set, and 7 and 8 were a symmetrical pair. Thus the elements of Ω must be completely symmetrical in all eight suffixes.

The locus of the points $y_\lambda = 0$ on Ω is the fourfold $\omega_{\lambda 7}$ in $\Delta_{\lambda 7}$, and therefore, using the symmetry of the figure, the fourfold ω_{rs} $(r, s = 1, \ldots, 7)$ is represented by the prime Δ'_{rs} in R' defined by the five points W_ρ' other than W_r' and W_s'. The representation of ω_{rs} on Δ_{rs}, say in particular of ω_{67} on Δ_{67}, is not now by quadrics through six points, but by cubics with five fixed double points $W_1', \ldots, W_5'$, and one fixed simple

* This corresponds to the representation of Segre's cubic manifold by quadric surfaces through five fixed points in [3], *cf.* (2), 154.

† We use ρ, σ, τ, when the range of the suffixes is 1, ... , 7.

point $D'_{67} = (\Delta'_{67}, W_6', W_7')$. The conics ϑ_{67} are represented by the lines through D'_{67}, and the ∞^6 system of conics, corresponding in the representation by quadrics to the lines of the [4], by the normal rational curves through the five points $W_1', \ldots, W_5'$. The systems of lines are represented by the lines through $W_1', \ldots, W_5'$, respectively, and by the normal rational curves through all six base points.

We have seen that the four [8]'s $\Delta_{12}, \Delta_{34}, \Delta_{56}, \Delta_{78}$ lie in a prime, whose complete intersection with Ω, since it is to be of order 40, consists of the four fourfolds $\omega_{12}, \omega_{34}, \omega_{56}, \omega_{78}$, each of order 10. In the representation the three primes $\Delta'_{12}, \Delta'_{34}, \Delta'_{56}$, together with the point W_7' form a manifold Σ'.

Consider next the solids on Ω; every point at which

$$y_1 = y_2 = y_3 = y_4 = 0$$

lies on Ω, so that the line $W_5' W_6'$ represents a solid, viz. Π_{568}, on Ω. The points for which $y_1 = y_2 = 0$, viz. $y_3^{-1}\mathbf{7}_{123} + y_4^{-1}\mathbf{7}_{124} + y_5^{-1}\mathbf{7}_{125} + y_6^{-1}\mathbf{7}_{126}$, generate the solid Π_{127}, so that Π_{127} is represented by the solid $\Pi'_{127} = [W_3', W_4', W_5', W_6']$. To planes in Π_{127} there correspond cubic surfaces with four fixed double points in Π'_{127}, and *vice versa*. The fifty-six solids on Ω are represented thus:

The set of twenty-one, $\Pi_{\rho\sigma 8}$ by lines $W_\rho' W_\sigma'$ $(\rho, \sigma = 1, \ldots, 7)$,

The set of thirty-five, $\Pi_{\rho\sigma\tau}$ by solids $\Pi'_{\rho\sigma\tau} = [W'_{\rho'}, W'_{\sigma'}, W'_{\tau'}, W_\kappa']$.

The parameters y_α at the point $A_{\alpha\beta\gamma 7} = \mathbf{7}_{\alpha\beta\gamma}$ on Ω satisfy the equations $y_\alpha = y_\beta = y_\gamma = 0$, so that the point is represented on R' by the plane $W'_{\alpha'}, W'_{\beta'}, W'_{\gamma'}$. That is, fixing the suffix 8 instead of 7, and using the symmetry of the figure, any of the thirty-five singular points $A_{\alpha\beta\gamma 8}$ on Ω is represented by the corresponding plane $W_\alpha' W_\beta' W_\gamma'$.

8. *The manifolds on Ω corresponding to lines, planes, etc., of R'.*

(i) Lines of R': in general a twisted cubic curve on Ω. If the line passes through W_r', it represents a line l_r. We denote this system of lines in R' by l_r'. The representatives of the lines l_8 on Ω are the normal rational (quintic) curves l_8' through the seven base points.

(ii) Planes of R': in general a del Pezzo surface (of order 9 in [9]) on Ω. If the plane passes through one point W_r', we get a normal quintic rational scroll in [6], whose generators are lines l_r. If it passes through W_r' and W_s', we get a quadric surface, which we shall denote by $\{l_r, l_s\}$, whose two systems of generators are lines l_r and l_s respectively.

(iii) Solids of R': in general a threefold of order 27 which is the projection from a general [5] of the manifold in [19] whose prime sections correspond to all the cubic surfaces in [3]. If the solid passes through W_r', the corresponding locus is of order 19, and consists of lines l_r; if through $W_r'\,W_s'$, the locus is of order 12, lies in [11], and consists of lines l_r and l_s, and if through $W_r'\,W_s'\,W_t'$, the locus is of order 6, lies in [7], and consists of lines l_r, l_s, and l_t. This last manifold we denote by $\{l_r, l_s, l_t\}$. $\{l_r, l_s\}$ consists of all the lines l_s which meet a given line l_r, and $\{l_r, l_s, l_t\}$ of all the lines l_t which meet a given surface $\{l_r, l_s\}$, the relation being quite symmetrical among the three families of lines. $\{l_r, l_s, l_t\}$ is the well-known locus that may be defined in this way; in each of two distinct solids in [7] take a quadric surface, and establish a projective relation between the solids in such a way that the two quadrics correspond. The joins of corresponding points on the two surfaces are the generators of one system of the locus. We can write the points of the locus parametrically in the form $\Sigma\,\xi_\alpha\eta_\beta\zeta_\gamma G_{\alpha\beta\gamma}$ ($\alpha, \beta, \gamma = 1, 2$ independently), using the eight points $G_{\alpha\beta\gamma}$ as the simplex of reference. The lines of the three systems are obtained by keeping each pair of parameters constant in turn. The usual representation of this locus is by cubic surfaces, in [3], with three fixed double points, namely that obtained above from the representation of Ω. It may be obtained by projecting the locus from the solid containing three concurrent lines (the tangent solid at the point), as, *e.g.*,

$$[G_{111}, G_{211}, G_{121}, G_{112}].$$

This corresponds to the algebraic representation by the equations

$$\xi_1/\xi_2 :\ \eta_1/\eta_2 :\ \zeta_1/\zeta_2 = x/t :\ y/t :\ z/t.$$

(iv) Finally, a prime of R': in general, since the prime meets the twenty-one lines $W_\rho'\,W_\sigma'$, it represents a fourfold of order 60. The intermediate cases may be worked out as above, but we pass immediately to the consideration of the manifold on Ω represented by a prime through four of the base points, say W_1', W_2', W_3', W_4'. It consists of all the lines l_4 which meet a given manifold $\{l_1, l_2, l_3\}$, and we denote it by $\{l_1, l_2, l_3, l_4\}$. It is represented on [4] by cubic threefolds that have four fixed double points W_1', ..., W_4', and three fixed collinear simple points, where the [4] meets the lines $W_6'\,W_7'$, $W_7'\,W_5'$, and $W_5'\,W_6'$. The four-times ruled manifold corresponding to cubic primals of [4] with four fixed double points only, lies in [14] and is of order 23. It has on it ∞^6 twisted cubic curves, corresponding to the lines of the [4], and $\{l_1, l_2, l_3, l_4\}$ is the projection of such a manifold on to [11] from three points which lie on the same twisted cubic curve. It is thus of order 20, and has on it a singular

point, actually A_{1234}, arising from the twisted cubic curve*. The most general four-times ruled locus of this type is that which is obtained by collineating two three-times ruled manifolds such as $\{l_\alpha, l_\beta, l_\gamma\}$ which lie in distinct [7]'s and joining corresponding points. Such a manifold lies in [15], is of order 24, and, in the same way as was the three-times ruled locus, it can be represented on [4] by quartic primals with four fixed triple points. Among these quartic primals is the family consisting of the solid containing the four points, and the cubics on which the four points are double. The solid represents a quite general point on the manifold (namely, the point of contact of the osculating [10] from which the projection is made to get the representation), so that the manifold represented by cubics in [4] with four fixed double points is the projection of this "general manifold" from any one of its points.

The manifold of R' representing any manifold $\{l_5, l_6, l_7, l_8\}$ is a quadric cone whose vertex is the plane $W_5' W_6' W_7'$ passing through $W_1', \ldots, W_4'$. This cone contains the four solids Π'_{234}, Π'_{341}, Π'_{412}, Π'_{123}, and therefore $\{l_5, l_6, l_7, l_8\}$ contains Π_{234}, Π_{341}, Π_{412}, Π_{123}, which pass through A_{1234} and define ∇_{1234}. Symmetrically, $\{l_1, l_2, l_3, l_4\}$ contains Π_{567}, Π_{678}, Π_{758}, Π_{568}; the first is clear from the representation, the other three are represented one by each of the three collinear simple base points. One of the quadric cones, together with any prime through Π'_{567}, forms a primal Σ', and thus any $\{l_5, l_6, l_7, l_8\}$ and any $\{l_1, l_2, l_3, l_4\}$ together form a complete prime section of Ω. Conversely, any prime through a manifold of either family cuts Ω residually in a manifold of the other. Each manifold lies in a [11], and since the two together lie in a prime, a [12], their intersection must lie in a [10], which, we have seen, contains the [9] ∇_{1234}.

9. *The representation of sections of Ω by tangent primes.*

Any prime section is represented by a cubic primal Σ' in R' which has seven double points W_ρ', and which, in consequence, contains the twenty-one lines $W_\rho' W_\sigma'$. Consider a prime in [13] that contains the tangent [5] $\mathfrak{P}$ at a point P of Ω. The corresponding primal Σ' will have an additional

* The point is sixfold, as may be proved thus: The projection of $\{l_1, l_2, l_3, l_4\}$ from a tangent [4] to itself is represented by cubic threefolds with five double points and three collinear simple points. It is the same manifold as is obtained by projecting $\omega_{\alpha\beta}$ from two points on a conic $\mathfrak{d}_{\alpha\beta}$. If we refer the cubic threefolds to the five double points as simplex of coordinates, and then write $1/x$ for each of the coordinates x in the equations, we obtain quadrics with eight fixed simple points on a rational quartic curve θ', θ' representing the projection of the singular point. Three quadrics through θ' meet further in another rational quartic with six points on θ'. Thus the point is sixfold on the projected manifold, and therefore also on $\{l_1, l_2, l_3, l_4\}$. *Cf.* §14.

double point at the point P' corresponding to P; the eight lines l_α, which pass through P, and lie in $\mathfrak{P}$, are represented by the seven lines $P'W_\rho'$, and by the normal rational curve through the seven base points and P', which clearly all lie on Σ'. Since $\mathfrak{P}$ meets Δ_{67} in a line, viz. $P_{6,7}P_{7,6}$, one of the primes through $\mathfrak{P}$ is $[\mathfrak{P}, \Delta_{67}]$. The representation of the section by this prime is the [4] Δ_{67}', together with the quadric [2]-cone, say Σ'', whose vertex is the plane $W_6'W_7'P'$, which passes through the remaining base points. The plane-vertex of this cone represents a quadric surface of lines l_6 and l_7 on Ω, at every point P of which this prime contains the tangent [5] $\mathfrak{P}$. Σ'' is composed of ∞^1 solids through the vertex, among which are the five $[W_\lambda', W_6', W_7', P']$, $\lambda = 1, \ldots, 5$. Each of these particular solids represents a three-times ruled locus $\{l_\lambda, l_6, l_7\}$. The section by the prime through Δ_{78} and $\mathfrak{P}$ is represented by a cubic primal having a triple point at W_7', and double points at $W_1', \ldots, W_6'$, and P'. The primal is therefore a cone, and consists, in fact, of the trisecant planes through W_7' of the quintic curve l_8' through P'.

At any of the points $A_{\alpha\beta\gamma\delta}$ there is a definite tangent [9], $\nabla_{\alpha\beta\gamma\delta}$, which we have seen lies in a prime with any tangent [5] $\mathfrak{P}$. The equation of the manifold Σ' representing the section by a prime through ∇_{1238} and $\mathfrak{P}$ can be written down at once by substituting $y_\alpha y_\beta y_\gamma$ for $x_{\alpha\beta\gamma}$ in the equation (7.1) of the prime. The equation becomes

$$\{(a_2-a_3)y_1+(a_3-a_1)y_2+(a_1-a_2)y_3\}$$
$$\times\{a_4(a_5-a_6)y_5y_6+a_5(a_6-a_4)y_6y_4+a_6(a_4-a_5)y_4y_5\}=0.$$

Thus Σ' breaks up into the prime through W_4', W_5', W_6', W_7', and P', and the quadric cone with vertex $W_1'W_2'W_3'$ through the same five points. The prime contains the manifolds $\{l_1, l_2, l_3, l_8\}$ and $\{l_4, l_5, l_6, l_7\}$ through P. The section by a general prime through ∇_{1238} is represented by a cubic fourfold on which the plane $W_1'W_2'W_3'$ is double, *cf.* equation (7.2).

We have seen that there is one particular prime $\mathfrak{P}$ through P which contains all the second neighbourhoods of the point. This will be represented by a cubic primal Σ' which has a *triple* point at the corresponding point P', in addition to double points at the base points. It is therefore necessarily a cubic cone, and is, in fact, just the locus of trisecant planes through P' to the normal rational curve l_8' through P'. On this cone, the quartic conical sheet of chords of l_8' through P' is double, so that $\mathfrak{P}$ touches Ω, *i.e.* contains the tangent [5], at every point of a certain surface ϕ. We denote by ϕ' the conical sheet on which ϕ is represented.

The complete intersection of any cubic primal Σ' that has a double point at P' with ϕ' is the seven lines l_ρ', and the quintic curve l_8' through P'

(verifying that the complete intersection of $\mathscr{S}$ with Ω is the eight lines l_α), and thus any such primal that is made to pass through a further point of ϕ' contains the sheet entirely. Thus the surface ϕ lies in [6].

Since ϕ is rational its properties are most easily derived from its plane representation, which we may obtain by first representing ϕ' on the plane. This is done by projecting ϕ' from any three points of itself, so that its prime sections become plane quartic curves with a fixed triple point, and fixed tangents at it. If the three points of projection are points W_ρ', the sections of ϕ' by the primals Σ' project into sextic curves with a fixed triple point, and fixed tangents there, and four fixed double points. Taking as triangle of reference the triple point, and two of the double points, the equation of such a curve may be written

$$z^3 k_3 + z^2 u_4 + xyz\, u_3 + x^2 y^2 u_2 = 0,$$

where k_i and u_i are polynomials of degree i in x and y, the k_i being fixed and the u_i variable, but subject to the condition that there are two other fixed double points on the curve. On inverting* with regard to the triangle of reference, this equation is transformed into

$$xy\, k_3' + z u_4' + z^2 u_3' + z^3 u_2' = 0,$$

there being still two assigned double points other than Z. This is the equation of a quintic curve of a family having three fixed double points, and five fixed collinear simple points. A further inversion, using the three double points as triangle of reference, reduces the family to the system of quartic curves σ'', with eight fixed simple points, $L_1'', \ldots, L_8''$, that lie on a conic λ''. This provides the simplest representation of ϕ on the plane, and from it we verify that ϕ lies in [6] and is normal, and find further that its order is 8, and that P, which is represented by λ'', is a fourfold point. The eight fixed points represent the lines $l_1, \ldots, l_8$ through P, so that in calculating the freedom of ϕ no allowance has to be made for the possible variation of the elements of the surface that correspond to the base points in the representation; since the figure in the plane (eight points on a conic) has freedom 13, the freedom of ϕ in [6] is

$$7^2 - 3^2 + 13 = 53\dagger.$$

* By "inversion" is meant the transformation $x' : y' : z' = 1/x : 1/y : 1/z$, or, generally, $\rho x_i' = 1/x_i$.

† *Cf.* (8), 527. We can check this value of the freedom by comparing the freedoms of Φ and ϕ' and the fixed elements on them. We have ${}_{[5]}f_{\phi'} = 26$, and there are seven arbitrary fixed points on it, corresponding to conics of a finite family on ϕ. Using this, we get ${}_{[6]}f_\phi = 7^2 - 6^2 + 26 + 7.2 = 53$.

Any prime section of ϕ through the multiple point P is represented by a quartic curve σ'' which has altogether nine points on λ'', and which consists therefore of λ'' and some other conic of the plane. Thus the projection of ϕ from P is a Veronese quartic surface.

The principal properties of ϕ are: it is a normal surface of order 8 in [6], represented on the plane by quartic curves with eight fixed points lying on a conic. It has on it a fourfold point P, through which pass eight lines $l_1, \ldots, l_8$. Corresponding to the lines of the plane through 0, 1, and 2 of the base points, we have:

One ∞^2 family of rational quartic curves in [3], each curve having a double point at P, any two of the curves forming a complete prime section.

Eight ∞^1 families of rational cubics, all passing through P, the members of each family meeting one of the lines l_α.

Twenty-eight conics, one meeting any pair of the lines. That which meets l_α and l_β lies on $\omega_{\alpha\beta}$ and belongs to the family other than $\vartheta_{\alpha\beta}$.

10. *The elliptic quartic curves on* Ω.

Through the seven points $W_1', \ldots, W_7'$ it is possible to draw ∞^8 normal elliptic sextic curves e', of which family one member exactly passes through two further general points of the [5]. Such a curve meets any primal Σ' in four variable points, and consequently represents an elliptic quartic curve, say e, on Ω. A primal Σ' contains any given curve e' if it is made to pass through four arbitrary points of the curve, and there are thus ten independent primals through the curve, and consequently ten independent primes in [13] through e. That is, each of the curves e lies in a solid, as we expect. Since e' passes through the point W_ρ', e meets the [8] $\Delta_{\rho 8}$, necessarily in a point of $\omega_{\rho 8}$; since also e' meets $\Delta'_{\rho\sigma}$ in six points, only five of which are base points, e meets $\omega_{\rho\sigma}$. Thus every curve e meets all twenty-eight of the [8]'s $\Delta_{\alpha\beta}$, and if an arbitrary point is taken on each of two of the fourfolds ω, exactly one curve e can be found to pass through them both.

Through any elliptic sextic curve in [5] we can draw a pencil of cubic primals on which the curve is double*, whose common intersection is the manifold of chords of the curve. Each of these primals, which we denote by Ξ', consists of two ∞^2 families of planes, trisecants of the curve e', in which two planes of the same family do not meet, save exceptionally in a point of e', and two planes of different families always have a common

* All the results concerning this family of primals are quoted from Veneroni, (5).

point, and may exceptionally have common a chord of e'. The equation of any Ξ' can be written as the vanishing of a determinant of order 3 whose elements are linear functions of the coordinates.

If we take elliptic parameters on the curve e', the trisecant planes of the two systems on a manifold Ξ' are the joins of the sets of three points of parameters ξ_i and η_i, where

$$\xi_1+\xi_2+\xi_3 \equiv -(\eta_1+\eta_2+\eta_3) \equiv \text{constant (mod period)}.$$

Special among these manifolds are those, four in number, for which the constant is a half-period, on which manifolds the two families of planes coincide. The determinant which represents one of these manifolds is symmetrical, and the manifold consists just of the planes which cut a certain Veronese surface, say F', in conics. The curve e', of course, lies on F', and there are thus four Veronese surfaces through e'.

Since e' is double on Ξ', and passes through the seven base points of the representation, Ξ' belongs to the family of primals Σ', and represents a prime section of Ω. This prime touches Ω at every point of the corresponding curve e, and belongs to a pencil of primes that have this property. Thus there is an [11] which touches Ω (*i.e.* contains the tangent [5]) at every point of any curve e.

We do not stop here to investigate the many properties of this type of prime section, but pass on to the more interesting problem of the nature of the surfaces F which are represented by the Veronese surfaces F' that can be drawn through the seven base points. We can get a simple representation of F in two stages, first of all of F itself on F', and then of F' on a plane F'' in the usual way by all the conics of the plane. A prime section of F corresponds to the curve of intersection of F' with a primal Σ', namely a curve of order 12 with seven fixed double points. This curve is represented on F'' by a curve of order 6, with seven fixed double points, say W_1'', ..., W_7''. Two of these plane curves have eight variable intersections, and the freedom of the system is six; thus F lies in [6] and is normal, and is of order 8. Since the plane representation is defined by seven arbitrary points, and thus has six invariants and no self-collineations, the freedom of F as a surface in [6] is 54. Through W_1'', ..., W_7'' we can draw ∞^2 elliptic cubic curves e'', so that there are on F' ∞^2 of the elliptic sextics e', and on F ∞^2 of the elliptic quartics e. Further, all cubics of the system through a fixed point P'' of F'' pass also through another fixed point Q'' (the ninth associated point). Thus all curves e' through P' on F' pass also through Q', and all curves e on F through P pass also through Q. We have, in fact, corresponding involutions on F'', on F', and on F, but, whereas there are seven base

points on both F'' and F', the involution on F is free of base points. The involution on F has this rather remarkable property, that the tangent planes to the surface at two conjugate points meet in a line. For consider any two of the elliptic quartics e through P and Q: the solids that contain them have common the line PQ, and therefore lie in a prime of the [6] in which F lies. The complete intersection of this prime with the surface is the two curves e. Further, the prime contains the tangent lines to the two curves at P and Q, and therefore contains the tangent planes to the surface at the two points. But there are ∞^1 of these curves through P and Q, and therefore ∞^2 of these primes, no two of which coincide, and any of which contains the two tangent planes. Thus the two tangent planes must lie in a solid.

The other families of curves on F that deserve notice are:

One ∞^2 family of normal rational sextics, corresponding to the lines of the plane F'';

Altogether fifty-six conics, corresponding to the lines $W_\rho'' W_\sigma''$ (twenty-one), the cubics with one double base point and six simple base points (seven), the conics through sets of five of the base points (twenty-one), and the base points themselves (seven). The last twenty-eight of these conics lie one on each of the manifolds $\omega_{\alpha\beta}$, but do not belong to the systems $\vartheta_{\alpha\beta}$.

Any of the manifolds Ξ' through a curve e' contains all four Veronese surfaces through the curve, and thus the four surfaces F through a curve e all lie in the [11] that touches Ω all along e. Further, the manifold of conic planes of one of the Veronese surfaces F'' belongs to the family Ξ', and represents, therefore, a section of Ω by a prime that contains the tangent [5] at every point of F. We denote a prime of this family by $\mathfrak{F}$.

We have seen that through two general points of Ω we can draw four of the surfaces F, all containing the curve e through the two points. If the two points lie on the same line l_α the whole character of the surface changes; it becomes, in fact, one of the surfaces ϕ introduced in § 9. For consider the nature of the curve e that can be drawn through such a pair of points. The curve e' representing it has l_α' as a trisecant line, and must therefore consist of this line, together with a rational quintic curve of which the line is a chord. One quintic curve can be drawn through the six base points other than W_α', and the two points on l_α'; it represents a twisted cubic on Ω which, with the line l_α, forms the curve e required.

A proper Veronese surface contains no curves of odd order, and there is no such surface therefore through the line l_α', and consequently none such through this particular pair of points; instead of four Veronese surfaces through the points, we get two quartic cones, one having its

vertex at each of the points. That is, instead of four surfaces F through the two points on Ω we get two surfaces ϕ, each point being singular on one of them. The net of elliptic quartics on a surface F is replaced by a net of rational quartics with a fixed double point on ϕ*.

11. *Summary of the properties of* Ω *and their representations.*

Ω is an eight-times ruled fivefold of order 40 in [13], representable by cubic fourfolds Σ' in [5] that have seven fixed double-points $W_1', \ldots, W_7'$. At every point P of Ω there is a tangent [5] ρ' whose complete intersection with Ω is the eight lines through the point, and an osculating prime π. In the table below the elements of Ω stand first, and are associated with their representations by an arrow $\rightarrow$.

Thirty-five singular points

$$A_{\alpha\beta\gamma\delta} \text{ or } A_{\alpha'\beta'\gamma'\delta'} \ (\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' = 1, \ldots, 8):$$

$$A_{\alpha\beta\gamma 8} \rightarrow \text{the plane } W_\alpha' W_\beta' W_\gamma'.$$

Fifty-six solids $\Pi_{\alpha\beta\gamma}$ lying entirely on Ω: thirty-five without suffix 8, $\Pi_{rst} \rightarrow$ the solid $\Pi'_{rst} = [W'_{r'}, W'_{s'}, W'_{t'}, W'_{u'}]$; twenty-one with suffix 8, $\Pi_{rs8} \rightarrow$ the line $W_r' W_s'$.

Twenty-eight [8]'s $\Delta_{\alpha\beta}$, each cutting Ω in a six-times ruled fourfold $\omega_{\alpha\beta}$ (representable by quadrics in [4] with six fixed base points): twenty-one without suffix 8, $\omega_{rs} \rightarrow$ the [4] Δ'_{rs} through the five points W_ρ' other than W_r' and W_s'; seven with suffix 8, $\omega_{r8} \rightarrow$ the point W_r'.

Eight ∞^4 families of lines l_α: $l_1, \ldots, l_7 \rightarrow$ lines $l_1', \ldots, l_7'$ through $W_1', \ldots, W_7'$ respectively; $l_8 \rightarrow$ a normal rational quintic curve l_8' through $W_1', \ldots, W_7'$. The lines l_β which meet a given line l_α form a quadric surface $\{l_\alpha, l_\beta\}$; l_γ which meet $\{l_\alpha, l_\beta\}$ form a sextic threefold $\{l_\alpha, l_\beta, l_\gamma\}$ in [7]; l_δ which meet $\{l_\alpha, l_\beta, l_\gamma\}$ form a fourfold of order 20 in [11] $\{l_\alpha, l_\beta, l_\gamma, l_\delta\}$.

* We notice that F has freedom 54 in [6] and 6 on Ω, while ϕ has freedom 53 in [6] and 5 on Ω, which is some indication that the two types of surfaces may be derived from surfaces of the same family on some manifold in higher space of which Ω is a projection. The close relation between the two surfaces is also brought out by their representations on a plane: F is represented by sextic curves that have seven fixed double points in general position, while in the corresponding representation of ϕ three of these double points coincide (giving a triple point with fixed tangents). The curves e on F are represented on the plane by cubics through the seven base points, while those on ϕ are represented by cubics each with a double point at the triple base point, and simple points at the other four base points. In the representation on R', if the elliptic sextic curve e' is made to have a double point, of the four Veronese surfaces through it only two are proper, while the other two are replaced by the quartic cone of chords through the double point.

One ∞^8 family of elliptic quartic curves e, each curve meeting all twenty-eight manifolds $\omega_{\alpha\beta} \rightarrow$ elliptic sextics e' through $W_1', \ldots, W_7'$. The tangent [5]'s to Ω along any curve e all lie in a [11].

One ∞^6 family of rational surfaces F of order 8, each in a [6], four passing through each curve $e \rightarrow$ Veronese surfaces F' through $W_1', \ldots, W_7'$. (The surface F is represented on a plane by sextic curves with seven fixed double points.) Section of Ω by the prime $\mathfrak{P}$ containing the tangent [5] at every point of $F \rightarrow$ cubic primal of conic-planes of F'.

One ∞^5 family of rational surfaces ϕ of order 8, each in a [6], one having a fourfold point at any point P of Ω and containing the eight lines through $P \rightarrow$ the quartic conical sheet with vertex P' passing through $W_1', \ldots, W_7'$. (The surface ϕ is represented on a plane by quartic curves having eight fixed points on a conic.) The osculating prime $\mathfrak{P}$ to Ω at P contains the tangent [5] to Ω at every point of ϕ.

Section by a prime through the tangent [5] at $P \rightarrow$ cubic primal Σ' with an additional double point at P'.

Section by a prime through the tangent [9] ∇_{rst8} at $A_{rst8} \rightarrow$ cubic primal Σ' on which $[W_r', W_s', W_t']$ is double; ∞^2 of these consist of a prime through Π'_{rsl}, and a quadric cone whose vertex is the plane $W_r' W_s' W_t'$, and which passes through the other four base points.

II. The analogue in [7] of Kummer's 16_6 figure.

12. *The aspect of the Kummer figure on which the analogy is based*.*

If the Segre cubic threefold Ω in [4] is projected from a point V of itself, we obtain a solid Υ of which every point is the projection of two points P_1 and P_2 of Ω, collinear with V. The pairs of points P_1, P_2 which coincide, *i.e.* the points of contact of the tangents to Ω from V, project into a Kummer surface Ψ, whose sixteen nodes arise ten from those of Ω, and the remaining six from the lines of Ω which pass through V, and whose sixteen tropes arise from the fifteen planes on Ω and the tangent solid to Ω at V. The six systems of lines on Ω project into the six quadratic congruences of bitangents to Ψ.

The Segre manifold is represented on a [3] R' by quadric surfaces through *five* base points, and thus its projection from V, namely the repeated solid Υ, is represented on R' by quadrics with *six* base points. The representation will be such that to every point P of Υ there correspond

* To make clearer the work which follows we summarize here some of the well-known properties of the Kummer surface, which may be found, *e.g.*, in Baker (2), 138–140 and 156–160, using a notation in which the letters are the same as those that are to be used for the analogous constructs.

two points P_1', P_2' of R', *i.e.* to every locus in Υ there will correspond two loci, which may coincide exceptionally, in R'. If P_1' and P_2' coincide, then P is a point of the Kummer surface Ψ, whilst the locus of these coincident point-pairs in R' is a Weddle surface Ψ'. To a line of one of the six quadratic congruences in Υ there correspond both a line through one of the base points in R', and the twisted cubic curve through the remaining five base points having the line as chord. The points common to the line and the curve are points of Ψ' and correspond to the points of contact of the line in Υ with Ψ.

A general line of Υ is represented by an elliptic quartic curve through the six base points, each point of it being represented by a pair of points which completes with the six base points a set of eight associated points. Four of the quadrics through the base points touch any one of these quartics, these being the coincident point pairs on the curve (*i.e.* the points other than base points in which it meets Ψ'). To them correspond the points in which the line in Υ meets the Kummer surface.

The sixteen nodes and sixteen tropes of Ψ form a 16_6 figure which is transformed into itself by sixteen collineations (including identity) and sixteen correlations. These transformations form a system determined by six mutually apolar linear complexes S_1, ..., S_6, the collineations being the transformations $S_\alpha S_\beta$ (and **1**), and the remaining correlations, which are polarities with respect to quadrics, being $S_\alpha S_\beta S_\gamma$ (which is the same as $S_{\alpha'} S_{\beta'} S_{\gamma'}$)*.

In the notation of § 1, a point of R' is (y_1, y_2, y_3, y_4), and the corresponding point of Υ is

$$\Sigma y_\alpha y_\beta A_{\alpha\beta 6} \quad \text{or} \quad \Sigma y_\alpha^{-1} y_\beta^{-1} A_{\alpha\beta 5} \quad (\alpha, \beta = 1, \ldots, 4),$$

where

$$\Sigma A_{\alpha\beta 6} = 0,$$

and now, in addition,

$$\Sigma a_\alpha a_\beta A_{\alpha\beta 6} = 0,$$

a_α being the parameters of V†. The transformation

$$\rho y_\alpha = a_\alpha / y_\alpha'$$

is clearly a collineation of Υ into itself, and is in fact, the suffixes of S_α being chosen to agree with those of W_α', the collineation $S_5 S_6$. As a transformation in R' it is the inversion in which $W_1' W_2' W_3' W_4'$ is the base tetrahedron, and W_5' is conjugate to W_6'. Thus, by symmetry, the collineation $S_\alpha S_\beta$ of Υ into itself corresponds to the inversion $\tau_{\alpha\beta}$ in which

* Hudson (1), 40.

† The quadrics in R' pass through the six points W_1', ..., W_4', $W_5' = (1, 1, 1, 1)$, and $W_6' = (a_1, a_2, a_3, a_4)$.

W_α' becomes W_β', and the remaining four points form the base tetrahedron. Now $S_1 S_2 S_3 S_4 S_5 S_6 = \mathbf{1}$, but the corresponding transformation

$$\tau_{16}\tau_{26}\tau_{36}\tau_{46}\tau_{56}$$

(or any similar product) is clearly *not* an identity. Thus, corresponding to the points P and $S_1 \dots S_6 . P$ on Υ, which coincide, we have the two points P', and $P'' = \tau_{16} \dots \tau_{56} . P'$ in R' which do not coincide. That is, P', P'' is the point pair in R' corresponding to the point P of Υ. The locus of the points of R' where $P' = P''$ is the Weddle surface.

13. *The* 64_8 *figure.*

We consider the arrangement of points, solids, and primes in [7] that are derived from various loci connected with Ω, on projection from the tangent [5] V to Ω at some point V into a manifold Υ in [7]. These spaces are denoted by the same symbols as were used for the corresponding spaces on Ω. It appears that there are on Υ sixty-four singular points, sixty-four solids, and sixty-four special prime sections, forming a highly symmetrical configuration, the relations among whose elements may be indicated in the same way as are those among the points and planes of the 16_6 figure, when this figure is considered as being derived from a single point in [3] by successive polarities with respect to six mutually apolar linear complexes*.

None of the points $A_{\alpha\beta\gamma\delta}$ on Ω lies in V, and none of the solids $\Pi_{\alpha\beta\gamma}$ meets V, so that we have on Υ thirty-five points and fifty-six solids with the same incidences (at least) as they had in [13]. Then V meets any of the [8]'s $\Delta_{\alpha\beta}$ in a line, so that from these we get twenty-eight primes ([6]'s) $\Delta_{\alpha\beta}$.

New spaces are derived in the projected figure from two sources: (i) the multiply ruled loci defined by two or more of the lines v_α that lie on Ω and pass through V, (ii) from spaces of high tangency connected with V.

Type (i): $\{v_1, v_2\}$ is a quadric surface which lies in a solid meeting V in a plane, so that from it we obtain an additional singular point A_{12}. Further, $\{v_1, v_2\}$ meets ω_{12} in a conic, say θ_{12}, of the system ϑ_{12}, which meets each of the six solids $\Pi_{12\alpha}$, say in the points V_{12}^α. These six points all project into the same point A_{12}, and thus Π_{123} and Π_{124}, which formerly met only in the point A_{1234}, now meet in the line $A_{12}A_{1234}$ at least. Moreover, the eight points $A_{123\alpha}$, A_{23}, A_{31}, A_{12} in Π_{123}, being projections of $A_{123\alpha}$, V_{23}^1, V_{31}^2, V_{12}^3, lie on a twisted cubic curve γ_{123} (*cf.* § 4).

* *E.g.*, Hudson (1), 45.

Next, $\{v_1, v_2, v_3\}$, since it is a threefold lying in a [7] which meets Υ in a solid, projects into a solid. But it already contains three points of Π_{123}, namely V^1_{23}, V^2_{31}, V^3_{12}, and it will appear later from the representation that the [9] which joins Υ to the [7], containing $\{v_1, v_2, v_3\}$, contains the whole of Π_{123}. Thus, although the fifty-six manifolds $\{v_\alpha, v_\beta, v_\gamma\}$ project from Υ into solids, these solids coincide with the projections of $\Pi_{\alpha\beta\gamma}$, and we obtain in this way no new solids, but only more complicated intersections among the projections of the solids of the original set.

Lastly, $\{v_1, v_2, v_3, v_4\}$, since it is fourfold lying in a [11] which meets Υ in a [4], projects into a fourfold in [6], say v_{1234} in Δ_{1234}. Since $\{v_1, v_2, v_3, v_4\}$ contains $\{v_\alpha, v_\beta, v_\gamma\}$, Δ_{1234} contains $\Pi_{\alpha\beta\gamma}$, for $\alpha, \beta, \gamma = 1, \ldots, 4$; the four solids on Ω, of which these are the projections, all pass through A_{1234}, and define the tangent [9] Δ_{1234} to Ω at the point. But ∇_{1234} meets Υ in a plane, and thus projects into the same [6] as does the [11] containing $\{v_1, v_2, v_3, v_4\}$. Further, ∇_{1234} also contains the four solids $\Pi_{\alpha'\beta'\gamma'}$ $(\alpha', \beta', \gamma' = 5, \ldots, 8)$ on Ω, which project into solids lying in the projection on $\{v_1, v_2, v_3, v_4\}$. Thus the [11]'s containing $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7, v_8\}$ project into the same [6], $\Delta_{1234} = \Delta_{5678}$. We obtain in this way, therefore, only thirty-five new primes in the [7].

Type (ii). The osculating prime $\mathfrak{V}$ at V touches Ω at every point of a surface ϕ which contains the eight lines v_α. Let T_1 be any other point of v_1, and let $\mathfrak{T}_1$ be the tangent [5] there. $\mathfrak{T}_1$ and Υ have a common line, and therefore define a [9], $[\mathfrak{T}_1, \Upsilon]$. $[\mathfrak{T}_1, \Upsilon]$ contains v_2, and the line of the family l_2 that passes through T_1, and therefore all the lines l_2 that meet v_1. That is, $[\mathfrak{T}_1, \Upsilon]$ contains all the lines l_α at any point of v_1, and therefore contains the tangent [5] to Ω at any point of v_1.

Further, $[\mathfrak{T}_1, \Upsilon]$ contains points of ϕ which do not lie in Υ, and therefore contains the [6] in which ϕ lies. Also, we have seen that $[\mathfrak{T}_1, \Upsilon]$ lies in $\mathfrak{V}$. On projection from Υ we get in the [7]:

from ϕ a new singular point, A,

from the eight [9]'s, $[\mathfrak{T}_\alpha, \Upsilon]$, eight new solids Π_α,

from the osculating prime $\mathfrak{V}$, a new prime Δ.

We have now altogether in [7] (the suffixes taking the values 1, ..., 8 in each case)

Sixty-four points: one A, twenty-eight $A_{\alpha\beta}$, thirty-five $A_{\alpha\beta\gamma\delta} = A_{\alpha'\beta'\gamma'\delta'}$;

Sixty-four solids: eight Π_α, fifty-six $\Pi_{\alpha\beta\gamma}$;

Sixty-four primes: one Δ, twenty-eight $\Delta_{\alpha\beta}$, thirty-five $\Delta_{\alpha\beta\gamma\delta} = \Delta_{\alpha'\beta'\gamma'\delta'}$;

such that each point lies in eight solids, each solid contains eight points which lie on a twisted cubic and lies in eight primes which osculate a rational cubic cone*, and each prime contains eight solids. The incidences are typified by:

$$\left.\begin{matrix} A \text{ lies in} \\ \Delta \text{ contains} \end{matrix}\right\} \quad \text{all } \Pi_\alpha.$$

$$\left.\begin{matrix} A_{12} \text{ lies in} \\ \Delta_{12} \text{ contains} \end{matrix}\right\} \quad \Pi_1, \Pi_2, \Pi_{12\alpha},$$

$$\left.\begin{matrix} A_{1234} \text{ lies in} \\ \Delta_{1234} \text{ contains} \end{matrix}\right\} \quad \Pi_{234}, \Pi_{341}, \Pi_{412}, \Pi_{123}, \Pi_{678}, \Pi_{785}, \Pi_{856}, \Pi_{567}.$$

We denote (as in § 4) the twisted cubics containing the eight points in Π_α and $\Pi_{\alpha\beta\gamma}$ by γ_α and $\gamma_{\alpha\beta\gamma}$, respectively.

14. *The projection v_{12} of the fourfold ω_{12}.*

The fourfold ω_{12} in which the [8] Δ_{12} cuts Ω is met by Υ in the two points $V_{1,2}$, $V_{2,1}$ in which it meets v_1, v_2, and we have seen that these points lie on the same conic θ_{12} of the system ϑ_{12} on ω_{12}. Thus ω_{12} projects into an octavic fourfold v_{12} in [6], on which the conic θ_{12} gives rise to a singular point, which, since θ_{12} lies in the solid containing $\{v_1, v_2\}$, is A_{12}. In the present discussion we omit the suffixes 12.

ω is represented in [4] either by cubics with five fixed double points and one fixed simple point, in which case the conic θ corresponds to a line through the simple point, or by quadrics through six points, in which case θ is represented by a normal rational curve θ' through the base points. In discussing the properties of ω or v alone, this second representation is the simpler. Since v is the projection of ω from two points θ, v is represented by quadrics, say Q, which have eight fixed base points, say $H_1', \ldots, H_8'$, which lie on a rational quartic curve θ'. The points H_α' represent solids Π_α', and θ' represents a singular point A through which they all pass. Since four quadrics, which have common a rational quartic curve, have only two further intersections, the point A is sixfold on v.

There are twenty-eight other singular points $A_{\alpha\beta}$ on v corresponding to the joins of pairs of the points H_α'. Since three quadrics in [4] which

* This statement has not yet been proved: it is inserted here to emphasize the duality of the figure. We prove, later, also that the eight solids, which lie in a prime and pass through a point, lie on a quadric cone.

have a common line meet further in a curve of which the line is a trisecant. the points $A_{\alpha\beta}$ are triple on v. Any line h_α' through H_α' represents a line l_α on v, so that v is eight-times ruled. ∞^1 of the lines of each family pass through A, corresponding to the chords of θ' through the various points H_α'. The plane $H_\alpha' H_\beta' H_\gamma'$ of the representation corresponds to the plane $A_{\beta\gamma} A_{\gamma\alpha} A_{\alpha\beta}$ which lies on v. We have on v, therefore, in addition to the eight solids, fifty-six planes, such that two solids and six planes (which all lie in the same prime) pass through each point $A_{\alpha\beta}$. The trisecant planes of θ' correspond to Steiner surfaces on v whose triple point is at A, and the normal cubic scrolls through θ' (of which there are ∞^3) correspond to normal rational quartic surfaces in [3] on v with triple points at A, which are representable on a plane by quartic curves having one fixed double point and eight fixed simple points.

Of the ∞^6 family of quadrics Q', ∞^5 contain θ' entirely, corresponding to the prime sections through A. Thus the projection of v from A is a locus represented by quadrics through a normal rational quartic, *i.e.* a general quadric fourfold of [5]. Hence, since v is of order 8, the point A, as we have seen already, has multiplicitly 6 on it. A quadric through θ' cuts the cone of chords of θ' vertex H_1' in θ' and residually in two lines, so that the lines l_1 through A form a quadric conical sheet in [3]. Since the cone contains the seven points $A_{\alpha 1}$ (which, as we have seen earlier, lie with A on a twisted cubic), the [3] in which it lies is Π_1. Any two of these eight cones through A have common one of the lines $AA_{\alpha\beta}$.

There is on v another family of lines, namely that represented by the chords g' of θ'. For consider the quadrics that pass through θ'; any that is made to pass through one other point of g' contains the line entirely, and therefore the whole family of quadrics Q' cuts the line in pairs of points of an involution, of which the points in which it meets θ' form one pair. Thus each of these lines represents a line g on v, which passes through A, and whose points are in (1, 2) correspondence with those of g'. A line g is the projection of a conic ϑ on ω whose plane meets the line from which the projection is to be made.

Considering again the quadric in [5], into which v projects from A, represented by quadrics through θ', we see that the projection from A of the locus of lines g is a Veronese surface. On each of the lines g' there are two self-corresponding points (the double points of the involution), and correspondingly two points on g, the projections of the points of contact of the tangents to the original conic ϑ that meet the line of projection. A is one of the points of this pair on a line g if g' is a tangent to θ', and the tangents to θ' correspond to the generators of the quartic cone joining A to a certain [4]-section of the Veronese surface, namely a certain normal

rational quartic curve. We see then that the locus ψ of the pairs of points on the lines g has these properties: (i) A is a fourfold point, (ii) the projection from A is a repeated Veronese surface. Thus ψ is a surface of order 12*.

Clearly the lines l_a which pass through A belong to the family g. On the line g', corresponding to any of these, the involution is such that H_a' is a member of every pair, and is therefore a repeated self-corresponding point. It follows that the lines l_x through A touch ψ all along a curve, which, since it contains the points $A_{\alpha\beta}$, and meets each generator in one point other than A, which is a simple point (corresponding to the tangent to θ' at H_a'), is the twisted cubic curve γ.

15. *The projections of the octavic surfaces F and of the quartic curves e.*

There are through every point of Ω ∞^3 octavic surfaces F, each lying in a [6], and representable on a plane by sextic curves with seven fixed double points. We want to find the nature of the projection from Υ of one of these that passes through V. Since Υ is the locus of tangent lines to Ω at V, it contains the tangent plane to F at V, and thus F projects into a surface Γ in [3] represented by sextic curves with eight fixed double points. Γ is therefore of (apparent) order 4. Through the point V of F there pass ∞^1 curves e, all of which have one other common point, say N, the conjugate of V in the involution on F. Since these curves all touch the tangent plane to F at V, they project into *repeated lines* on Υ, forming a cone whose vertex is the projection of N, which point we denote also by N. *I.e.* Γ is a repeated quadric cone whose vertex is N†.

In the representation of Γ on a plane, these lines correspond to the cubics e'' through the eight base points, and N is represented by N'', the ninth associated point of the eight. On any of these cubics the sextics cut out an involution which has four double points, of which one is N''. These correspond to the points on Γ in which the two sheets wind into each other (*i.e.* to the projections of the points of contact of [3]'s which pass through the tangent plane to F at V and touch the surface elsewhere). Since plane sextics with eight double points are of genus 2, plane sections of Γ must be of genus 2, and therefore the repeated conic which constitutes a plane section must have on it six branch points. In fact, we find without difficulty, that the two sheets of Γ meet along a sextic curve of

* To connect this with the subsequent work it may be stated now that the lines g belong to the family ϵ, and that ψ (repeated) is the section of Ψ by Δ.

† [*Added* 10 *April*, 1934.] *Cf.*, *e.g.*, Du Val, *Proc. London Math. Soc.* (2), 35 (1933), 48.

genus 4, say χ, represented by a plane curve or order 9 with eight triple points (but not passing through N'')*.

On Ω one curve e can be drawn through V and any other point; when such a curve is projected from Υ we get a line, say ϵ, of which every point is the projection of two points of e. Thus Ω projects into Υ in such a way that every point of Υ is the projection of two points on Ω, and there is on Υ a ninth ∞^4 family of lines, ϵ, quite distinct from the eight families l_a†.

Through any curve e there are four surfaces F, and on each of these associated with the point V of e there is the conjugate point N, such that all curves e on F through V pass through N. When we project, we get through the repeated line ϵ four repeated quadric cones Γ, whose vertices lie one at each of the points where the two branches of the line cut. Since there are ∞^4 lines ϵ, a finite number of points N on each of them, and ∞^1 lines ϵ through any point N, there must be ∞^3 points N, forming a threefold which we denote by Ψ. Ψ is the projection of the locus of points on Ω the tangent [5]'s $\mathfrak{U}$ at which meet Υ in lines. The projection of $\mathfrak{U}$ is a tangent [3] to Ψ.

We have found therefore that

Each point of Υ is the projection of a pair of points of an involution on Ω, whose coincident point pairs project into the points of Ψ. Ψ has ∞^4 quadrisecant lines ϵ, which, taken twice each, generate Υ. One (repeated) line ϵ passes through any point of Υ, while ∞^1 forming a quadric cone Γ pass through any point N of Ψ. Γ cuts Ψ in N and a sextic curve χ of genus 4.

We denote by Υ_1 the simple manifold, which, taken twice, forms Υ, *i.e.* Υ_1 is a manifold whose points are in (1, 2) correspondence with those of Ω.

We found in § 10 that the tangent [5]'s to Ω at the points of a curve e all lie in a [11]; if e passes through V it projects into a line ϵ, and the tangent [5]'s to Ω at points of e project into the tangent [5]'s to Υ (and Υ_1) at points of ϵ. But Υ is one of these tangent [5]'s to Ω, and thus the

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* *Cf.* Coxeter (6), 169. It is, perhaps, of interest to check the freedoms of the surfaces: F has freedom 54 in [6] and is normal, so that the freedom of its projection from a tangent plane (*i.e.* a plane of a certain ∞^2 family) is $54-3.7-(12-2)=23$. Γ is essentially a quadric cone in [3], with a definite sextic curve of genus 4 on it, so that its freedom is $8+(19-4)=23$.

† This corresponds exactly to the derivation of the repeated solid from the Segre manifold, the repeated lines of the solid being the projections of the elliptic cubics (*i.e.* plane sections) on the Segre manifold which pass through the point of projection. The pairs of points on Ω which project into the same point of Υ are slightly more difficult to obtain, for, to find P_2, which is to project into the same point as does P_1, we have to draw the curve e that passes through V and P_1; then P is the fourth intersection with e of the plane through P_1 and the tangent to e at V.

tangent [5]'s all along ϵ coincide. That is, associated with any line ϵ of Υ_1 there is a [5], say $\mathfrak{E}$, touching Υ_1 all along the line.

We found, also, that the tangent [5]'s at points of a surface F all lie in a prime $\mathfrak{F}$ of the [13], which, if F passes through V, projects into a prime of the [7], *i.e.* the tangent [5]'s to Υ_1 at points of a cone Γ, that is, the [5]'s $\mathfrak{e}$ associated with the generators of Γ, all lie in a prime, say $\mathfrak{P}$. Further, $\mathfrak{P}$ contains the tangent [3] to Ψ at every point common to Γ and Ψ, namely at the vertex N of the cone, and along the sextic curve χ.

The [5] $\mathfrak{E}$ contains the four cones Γ of which the corresponding line ϵ is a generator, and thus part at least of the intersection of Γ with Ψ is the four sextic curves χ. We prove, later, that the order of Ψ is 24, so that this, in fact, is the whole section.

We have found that the surfaces ϕ are to be included in the family F, so that among the surfaces Γ there will be some, that we denote by Γ^1, derived from surfaces ϕ that pass through V. Since Υ contains the tangent plane to such a surface, say ϕ^1, at V, and ϕ^1 lies in a [6], the tangent [5] to Ω at the singular point of ϕ^1 meets Υ in a line; *i.e.* the singular point is one of the points N that projects into a point of Ψ. There are ∞^2 surfaces ϕ^1 through V, and on each of them ∞^1 rational quartic curves with double points at N and simple points at V. These curves project into repeated lines on the surfaces Γ^1, on each of which there two coincidence points, other than N. The lines belong to the family ϵ, but, to distinguish them from general members, we denote them by ϵ^1. Γ^1 is a repeated quadric cone whose two sheets wind into each other along a sextic curve cut out on the cone by a cubic surface through the vertex.

Γ^1 is representable on the plane by quartic curves with eight fixed simple points L_a'', lying on a conic λ'', corresponding to the vertex of the cone, and one fixed double point V'', corresponding to a certain plane section. The generators of Γ^1 are represented by the lines through V'', these being cut by the quartics in pairs of points of an involution. The locus of the double points of these involutions is a quintic curve, with a triple point at V'', passing through L_a'', and touching the lines $L_a'' V''$ there. It corresponds to the curve along which the two sheets of Γ^1 wind; the plane section corresponding to V'' is tritangent to the curve, and we see from the representation, as also from the way in which Γ^1 arises from ϕ^1 by projection, that eight generators of Γ^1 touch the curve, namely the projections of the eight lines v_a on ϕ^1.

An exceptional type of surface ϕ^1, which we denote by ϕ^2, is that in which V lies on one of the lines of the surface through N, *i.e.* in which one of the lines v_a through V lies on ϕ^1. The quartic curves e^1 on it that pass through V, then reduce to v_a plus twisted cubics $e_a{}^2$ through N, meeting v_a

in one other point. There are ∞^1 of these twisted cubics. A surface ϕ_α^2 containing v_α lies entirely in the [9] which contains the tangent [5]'s to Ω at points of v_α, and projects therefore into a quadric cone Γ_α^2 lying entirely in Π_α. Since each of the cubics e_α^2 is projected from a chord, we obtain from it a *simple* line, so that the surface ϕ_α^2 itself projects into a simple quadric cone; on the other hand, points of the surface in the neighbourhood of v_α project into the same quadric cone. Thus, in this case, although the surface Γ_α^2 is a repeated quadric cone, the two sheets are quite distinct, and touch all along a certain curve, which is the projection of the cone of tangent lines to ϕ_α^2 at N. Since these lines form a quartic conical sheet of which v_α is one generator, the curve along which the two sheets touch is a twisted cubic, in fact, as we prove immediately, γ_α.

Among the curves e there is included one consisting of the line l_β through N, and a conic meeting l_β and v_α. The conic projects into a simple line, and l_β projects into the point on it, other than N, where the two sheets of Γ_α^2 touch. But l_β lies on $\{v_\alpha, v_\beta\}$ and therefore projects into $A_{\alpha\beta}$. Thus the twisted cubic along which the two sheets touch passes through the seven points $A_{\alpha\beta}$, and thus is γ_α.

16. *The representation on* Υ *on* [5].

The section of Ω by a prime that passes through the tangent [5] Υ at V is represented by a cubic primal Σ' in [5] which has an additional double point at the point, say W_0', corresponding to V. Thus Υ is represented on a [5] R' by cubic primals Σ' with eight fixed double points $W_1', \ldots, W_7', W_0'$. Although the parts played by the eight points are indistinguishable, to preserve the notation that has so far been adopted, it is necessary to treat elements that correspond to W_0' on a different footing from the others*. Since the thirty-five points, fifty-six solids, and twenty-eight primes in [7] corresponding to the various loci defined by seven of the base points form a symmetrical system, and the eighth

* For this reason we use 0 as the suffix instead of 8. In the present notation points and primes in the [7] are denoted by symbols with even numbers of suffixes, while symbols for solids have odd numbers of suffixes. Contrarily, in the representation, odd numbers of base points define the loci which correspond to points or primes, and even numbers to these which define the solids. To obtain a notation for the spaces in [7] which preserves the symmetry of the eight base points of the representation, we proceed as follows: suppress the suffix 8 in any symbol in which it occurs, and add the suffix 0 to any symbol in which the suffix 8 does not appear. *E.g.*, instead of

$$A_{1234} = A_{5678},\ A_{12},\ A_{18},\ \Pi_8,\ \Pi_1,\ \Pi_{128},\ \Pi_{123},$$

we write, respectively,

$$(A_{01234}) = A_{567},\ A_{120},\ A_1,\ \Pi,\ \Pi_{10},\ \Pi_{12},\ \Pi_{1230} = \Pi_{4567}.$$

base point is no different from the others, the whole figure of sixty-four points, sixty-four solids, and sixty-four primes that we now obtain must be completely symmetrical.

The representation of Υ is completely defined by the eight base points, and these points correspond to loci of a finite family of Υ, so that the freedom of Υ is $7.9+5=68$. Further, since any of the manifolds Υ, Υ_1, and Ψ uniquely defines the other two, their freedoms must be equal.

Since every point of Υ is the projection of a pair of points of an involution on Ω, to every point of Υ there will correspond a pair of points of an involution in R', and to any locus on Υ there will correspond two loci, generally distinct, in R'. The curves of most importance in the representation are those corresponding to the (repeated) lines ϵ on Υ, namely the elliptic sextic curves ϵ' through the eight base points. These are met by the primals Σ' in pairs of points of an involution, while there are four points at which primals of the family touch any of them. The pairs of the involution correspond to the points of ϵ, and the points of contact to the points where ϵ meets Ψ.

One normal rational quintic curve, say w', can be drawn through the eight points $W_0', \ldots, W_7'$, and, since it passes through eight double points of any manifold Σ', it must lie on the manifold. Thus the complete intersection of four primals Σ' is: w' plus twenty-eight lines $W_\rho' W_\sigma'$ plus a curve of order 48 with eight eightfold points; and, therefore, five primals Σ' have sixteen variable points of intersection, which, since Υ is the projection of Ω, must consist of eight pairs of the involution. That is, Υ_1, the simple manifold, is of order 8.

Since the points $A_{\alpha\beta\gamma\delta}$, and by symmetry also $A_{\alpha\beta}$ and A, are sixfold on Ω, they are sixfold on Υ, and therefore triple on Υ_1, *i.e.* the tangent cone to Υ_1 at any of these points is a cubic fivefold. Since Δ is the tangent prime to Υ_1 at A, the tangent cone contains the tangent to v at A, on which the cone formed of the ∞^2 lines ϵ, or g, through A, namely $[A, \psi]$, is double. But $[A, \psi]$ is the join of A to a Veronese surface in a [5] not containing A. It follows from this that the tangent cone to Υ_1 at A consists of the solids which join to A the conic planes of a certain Veronese surface. These form a manifold represented by the vanishing of a symmetrical cubic determinant whose elements are linear functions of the coordinates in the [6].

A line l_α on Υ is the projection both of a line l_α on Ω, and of some twisted cubic curve through V; it is represented, therefore, by a line l_α' through W_α' and a rational quintic curve l_α'' through the seven remaining base points. There are sixteen lines through each point of Υ_1, two of

each family, in addition to the one line ϵ. The line and the cubic curve on Ω have usually no common points, so that l_a does not meet Ψ. If the line and the cubic do meet, then they do so in a point N, the tangent [5] $\mathfrak{N}$

Elements in [7].	Suffixes in the alternative notation*.	Representations in the [5] R'. (Eight base points $W_0', \ldots, W_7'$, and the normal rational curve w' through them.)
Points		
A_{1238} or A_{4567}	123	Plane $W_1'W_2'W_3'$.
A_{12}	120	Plane $W_1'W_2'W_0'$.
A_{18}	1	Quartic cone of chords of w' vertex W_1'.
A	0	Quartic cone of chords of w' vertex W_0'.
Solids		
Π_{123}	1230 or 4567	Solids $\Pi'_{123} = [W_0', W_1', W_2', W_3']$ and $\Pi''_{123} = [W_4', W_5', W_6', W_7']$†.
Π_{128}	12	Line W_1', W_2', and the cubic cone of trisecant planes of w' vertex $W_1'\,W_2'$.
Π_1	10	Line $W_0'\,W_1'$, and the cubic cone of trisecant planes of w' vertex $W_0'\,W_1'$.
Π_8	No suffix	w' and the sextic locus of chords of w'.
Primes		
ν_{12} in Δ_{12}	120	[4], $\Delta'_{12} = [W_3', W_4', W_5', W_6', W_7']$, and the quadric cone Δ''_{12} of quadrisecant solids of w' vertex $[W_0', W_1', W_2']$.
ν_{1238} in Δ_{1238} or Δ_{4567}	123	[4], $[W_4', W_5', W_6', W_7', W_0']$, and the quadric cone of quadrisecant solids of w' vertex $[W_1', W_2', W_3']$.
ν_{18} in Δ_{18}	1	Point W_1', and the cubic cone of trisecant planes of w' vertex W_1'.
ν in Δ	0	Point W_0', and the cubic cone of trisecant planes of w' vertex W_0'.

to Ω at which meets ∇ in a line. Since the line necessarily lies in $\mathfrak{N}$ it projects into a tangent to Ψ. Conversely, insofar as any point of Ψ is the projection of only one point on Ω, there are only eight lines l_a, one of

* Suggested in the foot-note on p. 323.

† The two solids Π'_{123}, Π''_{123} meet in a line γ'_{123}, which represents the twisted cubic γ_{123} in Π_{123}, containing the eight points A_{23}, A_{31}, A_{12}, A_{123a}. An elliptic sextic with five points in either solid breaks up into two twisted cubics, one in each solid, having two common points on the line γ'_{123}. The corresponding line ϵ is therefore a chord of γ_{123}, and has a pair of repeated coincidence points, *i.e.* Π_{123} touches Ψ all along γ_{123}. The cones of chords of γ_{123} all belong to the family Γ^2.

each family, through it, all of them tangents to Ψ, and lying therefore in the tangent solid to Ψ at the point. We may define Ψ as the locus of points at which the two lines of any (and therefore all) of the families coincide, a definition which corresponds to that of the Kummer surface as the focal surface of the quadratic congruence.

Exceptionally, the line on Ω may be a chord of the conjugate cubic curve, in which case, it projects into a bitangent of Ψ, and, in fact, necessarily lies in one of the primes Δ. Lines of this type we discuss in more detail in the next section.

We conclude this section by summarizing the representations on R' of the loci on Υ, giving first in tabular form (see p. 325) the representatives of the elements of the 64_8 figure.

The representatives of the other principal loci are:

Line $l_\alpha \rightarrow$ line l_α' through W_α', and a rational quintic l_α'' through the remaining base points;

Repeated line ϵ, quadrisecant of $\Psi \rightarrow$ normal elliptic sextic curve ϵ' through the eight base points;

Section of Υ by the [5] & tangent along $\epsilon \rightarrow$ manifold of chords of ϵ';

Repeated quadric cone Γ of lines ϵ, the two sheets winding along a ${}_4c^6$, $\chi \rightarrow$ Veronese surface F' through the eight base points;

Section of Υ by a prime $\mathfrak{P}$ touching Υ over $\Gamma \rightarrow$ cubic primal of conic planes of F';

Repeated line ϵ tangent to $\Psi \rightarrow$ curve ϵ' has a double point;

Line l_α touching $\Psi \rightarrow$ line l_α' and conjugate quintic l_α'' have a common point;

Repeated quadric cone Γ^1 (the two sheets winding along a sextic curve with a double point at the vertex) whose generators are tangent to $\Psi \rightarrow$ quartic cone through the eight base points, but not containing w'.

Repeated quadric cone ${\Gamma_8}^2$ (the sheets touching along a twisted cubic) lying in Π_8, whose generators are chords of γ_8 and bitangents of $\Psi \rightarrow$ quartic cone of chords of w', whose vertex is (necessarily) on w'*.

17. *The lines ϵ corresponding to degenerate curves e'.*

Ψ is the locus of the projections of points N of Ω, the tangent [5]'s at which meet Υ, and the lines ϵ are the projections of curves e which meet Υ in two points. So far we have considered only one type of curve e, namely

* This is actually the only one of the twisted cubics not yet dealt with. The chords of $\gamma_{\rho\sigma\tau}$ ($\rho, \sigma, \tau = 1, \ldots, 7$) were discussed in the last foot-note, and those of γ_ρ and $\gamma_{\sigma\rho 8}$ at the end of § 13.

those which touch Υ at V, but Υ meets Ω altogether in the eight lines v_α, so that we have also to consider curves e which meet either (i) the same line v_α twice, or (ii) v_α and v_β each in a point.

Case (i) we have dealt with in effect already, since a curve e which meets v_1 twice is either v_1 itself together with a twisted cubic of which it is chord, or, if the two points on v_1 coincide, a quartic curve with a double point on v_1. In each of these cases the surface F is replaced by a surface ϕ whose multiple point is some point, say V_1, on v_1. The projection of this surface from Υ, which meets the [6] in which it lies in the tangent plane to it at V is a quadric conical sheet, and, since ϕ contains the seven lines $l_2, \ldots, l_8$ (which are not $v_2, \ldots, v_8$) through V_1, the cone passes through the points $A_{12}, \ldots, A_{18}$. *I.e.* it is one of the cones of chords of the twisted cubic on which the eight points in Π_1 lie. Thus case (i) adds nothing to what is already known.

Now consider case (ii) in which a curve e meets v_1 and v_2, say in V_1 and V_2. Such a curve is represented in R' (the representation is of Ω not Υ) by an elliptic sextic e' which passes through $W_1', \ldots, W_7'$, and, in addition, meets $V'W_1'$ in V_1', and $V'W_2'$ in V_2'. It therefore has four distinct points in the plane $V'W_1'W_2'$, and must consist of a conic ν' through $V_1'W_1'V_2'W_2'$ and a rational quartic μ' through $W_3', \ldots, W_7'$, bisecant to ν'. μ' lies in Δ_{12}', and Δ_{12}' meets ν' in two points, so that if ν' is fixed, seven points on μ' are known, and it is also fixed. ν' represents a plane section ν of the quadric surface $\{v_1, v_2\}$, and μ' represents a conic μ of the system other than ϑ_{12} on ω_{12}. What we have now on Ω is this: fixed generators v_1, v_2 of the quadric surface $\{v_1, v_2\}$, and a variable plane section ν; a fixed plane section θ of $\{v_1, v_2\}$ which is also a conic of the system ϑ_{12} on ω_{12}, passing through the two points $V_{1,2}, V_{2,1}$, where v_1 and v_2 meet ω_{12}; and a conic μ of the system, other than ϑ_{12} on ω_{12}, passing through the two points (θ, γ), and therefore fixed by ν. Since each line of the plane of θ (*i.e.* each pair of points on θ) gives only one conic μ, the conics μ, as v_1, v_2, and ν vary, form a family of freedom 2 only. The plane of μ meets Υ in the point where V_1V_2 meets $V_{1,2}V_{2,1}$, a point which does not generally lie on ω_{12}, and therefore not on μ; and thus μ projects into a repeated line, say ϵ_{12}, with two branch points, which, since μ meets $\{v_1, v_2\}$, passes through A_{12}. We have then in [7], through A_{12} and lying on v_{12}, ∞^2 lines ϵ_{12} belonging to the family ϵ, each with two branch points on it, neither of which in general coincides with A_{12}. These lines correspond exactly to the lines g on v which have already been discussed in § 14. They form a quartic cone whose prime section is a Veronese surface, and on which the branch points form a surface ψ_{12} of order 12 on which A_{12} is a fourfold point.

There is one other possible type of degenerate elliptic sextic curve e', namely that which consists of two twisted cubic curves with two common points, each curve passing through four of the base points. These correspond to the lines of the solid represented by the [3] which joins either set of points.

18. *The order of* Ψ.

Since both manifolds are of order 8 the complete intersection Δ_{12} with Υ_1 is v_{12}. We want now to find the complete intersection of Δ_{12} with Ψ. Part of it, at least, we have seen to be the surface ψ_{12} of order 12 with a fourfold point at A_{12}. Consider now the representation M' of a point M of Ψ that lies in Δ_{12}. M' must be such that there are ∞^1 elliptic sextic curves through it, each touched at that point by any primal Σ' through it. Among these primals is that composed of the prime Δ'_{12} and the quadric cone with a plane vertex Δ''_{12}. Since Δ'_{12} cannot touch the curve at M' (without the curve breaking up and lying partly in Δ'_{12}), M' must be a double point of this complete locus, namely a point of the quadric line-cone $(\Delta'_{12}, \Delta''_{12})$ whose vertex is the line $(\Delta'_{12}, [W_0' W_1' W_2'])$, say d'_{12}. But through any point M' of this cone we can draw one rational quartic curve that passes through $W_3', \ldots, W_7'$ and has d'_{12} as a chord, and this curve, together with the conic through W_0', W_1', W_2' that meets d'_{12} in the same two points, forms a curve of the family e'. Through this curve there can be drawn a Veronese surface (at least one and, in any case, a finite number). When this surface is represented on a plane, the quartic curve becomes a conic on which we have six points $W_3'', \ldots, W_7'', M''$, and the conic becomes a line on which we have three points W_0'', W_1'', W_2''. M' is any point of $(\Delta'_{12}, \Delta''_{12})$, and can thus be taken arbitrarily along the quartic, so that M'' can be any point of the conic. The nine points $W_0'', W_1'', \ldots, W_7'', M''$ are therefore not generally associated, and hence there is, in general, no other curve of the family e' through M' and on this particular Veronese surface, and thus, since the number of Veronese surfaces is finite, no other curves of the family at all. What we have proved so far is this: (i) any point of Ψ that lies in Δ_{12} is represented by a point of the quadric line-cone $(\Delta'_{12}, \Delta''_{12})$, (ii) not every point of $(\Delta'_{12}, \Delta''_{12})$, but only a finite number of the points on any of the ∞^2 family rational quartics that can be drawn through $W_3', \ldots, W_7'$ with d'_{12} as a chord, represents a point of Ψ.

Now consider what one of these rational quartics represents on v_{12}. A cubic primal through the three points on d'_{12}, with double points at $W_3', \ldots, W_7'$, meets the curve in two variable points which belong to an involution (since the curve belongs to the family e') of which the two points on d'_{12} are a pair. The curve, in fact, represents a repeated line ϵ_{12} through

A_{12}. We have found then that the only points of Ψ which lie in Δ_{12} are those which lie on the lines ϵ_{12} through A_{12}. But these points have already been discussed (§ 17); their locus is the surface ψ_{12}.

In § 14, v_{12} was obtained as the projection of a certain family of points on ω_{12}, but v_{12}, in addition to being the projection of ω_{12} (represented by Δ'_{12}), is the projection of some other quite distinct manifold on Ω, namely that represented by Δ''_{12}. From this second manifold also we should obtain ψ_{12} as the locus of branch points on the lines ϵ_{12}, whence it follows that a line ϵ_{12} (through A_{12}) is a quadrisecant of Ψ on which the points coincide in two pairs, *i.e.* ϵ_{12} is a double tangent of Ψ*. Thus the complete intersection of Ψ with Δ_{12} is the surface ψ_{12} of order 12 counted twice.

Ψ, the analogue of the Kummer surface, is a threefold of order 24 in [7], *on which there are sixty-four quadruple points. These lie by sets of eight on sixty-four twisted cubic curves that lie on the manifold, there being eight of the curves through each of the points. The solid containing any of the curves touches Ψ all along the curve. At each of the singular points there is an osculating prime, touching Ψ at every point of a surface of order* 12, *and containing the eight twisted cubics through the point, and therefore twenty-eight other of the singular points.*

19. *The manifold Ψ' corresponding to Ψ.*

Each point of Ψ corresponds to a pair of coincident points in the involution in R', so that Ψ, and the locus Ψ' of these points, are birationally related. Suppose that Ψ' is of order n, and that on it the eight points W_ρ' are r-ple, that the twenty-eight lines $W_\rho' W_\sigma'$, and the quintic curve w' is s-ple, and that the planes $W_\rho' W_\sigma' W_\tau'$, and the quartic cones of chords of w', vertices W_ρ', are simple.

(i) Since Ψ is of order 24, we get, by considering the variable intersections of Ψ' with three primals Σ',

$$24 = 3(9n-33s)-16(4r-8s).$$

(ii) Ψ is met by certain [5]'s in sets of four sextic curves of genus 4. These curves are represented by curves of order 18, on which the points W_ρ' are triple, and the four together must be equivalent to the residual intersection of Ψ' with two primals Σ', *i.e.* a curve of order $9n-33s$ on

* This is in addition to passing through the singular point A_{12} on ψ_{12}.

which the points W_ρ' are $(4r-8s)$-ple. We have, therefore,

$$72 = 9n-33s$$

and

$$12 = 4r-8s.$$

These two imply the equation above.

(iii) Δ_{12} cuts Ψ in a certain surface of order 12 repeated, which (as a simple surface) is represented by the section of Ψ' by Δ_{12}'. This consists in part of the ten planes $[W_\rho', W_\sigma', W_\tau']$ $(\rho, \sigma, \tau = 3, \ldots, 7)$, and for the rest of a surface of order $n-10$, on which the ten lines $W_\rho' W_\sigma'$ are $(s-3)$-ple, the five points W_ρ' are $(r-6)$-ple, the three collinear points $([W_1', W_0'], \Delta_{12}')$, etc., are s-ple, and the line on which they lie has an unknown multiplicity. This surface is to be met by two manifolds Σ' in twelve variable points, so that

$$12 = 3\{3(n-10)-10(s-3)\}-10(r-4s-6)-3s.$$

The three independent equations that we have obtained are solved by

$$n = 19, \quad r = 9, \quad s = 3.$$

Ψ', the analogue of the Weddle surface, is of order 19 *in* [5], *and has on it eight ninefold points, whose twenty-eight joins by pairs are triple lines, and whose fifty-six joins by sets of three are simple planes; further, the normal rational quintic curve through the eight points is triple, and those of its chords which pass through any of the eight points lie on Ψ'.*

20. *Collineations and correlations of the 64_8 figure into itself.*

If we consider the sixty-four points, solids, and primes as an abstract figure defined by the relations of incidence (*i.e.* Π_α contains $A_{\alpha\beta}$ and lies in $\Delta_{\alpha\beta}$, etc.), then, having named the point A, the symbols of all the other elements of the configuration are determined (save for a permutation of $1 \ldots 8$), thus, Δ is the prime paired with A, Π_α are the solids through A, etc. But any of the points could be named A, and more than nine of the points are independent (*i.e.* we can select in many ways eight points forming a simplex and a ninth point not lying in any prime of the simplex). Further, in two schemes in which different points have the symbol A, and suitable permutations of 1, ..., 8 for the suffixes are adopted, each point has two distinct names. Thus the 64_8 figure possesses sixty-four collineations into itself (including identity) in which none of the points remains fixed.

Equally we could begin by naming the prime Δ and construct a system of names for the elements in which the elements named at each step are the duals of those named in the original sequence. Thus the configuration has also sixty-four correlations into itself. These correlations are of three types:

(i) $$A \to \Delta_{12}, \quad A_{1\gamma} \to \Delta_{2\gamma}, \quad A_{34} \to \Delta_{1234},$$

(ii) $$A \to \Delta_{1234}, \quad A_{12} \to \Delta_{34}, \quad A_{1\delta} \to \Delta_{234\delta}, \quad A_{56} \to \Delta_{78}, \quad \ldots,$$

(iii) $$A \to \Delta, \quad A_{\alpha\beta} \to \Delta_{\alpha\beta}, \quad A_{\alpha\beta\gamma\delta} \to \Delta_{\alpha\beta\gamma\delta}.$$

We may shorten this description by making the convention that, if A_g stands for any of A, $A_{\alpha\beta}$, $A_{\alpha\beta\gamma\delta}$, then $A_{g\alpha\beta\ldots}$ is the point obtained by adding $\alpha\beta\ldots$ to the suffixes of the original point, where, if there now appear two equal suffixes, both are suppressed, and if there then remain more than four suffixes, these are replaced by the complementary set. We may now write

(i) $$A_g \to \Delta_{g12},$$

(ii) $$A_g \to \Delta_{g1234},$$

(iii) $$A_g \to \Delta_g.$$

In each case, where there is a correspondence $A_g \to \Delta_h$ there is also a correspondence $\Delta_g \to A_h$.

The sixty-four collineations are of the same types, and are obtained from the above by writing $A_g \to A_h$, and $\Delta_g \to \Delta_h$ instead of $A_g \to \Delta_h$.

In a correlation of type (i) each prime passes through the point of which it is the conjugate, while conjugate solids are not incident, so that the correlation is a polarity with respect to a null-system. In (ii) no pairs of conjugate elements are incident, while the conjugates of primes through a point are points in the conjugate prime, so that the correlation is a polarity with respect to a quadric. Finally, in (iii) every pair of conjugate elements are incident, each solid coinciding with its conjugate, so that the correlation is either a polarity with respect to a quadric on which all the solids lie, while the primes are the tangents to it at the corresponding points, or it is a polarity with respect to a null-system in which all the solids are self-conjugate. But null-systems in [7] have freedom 20, and six conditions are imposed in making a solid self-conjugate, so that if a figure contains four solids in general position it is not possible to find a null-system in which it is self-reciprocal. Such a set in the

64_8 figure is Π_1, Π_{234}, Π_{256}, Π_{278}. Thus there is a quadric, S, containing all the solids and touching all the primes Δ_g at the corresponding points A_g*.

If we denote by $T_{\alpha\beta}$ the collineation of type (i) in which $A_g \to A_{\alpha\beta g}$, we obtain a group of operations for which

$$T_{\alpha\beta}^2 = \mathbf{1},$$

$$T_{\alpha\beta} T_{\alpha\gamma} = T_{\beta\gamma},$$

$$T_{\alpha\beta} T_{\gamma\delta} = T_{\alpha\gamma} T_{\beta\delta} = T_{\alpha'\beta'} T_{\gamma'\delta'}$$

($\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ all different).

Thus we can express the collineation $T_{\alpha\beta}$ as the product of two collineations T_α and T_β which are commutative. *I.e.* we can express all the collineations in terms of the eight involutary collineations $T_1, \ldots, T_8$,

where $$T_\alpha T_\beta = T_{\alpha\beta},$$

$$T_\alpha{}^2 = \mathbf{1},$$

and $$T_1 T_2 \ldots T_8 = \mathbf{1}.$$

These collineations are not unique; we could use, *e.g.*, $\mathbf{1}, T_{12}, \ldots, T_{18}$; or $T_1, T_1 T_{12} = T_2, \ldots, T_1 T_{18} = T_8$, where T_1 is any involutary collineation. In fact, the most general system of collineations may be obtained from an arbitrary involutary collineation by means of a set of equations such as these. The figure consisting of the points $T_1 . A_g$ may be any other 64_8 figure with the same projective invariants as the original.

The system of collineations being established, to obtain any correlation S_g we first effect the corresponding collineation T_g, and then take the polar reciprocal with respect to S, *i.e.*

$$S_g = S T_g = T_g S,$$

where $S^2 = \mathbf{1}$.

Thus the whole group of correlations and collineations can be constructed from the set of eight involutary collineations T_α, and the polarity with respect to the quadric S. The system, written with general suffixes, is

$$T_\alpha \ldots T_\beta . A_g = A_{\alpha\beta \ldots g},$$

$$S T_\alpha \ldots T_\beta . A_g = \Delta_{\alpha\beta \ldots g},$$

$$T_\alpha \ldots T_\beta . \Pi_g = S T_\alpha \ldots T_\beta . \Pi_g = \Pi_{\alpha \ldots \beta g}.$$

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* We notice in confirmation of this that the eight solids which pass through a point all lie in a prime and on a quadric cone (§ 14). Also any two solids either have a common line, or do not meet, so that all the solids belong to the same system of generators.

From the nature of its intersections with the manifold it is clear that the 64_8 determines, uniquely, the manifold Υ_1 on which the points are quadruple, etc.* Thus any collineation of the 64_8 into itself, transforms also Ψ and Υ_1 into themselves, and any correlation establishes a correspondence between the points of the manifolds and certain primes having tangential relations to them. Consider first the correlation S. The quadric S contains all the solids Π_g, and therefore all the manifolds v_g, and thus, since any line l_a or ϵ meets all the v_g, S contains the whole manifold Υ_1†. The polar [5] of a line ϵ with respect to S is the tangent [5] to S along ϵ, and the tangent [5] to Υ_1 along ϵ is $\mathfrak{E}$. Thus $\mathfrak{E}$ must also be tangent to S‡, and is the polar [5] of ϵ with respect to S. In the same way the polar of a point N of Ψ is the prime $\mathfrak{N}$ touching Υ_1 over the quadric cone Γ of quadrisecants to Ψ through N. Further, since any of the correlations is the resultant of S and one of the collineations, its effect is to transform any line ϵ into some [5] $\mathfrak{E}$, and any point N into some prime $\mathfrak{N}$.

(i) *If N is a point of Ψ, each point $N_g = T_g N$ lies on Ψ and each prime $\mathfrak{N}_g = S_g N$ contains the tangent* [3] *to Ψ at N_g and at every point of the sextic curve χ on the quadric cone Γ whose vertex is N_g.*

(ii) *If P is a point of Υ_1, each point $P_g = T_g P$ lies on Υ_1 and each prime $\mathfrak{P}_g = S_g P$ is the tangent at P_g to the quadric S on which Υ_1 lies.*

It should be noticed that this construction gives no indication of the existence of solids containing sets of eight of the points N_g or P_g. Nor does it seem likely that these exist, at least in the case of the P_g; in fact, the figure formed, although a specialization of Wirtinger's 64_{28} figure, is less highly specialized than the 64_8. The relation between the 64_8 and Wirtinger's 64_{28} is investigated more closely in § 22.

21. *The inversions of R' corresponding to the collineations of Υ.*

The collineations of Υ into itself must be represented by Cremona transformations of R'. To obtain these transformations consider the

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* Consider first Δ_g: the solids in it define the quadric cone vertex A_g on which v_g lies, then the other twenty-eight points define the Veronese surface of projected lines ϵ_g, and thus, eventually, the whole manifold v_g. Finally, the sixty-four manifolds v_g cannot lie on more than one manifold Υ_1.

† $\mathfrak{E}$ contains the sixteen lines l_a at each point of ϵ; these all lie on S, and do not lie in a space of dimension less than 5 (except at points of Ψ). Thus the tangent prime at any point of ϵ passes through $\mathfrak{E}$.

‡ Since Υ_1 is a fivefold of order 8 in [7] it cannot lie on more than one quadric.

two equivalent forms for the points of Υ, namely

$$P \equiv \sum_{\lambda\mu\nu} y_\lambda y_\mu y_\nu \mathbf{8}_{\lambda\mu\nu} \equiv \prod_\kappa y_\kappa \Sigma y_\lambda^{-1} y_\mu^{-1} y_\nu^{-1} \mathbf{7}_{\lambda\mu\nu}, \tag{3.1}$$

where

$$\sum_{\lambda\mu} \mathbf{8}_{\lambda\mu n} = 0, \quad \Sigma \mathbf{7}_{\lambda\mu n} = 0, \tag{2.4}$$

and

$$\sum_{\lambda\mu} a_\lambda a_\mu \mathbf{8}_{\lambda\mu n} = 0, \quad \Sigma a_\lambda^{-1} a_\mu^{-1} \mathbf{7}_{\lambda\mu n} = 0,$$

the a_λ being the parameters of the point of projection V on Ω. Now effect the transformation

$$\rho y_\lambda = a_\lambda / y_\lambda'; \tag{21.1}$$

P becomes

$$\Sigma y_\lambda'^{-1} y_\mu'^{-1} y_\nu'^{-1} \mathbf{8}'_{\lambda\mu\nu},$$

where

$$\mathbf{8}'_{\lambda\mu\nu} = a_\lambda a_\mu a_\nu \mathbf{8}_{\lambda\mu\nu},$$

and the points $\mathbf{8}'_{\lambda\mu\nu}$ are connected by the relations

$$\Sigma a_\lambda^{-1} a_\mu^{-1} \mathbf{8}'_{\lambda\mu n} = 0,$$

and

$$\Sigma \mathbf{8}'_{\lambda\mu n} = 0.$$

Thus the effect of the transformation on Υ is to interchange the parts played by the points $\mathbf{8}_{\lambda\mu\nu}$ and $\mathbf{7}_{\lambda\mu\nu}$; the transformation is, in fact, the collineation $T_7 T_8$.

To avoid confusion of notation, take as the base points in R': W_1', ..., W_7', and W_8' (not W_0'), where $W_7' = W_1' + \ldots + W_6'$, and $W_8' = a_1 W_1' + \ldots + a_6 W_6'$. The transformation (21.1) which, as a transformation of R', we denote by τ_{78}, is the inversion in which $W_1' \ldots W_6'$ is the base simplex and W_7' is conjugate to W_8'. Thus, by symmetry, the collineation $T_\alpha T_\beta$ must be represented by the inversion $\tau_{\alpha\beta}$ in which W_α' becomes W_β', and the other six points form the base simplex.

With the same coordinate system the transformation τ_{81} is

$$\kappa y_1 = a_1 / y_1',$$

$$\kappa y_\lambda = (a_\lambda y_1' - a_1 y_\lambda') / y_1' (y_1' - y_\lambda'). \tag{21.2}$$

By means of the transformation $\tau = \tau_{81}\tau_{82}\tau_{83}\tau_{84}\tau_{85}\tau_{86}\tau_{87}$, we obtain from P' a point $P'' = \tau P'$ which does not, in general, coincide with P'. But the corresponding collineation of Υ_1 is the identical transformation, so that the pair of points of R' corresponding to the point P is P', P''. We

can now discard the elliptic curves as a basis of the involution in R', and establish the involution *ab initio* in this way*:

The ∞^7 cubic primals in a [5] R' *that have eight fixed double points can be put in* (1, 1) *correspondence with the primes of* [7] *in such a way that to a point of R' there corresponds a point of a manifold Υ. There are twenty-eight inversion transformations $\tau_{\alpha\beta}$ of this system of primals into itself, each corresponding to a collineation of Υ into itself. The transformation $\tau = \tau_{81}\tau_{82}\cdots\tau_{87}$ (or any similar product) is not an identity, while the corresponding collineation is. Thus to every point P of Υ there corresponds a pair of points P' and $P'' = \tau P'$ of an involution in R'. The locus of double points of the involution is Ψ', the analogue of the Weddle surface*†.

22. *The relation of the* 64_8 *to Wirtinger's* 64_{28}.

By considering ϑ functions of a triple variable Wirtinger‡ obtained a generalization of the Kummer configuration and surface of which the figure discussed here seems to be a special case. Wirtinger finds a threefold M_3^{24} of order 24 in [7] with sixty-four quadruple points, and sixty-four primes, each touching it over a surface M_2^{12}, twenty-eight of the points lying in each prime, and twenty-eight of the primes passing through each point, forming thus a 64_{28} configuration. This manifold bears a strong resemblance to Ψ', but differs from it in two important respects:

(i) There appear to be no solids corresponding to the Π_g.

(ii) Any of the primes contains only twenty-eight points, whereas in the 64_8 Δ_g contains not only the points $A_{g,\beta}$, but also A_g.

The 64_{28} and M_3^{24} have the same type of group of sixty-four collineations and correlations into themselves as has been found for the 64_8 and Ψ' (*i.e.* the sixty-four correlations are all polarities, twenty-eight of them with respect to null-systems, and thirty-six with respect to quadrics). Further, the correlatives of points of M_3^{24} are primes touching the manifold along sextic curves. The essential difference appears to be that one of the

* An alternative to the elliptic curves is essential in the more general cases, since the manifold of repeated lines ϵ derived from them is of dimension less than that of the manifold corresponding to Υ. There are, in fact, always ∞^1 of the curves, while the dimension of Υ and R' is $2r-1$.

† By equating the coordinates of P' and P'' [which are easily calculated from (21.2)] we can obtain equations for Ψ'.

‡ Wirtinger (3).

correlations is not distinguished from the rest by being a polarity with respect to a quadric on which the manifold lies.

Barrau found that there was a similar figure consisting of the sixty-four points obtained by taking each set of coordinates with each of the arrangements of signs in the following scheme*:

a	b	c	d	e	f	g	h	+	+	+	+	+	+	+	+
b	a	d	c	f	e	h	g	+	+	+	+	−	−	−	−
c	d	a	b	g	h	e	f	+	+	−	−	+	+	−	−
d	c	b	a	h	g	f	e	+	−	+	−	+	−	+	−
e	f	g	h	a	b	c	d	+	−	−	+	−	+	+	−
f	e	h	g	b	a	d	c	+	−	−	+	+	−	−	+
g	h	e	f	c	d	a	b	+	−	+	−	−	+	−	+
h	g	f	e	d	c	b	a	+	+	−	−	−	−	+	+

and of the sixty-four primes with the same coordinates†. Of the thirty-six quadrics with respect to which the figure is self-polar, eight have equations

$$\sum_{1}^{8} \pm x_i^2 = 0$$

(the sets of signs being those of the scheme above), while the equations of the others are of the form

$$x_1 x_2 \pm x_3 x_4 \pm x_5 x_6 \pm x_7 x_8 = 0.$$

The twenty-eight null-systems are typified by

$$p_{12} + p_{34} + p_{56} + p_{78} = 0.$$

If we make P lie on one of the quadrics, then each prime of the figure contains an extra point. For example, if P lies on

$$\Sigma x_i^2 = 0,$$

then each prime touches the quadric, its point of contact being that which

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* This table is taken from Barrau (9), 264.

† In the Kummer figure the corresponding scheme can be condensed by expressing the sixteen points as P, iP, Pi, iPj, etc., where P, the point whose coordinates are (a, b, c, d), is written as $a+ib+jc+kd$ and treated as a quaternion [Baker (2), 138]. A similar notation for the 64_{28} does not seem possible, since we have to find an algebra with eight units, similar to quaternion algebra, and associative in multiplication. This property of associativeness precludes, for example, Cayley's Algebra, for which see L. E. Dickson, *Linear algebras* (Cambridge, 1914), 14.

has the same coordinates as it, a point of the 64_{28} which it does not in the general case contain*. But, although we have now twenty-nine points in each prime, there is, as yet, no arrangement of the points in sets of eight lying in solids. This figure is, in fact, similar to that formed by the points P_g derived from any point P of Υ_1 (§ 20).

Barrau's system of quadrics and null-systems is completely defined by the coordinate system, while the coordinate system is defined by it, and his 64_{28} is derived from an arbitrary point of the [7]. Thus his 64_{28} has seven projective invariants. On the other hand, Wirtinger's 64_{28} has only six projective invariants (the periods of the ϑ-functions), and the 64_8 has only five.

If we assume, as is most probably true, that the system of collineations and correlations is the same in each case, then

(i) from a general point of [7] we derive Barrau's 64_{28},

(ii) from a point on one of the thirty-six quadrics we derive Barrau's 64_{29},

(iii) from a point of a certain primal we derive Wirtinger's 64_{28},

(iv) from a point of some fivefold that lies on one of the quadrics we derive the 64_8.

The probability is that (iv) is the result of (ii) and (iii) together; in fact, that the 64_8 is the special Wirtinger figure in which the ϑ-functions are hyperelliptic†.

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* If one of the points of the 16_6 is made to lie on one of the polarizing quadrics, then all the points lie on the quadric, and the figure reduces to the intersections of four generators of each system on the quadric, and the tangent planes at those points.

† This suggestion is due to Mr. Du Val, who also was the first to point out that Wirtinger's and Barrau's figures had different freedoms.

SOME THEOREMS ON MOMENT CONSTANTS AND FUNCTIONS

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1. Given a sequence $f_j(x)$ of bounded continuous functions defined in the finite or infinite interval (a, b), and a function $g(x)$ of bounded variation in that interval, we call

$$\int_a^b f_j\,dg$$

the sequence of moment constants of g with respect to the f_j. Given a function $K(x, y)$ which is bounded and continuous for x in (a, b) and y in (c, d), then

$$\int_a^b K(x, y)\,dg$$

is the moment function of g with respect to K.

The fundamental existence theorem when (a, b) is a finite interval is due to F. Riesz. If we are given n constants a_j $(j = 1, \ldots, n)$ and the n functions f_j, a necessary and sufficient condition that there shall exist a g whose total variation does not exceed 1, and which satisfies

$$a_j = \int_a^b f_j\,dg \quad (j = 1, \ldots, n), \tag{i}$$

is that, for all μ_j,

$$\left|\sum_1^n \mu_j a_j\right| \leqslant \max\left|\sum_1^n \mu_j f_j(x)\right| \quad (a \leqslant x \leqslant b).$$

Of this theorem, Riesz* and Helly† have given proofs which would not lead one to suspect that it can be established by a short and simple argument. This, however, is so. We venture to assert that the proof given in § 2 is definitive.

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* F. Riesz, *Annales Sci. de l'Ecole Normale*, 28 (1911), 33–62.

† Helly, *Wiener Sitzungsberichte*, 121 (1912), 265–297.

We investigate the properties of those functions $g(x)$ which satisfy (i) and whose total variation is least. For the equations

$$a_j = \frac{1}{\pi}\int_0^{2\pi} \cos jx\, dg, \qquad b_j = \frac{1}{\pi}\int_0^{2\pi} \sin jx\, dg \quad (j = 0, 1, \ldots, n),$$

we show that, except for trivial differences, there is only one such function.

To find a necessary and sufficient condition that

$$c(y) = \int_a^b K(x, y)\, dg \tag{ii}$$

shall have a solution g of bounded variation requires no special treatment when (a, b) is a finite interval, since (ii) is easily seen to be equivalent to the system

$$c(y_n) = \int_a^b K(x, y_n)\, dg \quad (n = 1, 2, \ldots),$$

where (y_n) are all the rational points in (c, d). Only when (a, b) is an unbounded interval is the problem of interest, and then, in the general case, is too difficult. What we consider is the special case presented by the equations

$$a(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos xt\, dg(t), \qquad b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin xt\, dg(t).$$

2. THEOREM I (F. Riesz). *Let f_j $(j = 1, \ldots, n)$ be a set of continuous functions defined in the finite interval (a, b). Let a_j be a set of constants such that, for every set of numbers μ_j, we have*

$$\left|\sum_1^n \mu_j a_j\right| \leqslant \max\left|\sum_1^n \mu_j f_j(x)\right| \quad (a \leqslant x \leqslant b). \tag{1}$$

Then there is a function $g(x)$, defined in (a, b), whose total variation does not exceed 1 and such that

$$a_j = \int_a^b f_j\, dg \quad (j = 1, \ldots, n). \tag{2}$$

Let E be the n-dimensional set of points $\xi \equiv (\xi_1, \ldots, \xi_n)$ such that to each ξ there corresponds a g_ξ which satisfies

$$\int_a^b |dg_\xi| \leqslant 1, \qquad \xi_j = \int_a^b f_j\, dg_\xi \quad (j = 1, \ldots, n).$$

If ξ and η belong to E, then so does $\{\lambda\xi_1 + (1-\lambda)\eta_1, \ldots\}$ for all $0 < \lambda < 1$.

The set E is closed. For let ξ^ν be a sequence of points of E which tends to ξ, and let g_ν be the corresponding sequence of functions which satisfy

$$\int_a^b |dg_\nu| \leqslant 1, \qquad \xi_j{}^\nu = \int_a^b f_j\, dg_\nu \quad (j = 1, \ldots, n).$$

We may suppose that $g_\nu(a) = 0$. Then the sequence g_ν is uniformly bounded. We can select a subsequence g_{ν_k} which converges for $x = a$, $x = b$ and for rational x in (a, b). We define a function $g(x)$ as the limit of the subsequence for these values of x; for irrational x within (a, b) we define $g(x)$ by the formula $g(x) = \overline{\lim}_{\xi \to x} g(\xi)$, where ξ is confined to rational values. Then

$$\int_a^b |dg| \leqslant 1.$$

We simplify the notation by writing k for ν_k. Then, for all k,

$$\int_a^b |dg_k| \leqslant 1, \qquad \xi_j{}^k = \int_a^b f_j\, dg_k \quad (j = 1, \ldots, n). \tag{3}$$

Let $\epsilon > 0$ and let $a = x_0 < x_1 < \ldots < x_m = b$ be a scale of numbers such that the oscillation of f_j $(j = 1, \ldots, n)$ in (x_i, x_{i+1}) is less than ϵ. The points $x_i\,(i = 1, \ldots, m-1)$, whether rational or not, are the limits of sequences $\xi_i^{(r)}$ of rational points such that $g(\xi_i^{(r)}) \to g(x_i)$. For all sufficiently large r, the oscillation of f_j in $(\xi_i^{(r)}, \xi_{i+1}^{(r)})$ is less than ϵ, where $\xi_0^{(r)} = a$, $\xi_m^{(r)} = b$. Hence, for such r,

$$\left| \int_a^b f_j\, dg_k - \sum_{i=0}^{m-1} f_j(\xi_i^{(r)}) \{g_k(\xi_{i+1}^{(r)}) - g_k(\xi_i^{(r)})\} \right|$$

does not exceed ϵ times the total variation of g_k. Hence, for all k, by (3),

$$\left| \xi_j{}^k - \sum_{i=0}^{m-1} f_j(\xi_i^{(r)}) \{g_k(\xi_{i+1}^{(r)}) - g_k(\xi_i^{(r)})\} \right| < \epsilon \quad (r \geqslant R).$$

Letting $k \to \infty$, and then $r \to \infty$, we obtain

$$\left| \xi_j - \sum_{i=0}^{m-1} f_j(x_i) \{g(x_{i+1}) - g(x_i)\} \right| \leqslant \epsilon.$$

Since ϵ is arbitrary, we have

$$\xi_j = \int_a^b f_j\, dg,$$

which proves that ξ belongs to E.

The relation (1) means that to each set of μ_j there corresponds a θ such that $\theta^2 \leqslant 1$, and

$$\sum_1^n \mu_j a_j = \theta \max \sum_1^n \mu_j f_j(x) = \theta \sum_1^n \mu_j f_j(\zeta)$$

for a suitable ζ. If $a < \zeta \leqslant b$, define a function $g(x)$ by $g(x) = 0$ $(a \leqslant x < \zeta)$, $g(x) = \theta$ $(\zeta \leqslant x \leqslant b)$. If $\zeta = a$, define $g(x)$ by $g(a) = 0$, $g(x) = \theta$ $(a < x \leqslant b)$. Then

$$\int_a^b |dg| \leqslant 1, \qquad \Sigma \mu_j a_j = \Sigma \mu_j c_j, \tag{4}$$

with

$$c_j = \int_a^b f_j \, dg.$$

Thus $C \equiv (c_1, \ldots, c_n)$ belongs to E, and (4) shows that every plane

$$\Sigma \mu_j (x_j - a_j) = 0$$

through $A \equiv (a_1, \ldots, a_n)$ has a point of E on it. If now A does not belong to E, there is a point $P \equiv (p_1, \ldots, p_n)$ of E whose distance from A is positive and a minimum. Consider the plane

$$\Sigma (p_j - a_j)(x_j - a_j) = 0.$$

By what precedes, this plane has a point $Q \equiv (q_1, \ldots, q_n)$ of E on it. Hence

$$\Sigma (p_j - a_j)(q_j - a_j) = 0. \tag{5}$$

The point $R(\lambda) \equiv \{\lambda p_1 + (1-\lambda) q_1, \ldots\}$ belongs to E for all λ such that $0 < \lambda < 1$. The distance $d(\lambda)$ of $R(\lambda)$ from A is given by

$$\begin{aligned} d^2(\lambda) &= \Sigma [\lambda (p_j - a_j) + (1-\lambda)(q_j - a_j)]^2 \\ &= \lambda^2 \Sigma (p_j - a_j)^2 + (1-\lambda)^2 \Sigma (q_j - a_j)^2 \end{aligned} \tag{6}$$

by (5); and, if

$$0 < 1 - \lambda < \frac{\Sigma (p_j - a_j)^2}{\Sigma (q_j - a_j)^2},$$

then (6) is less than $\Sigma (p_j - a_j)^2$. This contradiction shows that A belongs to E, and proves the theorem.

THEOREM II (F. Riesz). *Let f_j be a sequence of continuous functions defined in the interval (a, b), and a_j a sequence of constants such that, for every finite set of numbers $\mu_1, \ldots, \mu_n$, the condition (1) is satisfied. Then there is a function $g(x)$, defined in (a, b), whose total variation does not exceed 1 and such that*

$$a_j = \int_a^b f_j \, dg \quad (j = 1, 2, \ldots).$$

For every positive integer n, the conditions of Theorem I are satisfied. Hence there is a g_n such that

$$\int_a^b |dg_n| \leqslant 1, \qquad a_j = \int_a^b f_j dg_n \quad (j = 1, \ldots, n).$$

Applying the process used above, we obtain a sequence n_k giving rise to a function g whose total variation does not exceed 1 and such that

$$\lim_{k\to\infty} \int_a^b f_j dg_{n_k} = \int_a^b f_j dg$$

for all j. But

$$a_j = \int_a^b f_j dg_{n_k}$$

for all $n_k \geqslant j$. Hence the result.

3. The relation (1) has been shown to be sufficient for the conclusion of Theorem I to hold; and it is obviously a necessary condition. A similar remark is true of Theorem II. As an example, consider the following theorem of F. Riesz:

THEOREM III. *Let $A(f)$ be a linear functional whose field is the family of continuous functions $f(x)$ defined in the closed interval $(-1, 1)$. Then there is a function of bounded variation $g(x)$ such that*

$$A(f) = \int_{-1}^{1} f dg.$$

There is a constant $M > 0$ such that, for every continuous f,

$$|A(f)| \leqslant M \operatorname{Max}|f|.$$

If $\phi_n = x^{n-1}$ $(n = 1, 2, \ldots)$, then, writing $a_n = A(\phi_n)$, we have

$$\left| \sum_1^n \mu_j a_j \right| = |A(\Sigma \mu_j \theta_j)| \leqslant M \operatorname{Max}|\Sigma \mu_j \phi_j|.$$

By Theorem II, there is a $g(x)$ such that

$$\int_{-1}^{1} |dg| \leqslant M, \qquad a_j = \int_{-1}^{1} \phi_j dg.$$

If f is any continuous function, we can find a sequence P_n of polynomials which converges uniformly to f. Then

$$A(f) = \lim A(P_n) = \lim \int_{-1}^{1} P_n dg = \int_{-1}^{1} f dg.$$

As a further example, we shall prove

THEOREM IV. *Let λ_n be a sequence of positive integers which satisfy $\lambda_{n+1}/\lambda_n \geqslant q > 1$. Let (a, b) be an interval and let a_n, α_n be two sequences. If $|a_n| \leqslant 1$, there is a function $g(x)$ of bounded variation in (a, b) such that*

$$a_n = \int_a^b \cos(\lambda_n x + \alpha_n)\, dg$$

for all sufficiently large n.

We require the following lemma.

LEMMA 1. *There is a constant C, depending only on q and on*

$$\theta = \frac{b-a}{2\pi},$$

such that, for any numbers μ_n,

$$\max_{(a,b)} \sum_{n=m}^{M} \mu_n \cos(\lambda_n x + \alpha_n) > C \sum_{n=m}^{M} |\mu_n|$$

for $M > m \geqslant m_0$, where m_0 depends only on θ and q.

Assuming for the moment that the lemma is proved, we have

$$\left| \sum_{m_0}^{M} a_n \mu_n \right| \leqslant \Sigma |\mu_n| \leqslant \frac{1}{C} \max_{(a,b)} \left| \sum_{m_0}^{M} \mu_n \cos(\lambda_n x + \alpha_n) \right|,$$

and the conditions of Theorem II are satisfied. The result follows by that theorem; it will be seen, from Theorem XVII, that $g(x)$ may be taken to be monotone.

The lemma, however, is substantially equivalent to a theorem of Zygmund*. The following proof will be found simpler than his. We may suppose that $b-a < 2\pi$. Let k be an integer which satisfies

$$1 + \frac{1}{q^k - 1} < \frac{q+1}{2}, \quad 1 - \frac{1}{q^k - 1} > \frac{q+1}{2q}. \tag{7}$$

For a given m and M, we may suppose that there is an integer r such that $m + (k-1) + rk = M$. Consider the function

$$P_m(x) = \prod_{j=0}^{r} \{1 + \epsilon_{m+jk} \cos(\lambda_{m+jk} x + \alpha_{m+jk})\}$$

$$= 1 + \sum_{j=0}^{r} \epsilon_{m+jk} \cos(\lambda_{m+jk} x + \alpha_{m+jk}) + S_m(x),$$

* Zygmund, *Studia Math.*, 3 (1931), 77–91.

which is non-negative if the ϵ's do not exceed 1 in absolute value. Let $\mu = \mu(\theta, q)$ be a positive integer which will be defined below. The function $S_m(x)$ is a trigonometric polynomial, the order of whose terms is of the form

$$s = \lambda_{m+ks_\nu} + \sum_{l=1}^{\nu-1} (\pm\lambda_{m+ks_l}), \tag{8}$$

where $s_{l_1} < s_{l_2} < \ldots < s_{l_\nu}$ form a set of at least two numbers from 0, 1, ..., r. For $m \geqslant m(\mu)$, we must have

$$|s-\lambda_n| > \mu \tag{9}$$

for all s and n. For s lies between the numbers

$$\lambda_{m+ks_\nu}[1 \pm (q^{-k}+q^{-2k}+\ldots)] = \lambda_{m+ks_\nu}\left(1 \pm \frac{1}{q^k-1}\right).$$

Hence, taking account of (7),

(i) if $n > m+ks_\nu$, then

$$|s-\lambda_n| \geqslant q\lambda_{m+ks_\nu} - \frac{q+1}{2}\lambda_{m+ks_\nu} = \frac{q-1}{2}\lambda_{m+ks_\nu};$$

(ii) if $n = m+ks_\nu$, then

$$|s-\lambda_n| = \left|\sum_1^{\nu-1} \pm\lambda_{m+ks_l}\right| > \lambda_{m+ks_{\nu-1}}\left(1-\frac{1}{q^k-1}\right) > \frac{1+q}{2q}\lambda_{m+ks_{\nu-1}};$$

(iii) if $n < m+ks_\nu$, then

$$|s-\lambda_n| > \frac{1+q}{2q}\lambda_{m+ks_\nu} - \lambda_{m+ks_\nu-1} > \frac{q-1}{2q}\lambda_{m+ks_\nu}.$$

This proves the assertion. The same argument applies to P_{m+1}, ..., P_{m+k-1} obtained from P_m on replacing m by $m+1$, ..., $m+k-1$ respectively. Hence

$$\Pi(x) = \sum_{p=0}^{k-1} P_{m+p} = k + \sum_{n=m}^{M} \epsilon_n \cos(\lambda_n x + \alpha_n) + \Sigma(x),$$

where $\Sigma(x)$ is a trigonometric polynomial the orders of whose terms satisfy (9) for $m \geqslant m(\mu)$.

Let $\eta = \theta/(16k)$. Let (A, B) be an interval of length 2π which is concentric with (a, b). Let

$$T(x) = c_0 + \sum_{n=1}^{\mu} c_n \cos(nx+\beta_n)$$

lie between 0 and η in the intervals (A, a), (b, B); between $1-\eta$ and 1 in the interval $\{a+\frac{1}{3}(b-a),\ b-\frac{1}{3}(b-a)\}$; and between 0 and 1 in the remaining two intervals. Then μ depends only on θ and q, and

$$c_0 = \frac{1}{2\pi}\int_A^B T(x)\,dx > \frac{1}{2\pi}(1-\eta)\,\frac{b-a}{3} > \frac{\theta}{4}. \tag{10}$$

Further,

$$T(x)\,\Pi(x) = kc_0 + c_0 \sum_m^M \epsilon_n \cos(\lambda_n x + \alpha_n) + \Theta(x),$$

where $\Theta(x)$ is a trigonometric polynomial the orders of whose terms differ from every λ_n in virtue of (9), provided that $m \geqslant m(\mu)$ and also that $(q-1)\lambda_m > \mu$. Take $\epsilon_n = \operatorname{sgn}\mu_n$ and write

$$V(x) = \sum_m^M \mu_n \cos(\lambda_n x + \alpha_n).$$

Then
$$\pi c_0 \sum_m^M |\mu_n| = \int_A^B V(x)\,T(x)\,\Pi(x)\,dx. \tag{11}$$

Let $K = \Sigma|\mu_n|$. Since $T(x)$, $\Pi(x)$ are non-negative, the integral last written does not exceed

$$\eta \operatorname*{Max}_{(A,\,B)} V(x)\left[\int_A^a + \int_b^B \Pi(x)\,dx\right] + \operatorname*{Max}_{(a,\,b)} V(x)\int_a^b \Pi(x)\,dx$$

$$< \frac{\theta}{16k}\,K\,2\pi k + \operatorname*{Max}_{(a,\,b)} V(x)\,2\pi k.$$

Thus (11) gives

$$\pi c_0 K < \pi\theta K/8 + 2\pi k \operatorname*{Max}_{(a,\,b)} V(x),$$

and so, by (10),
$$\operatorname*{Max}_{(a,\,b)} V(x) > \frac{\theta}{16k}\,\Sigma|\mu_n|.$$

4. There are special cases in which Theorem I may be extended by allowing the functions f_j to be discontinuous. We must, however, explain the interpretation to be put on the Stieltjes integrals in such cases. If $g(x)$ is bounded and non-diminishing in the closed interval (a, b), and $f(x)$ is bounded and B-measurable in that interval, we define a function $F(\xi)$ in $g(a) \leqslant \xi \leqslant g(b)$ by

$$\left.\begin{aligned} F(\xi) &= f(a), \quad g(a) \leqslant \xi \leqslant g(a+0) \\ &= f(x), \quad g(x-0) \leqslant \xi \leqslant g(x+0) \\ &= f(b), \quad g(b-0) \leqslant \xi \leqslant g(b) \end{aligned}\right\} \quad (a < x < b)$$

The value of $$\int_a^b f\,dg \tag{12}$$

is defined to be the value of the Lebesgue integral

$$\int_{g(a)}^{g(b)} F(\xi)\,d\xi.$$

If $g(x)$ is of bounded variation in (a, b), we can write $g(x) = P(x) - N(x)$, where P and N are bounded and non-diminishing, and the value of (12) is defined to be

$$\int_a^b f\,dP - \int_a^b f\,dN.$$

With this convention, Theorem I holds when each f_j is B-measurable and takes only a finite set of values.

THEOREM V. *Let f_j $(j = 1, \ldots, n)$ be a set of B-measurable functions which take only a finite set of values in the closed intervals (a, b). Let a_j be a set of constants such that, for every set of μ_j, the relation* (1) *is satisfied. Then there is a function $g(x)$ whose total variation in (a, b) does not exceed* 1 *and which satisfies* (2). *This function is a step function with a finite set of discontinuities.*

We require the following lemma.

LEMMA 2. *Given m sets of n numbers*

$$\xi_1^{(i)}, \ \ldots, \ \xi_n^{(i)} \quad (i = 1, \ldots, m),$$

and a set of constants $a_1, \ldots, a_n$ such that, for every set of $\mu_1, \ldots, \mu_n$, we have

$$\left| \sum_1^n \mu_j a_j \right| \leqslant \operatorname*{Max}_{(i)} \left| \sum_1^n \mu_j \xi_j^{(i)} \right|, \tag{13}$$

there is a set of d_i such that

$$\sum_{(i)} |d_i| \leqslant 1, \quad a_j = \sum_{(i)} d_i \xi_j^{(i)} \quad (j = 1, \ldots, n).$$

We regard $\xi^{(i)} \equiv (\xi_1^{(i)}, \ldots, \xi_n^{(i)})$ as a point in n-dimensional space. We consider the set E of points ξ such that to each ξ there corresponds a set of δ_i for which

$$\sum_{(i)} |\delta_i| \leqslant 1, \quad \xi_j = \sum_{(i)} \delta_i \xi_j^{(i)} \quad (j = 1, \ldots, n).$$

The set E is closed; and if ξ and η belong to E, so does

$$\{\lambda\xi_1+(1-\lambda)\eta_1, \ldots\}$$

for all λ such that $0<\lambda<1$. The relation (13) means that to each set of μ_j there corresponds a θ satisfying $\theta^2 \leqslant 1$ and an i such that

$$\sum_1^n \mu_j a_j = \theta \sum_1^n \mu_j \xi_j^{(i)} = \sum_1^n \mu_j c_j$$

with $c_j = \theta\xi_j^{(i)}$. Thus $C \equiv (c_1, \ldots, c_n)$ belongs to E; and every plane

$$\Sigma \mu_j(x_j - a_j) = 0$$

through $A \equiv (a_1, \ldots, a_n)$ has a point of E on it. Hence, as in Theorem I, A belongs to E.

Turning to the proof of the theorem, we consider the complexes $\{f_1(x), \ldots, f_n(x)\}$ for all x in (a, b), regarding two complexes

$$\{f_1(\xi), \ldots\}, \quad \{f_1(\eta), \ldots\}$$

as identical if and only if $f_1(\xi) = f_1(\eta), \ldots, f_n(\xi) = f_n(\eta)$. Then there is only a finite number of such complexes, and a finite set of points, say $x_1, \ldots, x_m$, such that

$$\{f_1(x_i), \ldots, f_n(x_i)\} \quad (i = 1, \ldots, m)$$

represents all the distinct complexes. For any given set of μ_j, the function $\Sigma \mu_j f_j(x)$ can only take a finite set of values, and all these are included in the m values $\Sigma \mu_j f_j(x_i)$. If now (1) is satisfied, then, writing $\xi_j^{(i)} = f_j(x_i)$, we see that the conditions of Lemma 2 hold. By that lemma, there is a set of d_i such that

$$\sum_{(i)} |d_i| \leqslant 1, \quad a_j = \sum_{(i)} d_i f_j(x_i) \quad (j = 1, \ldots, n). \quad (14)$$

We now define a step function whose discontinuities occur only at the points x_i by

$$g(a) = 0, \quad g(x_i+0) - g(x_i-0) = d_i, \quad g(x_i) = \tfrac{1}{2}\{g(x_i+0)+g(x_i-0)\}$$

$$(a < x_i < b),$$

where we use the convention $a_i - 0 \equiv a_i$, $b_i + 0 \equiv b_i$. Then, in virtue of the definitions previously given, (14) becomes

$$\int_a^b |dg| \leqslant 1, \quad a_j = \int_a^b f_j\, dg \quad (j = 1, \ldots, n).$$

5. Just as the proof of Theorem I is a natural modification of the proof of Theorem I of a previous paper*, so the proof of the following theorem is a natural modification of the proof of Theorem III of that paper. We therefore state it without proof.

THEOREM VI. *Let f_j $(j=1, \ldots, n)$ be a set of continuous functions defined in (a, b). Let a_j be a set of constants. Then, if* (1) *is satisfied, either for* (i) *every set of*

$$\mu_j \geqslant 0,$$

or for (ii) *every set of*

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0,$$

or for (iii) *every set of*

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0, \quad \mu_1-\mu_2 \geqslant \mu_2-\mu_3 \geqslant \ldots \geqslant \mu_{n-1}-\mu_n \geqslant 0,$$

there is a function $g(x)$, whose total variation in (a, b) does not exceed 1, *such that*

in case (i) $$a_j \leqslant \int_a^b f_j\,dg \qquad (j=1, \ldots, n),$$

in case (ii) $$\sum_1^k a_j \leqslant \sum_1^k \int_a^b f_j\,dg \qquad (k=1, \ldots, n),$$

in case (iii) $$\sum_1^k (k+1-j)\,a_j \leqslant \sum_1^k (k+1-j) \int_a^b f_j\,dg \quad (k=1, \ldots, n).$$

There is a corresponding theorem for a sequence of f_j and of a_j.

6. Suppose that we are given n continuous functions f_j, defined in the closed interval (a, b), and n constants a_j. If a function $g(x)$ exists whose total variation does not exceed 1 and which satisfies the equations (2), we say that $g(x)$ belongs to the class G. We say that a set of μ_j is the null set if $\mu_j = 0$ $(j=1, \ldots, n)$. We have

THEOREM VII. *Let* 1 *be the lower bound of numbers M such that*

$$\left| \sum_1^n \mu_j a_j \right| \leqslant M \max \left| \sum_1^n \mu_j f_j \right| \qquad (15)$$

* Verblunsky, "Some theorems on generalized Fourier constants", *Proc. London Math. Soc.* (2), 37 (1934), 33–62.

for all μ_j. Suppose that, for every set of μ_j other than the null set, the function $F(\mu) = F(\mu, x) = |\Sigma \mu_j f_j|$ does not attain its maximum in (a, b) at more than a finite set of points. Then

(i) *G exists, and every member of it has its total variation equal to* 1;

(ii) *every $g \subset G$ satisfies*

$$\underline{\lim_{x \to \xi}}\, g(x) \leqslant g(\xi) \leqslant \overline{\lim_{x \to \xi}}\, g(x) \quad (a < \xi < b); \tag{16}$$

(iii) *there exist two finite sets S_1, S_2 with no points in common such that every member of G is constant in any interval which contains no point of $S = S_1 + S_2$, and is non-diminishing at every point of S_1, and non-increasing at every point of S_2.*

By saying that $g(x)$ is non-diminishing at ξ, we mean that

$$g(\xi+0) - g(\xi-0) \geqslant 0$$

if $a < \xi < b$, and that $g(a+0) - g(a) \geqslant 0$ if $\xi = a$, and that $g(b) - g(b-0) \geqslant 0$ if $\xi = b$. Similarly for the term non-increasing at ξ.

The first part of the enunciation means that the conditions of Theorem I are satisfied, but that for no $\epsilon > 0$ is the relation

$$|\Sigma \mu_j a_j| \leqslant (1-\epsilon) \max |\Sigma \mu_j f_j| \tag{17}$$

satisfied for all μ_j. By Theorem I, G exists. If $g \subset G$, then the total variation of g cannot be less than 1. For if it was $1-\epsilon$ with $\epsilon > 0$, then (17) would be satisfied for all μ_j. This proves (i). If now there is a $g \subset G$ for which (16) does not hold for some ξ, then the function $h(x)$, defined by $h(x) = g(x)$ $(x \neq \xi)$, $h(\xi) = g(\xi-0)$, is a member of G and its total variation is less than 1. This contradicts (i), and proves (ii).

The condition respecting $F(\mu)$ implies that the f_j are linearly independent in (a, b). Hence there is a set of μ_j, different from the null set, say μ_j^0, such that

$$|\Sigma \mu_j^0 a_j| = \max |\Sigma \mu_j^0 f_j|. \tag{18}$$

By hypothesis, $F(\mu^0)$ attains its greatest value m in $a \leqslant x \leqslant b$ at no more than a finite set of points, say the points $\xi_1 < \xi_2 < \ldots < \xi_\nu$. Let $g \subset G$, and let $V(\alpha, \beta)$ denote the total variation of $g(x)$ in (α, β). If $\delta \equiv (\alpha, \beta)$ be an interval completely interior to (ξ_i, ξ_{i+1}), where $i = 1, \ldots, \nu-1$, then there is a $\theta = \theta(\delta)$, satisfying $0 \leqslant \theta < 1$, such that

$$\max_{\alpha \leqslant x \leqslant \beta} F(\mu^0) \leqslant \theta m,$$

and $m > 0$. By (18),

$$m = |\Sigma \mu_j^0 a_j| = \left| \int_a^b \{\Sigma \mu_j^0 f_j\} dg \right|$$

$$= \left| \int_a^\alpha + \int_\alpha^\beta + \int_\beta^b \right|$$

$$\leqslant mV(a, \alpha) + \theta m V(\alpha, \beta) + m V(\beta, b).$$

But $$V(a, \alpha) + V(\alpha, \beta) + V(\beta, b) = 1.$$

Hence $V(\alpha, \beta) = 0$, so that g is constant in each of the open intervals (ξ_i, ξ_{i+1}). If $\xi_1 > a$, we can take for (α, β) an interval of the form $(a, \xi_1 - \epsilon)$, and it follows that $g(x)$ is constant in $a \leqslant x < \xi_1$. Similarly if $\xi_\nu < b$.

We take $S \equiv (\xi_1, \ldots, \xi_\nu)$. Consider a point ξ_i of S. Either (α) every function $g \subset G$ satisfies $g(\xi_i + 0) - g(\xi_i - 0) \geqslant 0$, in which case we put ξ_i in a set S_1; or (β) every $g \subset G$ satisfies $g(\xi_i + 0) - g(\xi_i - 0) \leqslant 0$, in which case we put ξ_i in a set S_2 unless $g(\xi_i + 0) - g(\xi_i - 0) = 0$ for every $g \subset G$, and then we put ξ_i in S_1; or (γ) there are two functions g and h of G which satisfy

$$g(\xi_j + 0) - g(\xi_i - 0) > 0, \quad h(\xi_i + 0) - h(\xi_i - 0) < 0. \tag{19}$$

We must show that case (γ) cannot occur. Write $d(\xi) = g(\xi + 0) - g(\xi - 0)$, and let $\delta(\xi)$ have a similar meaning relative to h. Then, since

$$\sum_S |d(\xi)| = 1, \quad \sum_S |\delta(\xi)| = 1, \tag{20}$$

the function $f = \frac{1}{2}(g + h)$, which belongs to G, has a total variation

$$\tfrac{1}{2} \sum_S |d(\xi) + \delta(\xi)| < 1$$

by (19) and (20). This contradicts (i) and proves the theorem.

Let f_j be n linearly independent functions continuous in the closed interval (a, b). If one of them, f_1 say, is a constant, which in virtue of the linear independence is different from zero, then the condition of the preceding theorem regarding $F(\mu)$ is not satisfied, since $|f_1(x)|$ attains its maximum at more than a finite set of points. In this case we have the following result, which we enunciate as a lemma.

LEMMA 3. *Given n linearly independent functions $f_1 = c$, $f_2, \ldots, f_n$, continuous in (a, b), and n constants a_j, let 1 be the lower bound of numbers M such that* (15) *holds for all μ_j. Suppose that, for all sets μ_j except sets of the*

form $(\mu_1, 0, \ldots, 0)$, *the function* $F(\mu)$ *attains its greatest value at no more than a finite set of points. Then, if* $|a_1| < |c|$, *the conclusion of Theorem VII holds.*

The relation (18) is satisfied for a set μ_j^0 of the μ_j not all zero, and this set cannot be of the form $(\mu_1^0, 0, \ldots, 0)$, since $|a_1| < |c|$. Hence $F(\mu^0)$ attains its greatest value at no more than a finite set of points and the preceding argument is applicable.

7. We shall now consider two cases in which, with a natural convention, there is only one solution of the equations (2) whose total variation is least. Given $2n+1$ numbers $a_0, a_1, b_1, \ldots, a_n, b_n$, we denote by F the class of functions $g(x)$ whose total variation in $(0, 2\pi)$ does not exceed π, and which satisfy

$$g(+0) = 0, \qquad a_j = \frac{1}{\pi}\int_0^{2\pi} \cos jx\, dg, \qquad b_j = \frac{1}{\pi}\int_0^{2\pi} \sin jx\, dg$$
$$(j = 0, 1, \ldots, n).$$

THEOREM VIII. *Let* $a_0, a_1, b_1, \ldots, a_n, b_n$ *be a set of numbers with* $|a_0| < 1$. *Let* 1 *be the lower bound of numbers* M *such that*

$$\left|\lambda_0 a_0 + \sum_1^n (\lambda_j a_j + \mu_j b_j)\right| \leqslant M \max \left|\lambda_0 + \sum_1^n (\lambda_j \cos jx + \mu_j \sin jx)\right|$$

for every set of λ'*s and* μ'*s. Then*

(i) F *exists and every* $g \subset F$ *has its total variation equal to* π *and is a step function with not more than* $2n$ *non-equivalent points of discontinuity,* 0 *and* 2π *being equivalent points*

(ii) *the values of* $g(x)$ *at its points of continuity are known; and if* $h \subset F$, *then* h *can only differ from* g *at the points of discontinuity of* g, *and all the possible values of* h *at these points are determinate.*

Allowing for the difference of notation, the conditions of Lemma 3 are satisfied, and so the conclusion of Theorem VII holds. This proves (i) except for the number of discontinuities. The set of discontinuities of g is contained in the set S of Theorem VII. We shall show that when $S \equiv (x_1, \ldots, x_\nu)$ does not contain $x = 0, 2\pi$, then $\nu \leqslant 2n$ and the numbers $\delta_i = g(x_i+0) - g(x_i-0)$ are uniquely determined. In this case, since $g(+0) = 0$, every member of F is known except at the points x_i, and they are identical except at these points; and by (16) the value at x_i may be taken arbitrarily in the closed interval whose end points are $g(x_i+0)$,

$g(x_i-0)$. When S contains $x=0,\ 2\pi$, say $x_1=0,\ x_\nu=2\pi$, we shall show that $\nu\leqslant 2n+1$ and also that the numbers $\delta_1+\delta_\nu,\ \delta_2,\ \ldots,\ \delta_{\nu-1}$ are uniquely determined, where $\delta_1=g(+0)-g(0),\ \delta_\nu=g(2\pi)-g(2\pi-0)$. In this case, every member of F has the same known values except at the points x_i; and the possible values at $x_2,\ \ldots,\ x_{\nu-1}$ are subjected only to the condition (16). Further, since $\Sigma|\delta_i|=\pi$, we have two equations of the form

$$|\delta_1|+|\delta_\nu|=\alpha,\quad \delta_1+\delta_\nu=\beta,$$

where α and β are known, and it is easy to see that all possible solutions of this system are determinate; and $g(0),\ g(2\pi)$ may have any values consistent with $g(+0)-g(0)=\delta_1,\ g(2\pi)-g(2\pi-0)=\delta_\nu$, because the integrals

$$\int_0^{2\pi}\cos jx\,dg,\qquad \int_0^{2\pi}\sin jx\,dg$$

are unaltered for varying δ_1 and δ_ν provided that $\delta_1+\delta_\nu$ remains constant.

The set S is the set in $0\leqslant x\leqslant 2\pi$ at which

$$|H(x)|=\left|\lambda_0^0+\sum_1^n(\lambda_j^0\cos jx+\mu_j^0\sin jx)\right|$$

attains its greatest value, where $\lambda_0^0,\ \lambda_j^0,\ \mu_j^0$ satisfy

$$|\lambda_0^0 a_0+\Sigma(\lambda_j^0 a_j+\mu_j^0 b_j)|=\max|\lambda_0^0+\Sigma(\lambda_j^0\cos jx+\mu_j^0\sin jx)|$$

and $\lambda_j^0,\ \mu_j^0$ are not all zero. Let $m>0$ be the greatest value of $|H(x)|$. Then $H(x)$ as a trigonometric polynomial of order n cannot take a particular value at more than $2n$ points in $0\leqslant x<2\pi$, multiple roots being counted according to their multiplicity. The value m being a maximum value of $H(x)$, it follows that every point ξ at which $H(\xi)=m$ is a multiple root. Hence $H(x)$ cannot take the value m at more than n distinct points in $0\leqslant x<2\pi$. Similarly for $-m$. Hence $|H(x)|$ cannot take the value m at more than $2n$ distinct points in $0\leqslant x<2\pi$. This proves (i). Let $x_1<x_2<\ldots<x_p,\ p\leqslant 2n$, be the distinct points in question. Suppose in the first place that $x_1>0$.

To every $g\subset F$ there correspond p numbers $d_1,\ \ldots,\ d_p$, where

$$d_i=g(x_i+0)-g(x_i-0),$$

such that

$$\left.\begin{aligned}\pi a_j&=\sum_{(i)}d_i\cos jx_i\\ \pi b_j&=\sum_{(i)}d_i\sin jx_i\end{aligned}\right\}\quad (j=0,\ 1,\ \ldots,\ n).\qquad (21)$$

All we need show is that (21) cannot have more than one solution in the d's.

This will follow if we show that the rank of the matrix

$$\begin{matrix} 1 & 1 & \dots & 1 \\ \cos x_1 & \cos x_2 & \dots & \cos x_p \\ \sin x_1 & \sin x_2 & \dots & \sin x_p \\ \dots & \dots & \dots & \dots \\ \cos nx_1 & \cos nx_2 & \dots & \cos nx_p \\ \sin nx_1 & \sin nx_2 & \dots & \sin nx_p \end{matrix}$$

is equal to p. If $x_{p+1}, \dots, x_{2n+1}$ denote $2n+1-p$ distinct points in $0 < x < 2\pi$ which are distinct from $x_1, \dots, x_p$, then the hypothesis that the rank of the matrix is less than p implies that the determinant obtained by adding to the matrix the $2n+1-p$ columns corresponding to the points $x_{p+1}, \dots, x_{2n+1}$ vanishes. This determinant, however, as is well known, differs from zero for any $2n+1$ non-equivalent points x_i.

It remains to consider the case $x_1 = 0$. Then $x = 2\pi$ is a point at which $|H(x)|$ attains its greatest value. Write $x_{p+1} = 2\pi$. To every $g \subset F$ there correspond $p+1$ numbers $\delta_1, \dots, \delta_{p+1}$ which satisfy

$$\pi a_j = \Sigma \delta_i \cos j x_i, \quad \pi b_i = \Sigma \delta_i \sin j x_i \quad (j = 0, \dots, n).$$

Writing $d_1 = \delta_1 + \delta_{p+1}$, $d_2 = \delta_2, \dots, d_p = \delta_p$, we obtain the equations (21) which have only one solution in the d's. This completes the proof.

Let $p_0(x) = 1, p_1(x), \dots, p_n(x)$ form a set of linearly independent polynomials of degree not exceeding n. Given $n+1$ numbers $a_0, a_1, \dots, a_n$, let P be the class of functions $g(x)$ whose total variation in (a, b) does not exceed 1 and which satisfies

$$g(a) = 0, \quad a_j = \int_a^b p_j \, dg \quad (j = 0, \dots, n).$$

THEOREM IX. *Let $a_0, a_1, \dots, a_n$ be a set of numbers with $|a_0| < 1$. Let 1 be the lower bound of numbers M such that*

$$\left| \sum_{j=0}^{n} \lambda_j a_j \right| \leqslant M \max |\Sigma \lambda_j p_j| \quad (a \leqslant x \leqslant b)$$

for all λ_j. Then

(i) *P exists, and every $g \subset P$ has its total variation equal to 1 and is a step function with not more than $n+1$ points of discontinuity in $a \leqslant x \leqslant b$;*

(ii) *the values of g at its points of discontinuity are known; and if $h \subset P$, then h can only differ from g at the points of discontinuity of g, and all possible values of h at these points are determinate.*

The proof is similar to that of the preceding theorem, and somewhat simpler. It should be observed that, if $S \equiv (x_1, \ldots, x_q)$, then all the numbers $\delta_i = g(x_i+0)-g(x_i-0)$, with the usual convention for $x_i = a, b$, are uniquely determined.

8. THEOREM X. *Given the even function $a(x)$, and the odd function $b(x)$, both continuous in $(-\infty, \infty)$, a necessary and sufficient condition for the existence of a function $g(x)$ whose total variation in $(-\infty, \infty)$ does not exceed π and which satisfies*

$$a(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos xt\, dg(t), \quad b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin xt\, dg(t) \quad (-\infty < x < \infty)$$

is that, for any finite set of numbers λ_j, μ_j $(j = 1, \ldots, n)$ and every finite set of points x_j, we have

$$\left| \sum_{j=1}^{n} \{\lambda_j a(x_j)+\mu_j b(x_j)\} \right| \leqslant \max \left| \sum_{j=1}^{n} (\lambda_j \cos x_j t+\mu_j \sin x_j t) \right| \quad (-\infty < t < \infty). \quad (22)$$

The condition is clearly necessary. To prove that it is sufficient, we observe, in the first place, that the condition implies that

$$|a(x)| \leqslant 1, \quad |b(x)| \leqslant 1 \quad (-\infty < x < \infty). \quad (23)$$

Next, the condition is the same as the following: for every set of complex numbers $\mu_{-n}, \mu_{-(n-1)}, \ldots, \mu_0, \mu_1, \ldots, \mu_n$ which satisfy $\mu_{-n} = \bar{\mu}_n$, and every set of points $x_{-n}, x_{-(n-1)}, \ldots, x_0, x_1, \ldots, x_n$ which satisfy $x_{-n} = -x_n$ (so that μ_0 is real and $x_0 = 0$), we have

$$\left| \sum_{-n}^{n} \mu_j c(x_j) \right| \leqslant \tfrac{1}{2} \max \left| \sum_{-n}^{n} \mu_j e^{-ix_j t} \right| \quad (-\infty < t < \infty), \quad (24)$$

where

$$c(x) = \tfrac{1}{2}\{a(x)-ib(x)\}. \quad (25)$$

We require three lemmas.

LEMMA 4. *Let $g_n(t)$ be a sequence of functions such that*

$$\int_{-\infty}^{\infty} |dg_n| \leqslant 1. \quad (26)$$

If the sequence

$$c_n(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg_n \quad (27)$$

converges for all x to a continuous function $c(x)$, then there exists a function $g(x)$ which satisfies

$$\int_{-\infty}^{\infty} |dg| \leqslant 1, \quad c(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg \quad (-\infty < x < \infty). \tag{28}$$

We may suppose that the functions $g_n(x)$ satisfy

$$|g_n(x)| \leqslant 1 \quad (-\infty < x < \infty). \tag{29}$$

In virtue of (26) and (29), we can find a subsequence n_k (as in the proof of Theorem I) which gives rise to a function $g(x)$ which satisfies the first relation in (28) and which, for all rational x, is equal to the limit of $g_{n_k}(x)$. Let ξ be an arbitrarily assigned number different from zero. Then

$$\int_0^{\xi} c_n(x)\, dx = \int_0^{\xi} dx \int_{-\infty}^{\infty} e^{-ixt}\, dg_n. \tag{30}$$

Let $A = A(n, \epsilon, \xi)$ and A_0 be rational numbers such that

$$A|\xi| > 1, \quad A_0|\xi| > 1, \quad A_0\epsilon > 1,$$

$$\int_{-\infty}^{-A} + \int_{A}^{\infty} |dg_n| < \frac{\epsilon}{|\xi|}, \quad \int_{-\infty}^{A_0} + \int_{A_0}^{\infty} |dg| < \frac{\epsilon}{|\xi|},$$

where $\epsilon > 0$ is arbitrarily assigned. By (30),

$$\left| \int_0^{\xi} c_n(x)\, dx - \int_0^{\xi} dx \int_{-A}^{A} e^{-ixt}\, dg_n \right| < \epsilon.$$

But

$$\int_0^{\xi} dx \int_{-A}^{A} e^{-ixt}\, dg_n = \int_{-A}^{A} \frac{e^{-i\xi t} - 1}{-it}\, dg_n$$

and

$$\left| \int_{-\infty}^{-A} + \int_{A}^{\infty} \frac{e^{-i\xi t} - 1}{-it}\, dg_n \right| < \frac{2}{A} \frac{\epsilon}{|\xi|} < 2\epsilon.$$

Hence, since ϵ is arbitrary, we have

$$\int_0^{\xi} c_n(x)\, dx = \int_{-\infty}^{\infty} \frac{e^{-i\xi t} - 1}{-it}\, dg_n. \tag{31}$$

By (27), $|c_n(x)| \leqslant 1$ for all x and n. Hence, as $n \to \infty$, the first member in (31) tends to

$$\int_0^{\xi} c(x)\, dx.$$

Let n have the values of the sequence n_k. Then, as in the proof of Theorem I,

$$\int_{-A_0}^{A_0} \frac{e^{-i\xi t} - 1}{-it}\, dg_{n_k} \to \int_{-A_0}^{A_0} \frac{e^{-i\xi t} - 1}{-it}\, dg.$$

Further, for all n,

$$\left|\int_{-\infty}^{-A_0}+\int_{A_0}^{\infty}\frac{e^{-i\xi t}-1}{-it}\,dg_n\right|<\frac{2}{A_0}\int_{-\infty}^{\infty}|dg_n|<2\epsilon.$$

Hence
$$\left|\int_0^{\xi}c(x)\,dx-\int_{-A_0}^{A_0}\frac{e^{-i\xi t}-1}{-it}\,dg\right|\leqslant 2\epsilon.$$

But
$$\left|\int_{-\infty}^{-A_0}+\int_{A_0}^{\infty}\frac{e^{-i\xi t}-1}{-it}\,dg\right|<\frac{2\epsilon}{A_0|\xi|}<2\epsilon.$$

Hence, since ϵ is arbitrary,

$$\int_0^{\xi}c(x)\,dx=\int_{-\infty}^{\infty}\frac{e^{-i\xi t}-1}{-it}\,dg. \tag{32}$$

Let
$$\gamma(x)=\int_{-\infty}^{\infty}e^{-ixt}\,dg. \tag{33}$$

Then, by the method used for proving (31), we have

$$\int_0^{\xi}\gamma(x)\,dx=\int_{-\infty}^{\infty}\frac{e^{-i\xi t}-1}{-it}\,dg.$$

Thus
$$\int_0^{\xi}c(x)\,dx=\int_0^{\xi}\gamma(x)\,dx$$

for all ξ. Since $c(x)$ is continuous by hypothesis, and $\gamma(x)$ is continuous, as is easily seen, by (33), we have $c(x)=\gamma(x)$ for all x, and this by (33) is the desired result.

This lemma for monotone functions is proved by Bochner* by a method which seems unnecessarily complicated.

LEMMA 5. *If $\alpha(x)=\alpha(-x)$, $\beta(x)=-\beta(x)$ are continuous in $(-\infty,\ \infty)$ and absolutely integrable in that interval, and if $\gamma(x)=\frac{1}{2}\{\alpha(x)-i\beta(x)\}$ satisfies*

$$\left|\sum_{-n}^{n}\mu_j\gamma(x_j)\right|\leqslant\tfrac{1}{2}\max\left|\sum_{-n}^{n}\mu_j e^{-ix_jt}\right| \quad (-\infty<t<\infty) \tag{34}$$

for all μ_j, x_j subject to the same restriction as in (24), then there is a function $\phi(x)$, whose total variation in $(-\infty,\ \infty)$ does not exceed π, such that

$$\alpha(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\cos xt\,d\phi, \qquad \beta(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\sin xt\,d\phi$$
$$(-\infty<x<\infty).$$

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* Bochner, *Vorlesungen über Fouriersche Integrale* (1932), 72.

Since $|\gamma(x)| \leqslant 1$ is continuous and absolutely integrable in $(-\infty, \infty)$, we have

$$\gamma(x) = \lim_{\eta \to 0} \gamma(\eta, x) \quad (\eta > 0, -\infty < x < \infty) \tag{35}$$

boundedly, where

$$\gamma(\eta, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\eta t^2} e^{-ixt} dt \int_{-\infty}^{\infty} e^{i\xi t} \gamma(\xi) d\xi.$$

We first prove that, for $\eta > 0$, we have

$$\left| \sum_{-n}^{n} \mu_j \gamma(\eta, x_j) \right| \leqslant \tfrac{1}{2} \max \left| \sum_{-n}^{n} \mu_j e^{-ix_j t} \right|. \tag{36}$$

We have

$$\begin{aligned} \gamma(\eta, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\xi) d\xi \int_{-\infty}^{\infty} e^{-\eta t^2} e^{i(\xi - x)t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\xi) \sqrt{\left(\frac{\pi}{\eta}\right)} e^{-(\xi - x)^2/(4\eta)} d\xi \\ &= \int_{-\infty}^{\infty} \gamma(x + \xi) P(\xi, \eta) d\xi, \end{aligned}$$

where

$$P(\xi, \eta) = \frac{1}{2\sqrt{(\pi\eta)}} e^{-\xi^2/(4\eta)}; \quad \int_{-\infty}^{\infty} P(\xi, \eta) d\xi = 1. \tag{37}$$

Hence

$$\sum_{-n}^{n} \mu_j \gamma(\eta, x_j) = \int_{-\infty}^{\infty} \{\Sigma \mu_j \gamma(x_j + \xi)\} P(\xi, \eta) d\xi.$$

The first member is real. Hence

$$\sum_{-n}^{n} \mu_j \gamma(\eta, x_j) = \int_{-\infty}^{\infty} \Re \{\Sigma \mu_j \gamma(x_j + \xi)\} P(\xi, \eta) d\xi. \tag{38}$$

Write $\mu_j = \mu_j' + i\mu_j''$. Remembering that $\mu_{-j} = \bar{\mu}_j$, $x_{-j} = -x_j$, $\gamma(-x) = \bar{\gamma}(x)$, we have

$$\begin{aligned} &\Re \sum_{-n}^{n} \mu_j \gamma(x_j + \xi) \\ &= \Re \Big[\tfrac{1}{2} \mu_0 \{a(\xi) - ib(\xi)\} + \tfrac{1}{2} \sum_{1}^{n} (\mu_j' + i\mu_j'') \{a(x_j + \xi) - ib(x_j + \xi)\} \\ &\qquad + \tfrac{1}{2} \sum_{1}^{n} (\mu_j' - i\mu_j'') \{a(\xi - x_j) - ib(\xi - x_j)\} \Big] \\ &= \tfrac{1}{2} \Big[\mu_0 a(\xi) + \sum_{1}^{n} \{\mu_j' a(x_j + \xi) + \mu_j'' b(x_j + \xi)\} + \sum_{1}^{n} \{\mu_j' a(\xi - x_j) - \mu_j'' b(\xi - x_j)\} \Big]. \end{aligned}$$

We now apply the property expressed by (22). We have

$$\left|\Re\sum_{-n}^{n}\mu_j\gamma(x_j+\xi)\right| \leqslant \tfrac{1}{2}\max_{(t)}\left|\mu_0\cos\xi t+\sum_{1}^{n}\{\mu_j'\cos(x_j+\xi)t+\mu_j''\sin(x_j+\xi)t\}\right.$$
$$\left.+\sum_{1}^{n}\{\mu_j'\cos(\xi-x_j)t-\mu_j''\sin(\xi-x_j)t\}\right|$$
$$\leqslant \tfrac{1}{2}\max_{(t)}\left|\cos\xi t\left\{\mu_0+2\sum_{1}^{n}(\mu_j'\cos x_j t+\mu_j''\sin x_j t)\right\}\right|$$
$$\leqslant \tfrac{1}{2}\max_{(t)}\left|\sum_{-n}^{n}\mu_j e^{-ix_j t}\right|.$$

Applying this to (38), and using (37), we obtain (36).

For fixed η, the real function

$$F(\eta, t)=\frac{1}{2\pi}e^{-\eta t^2}\int_{-\infty}^{\infty}e^{i\xi t}\gamma(\xi)\,d\xi$$

is absolutely integrable in $(-\infty, \infty)$. Given $\epsilon>0$, there is a positive integer $N=N(\eta, \epsilon)$ such that

$$\int_{-\infty}^{-2\pi N}+\int_{2\pi N}^{\infty}|F(\eta, t)|\,dt<\tfrac{1}{2}\epsilon. \tag{39}$$

Consider the function

$$\lambda(\eta, t)=\operatorname{sgn}F(\eta, t)\quad(-2\pi N\leqslant t\leqslant 2\pi N).$$

We can find a real trigonometric polynomial

$$P(t)=\sum_{-\nu}^{\nu}\mu_j e^{-ijt/(2N)}\quad(\mu_{-j}=\bar{\mu}_j)$$

which satisfies
$$|P(t)|\leqslant 1\quad(-\infty<t<\infty) \tag{40}$$
$$|P(t)-\lambda(\eta, t)|<\epsilon/(8\pi N),\quad t\subset(-2\pi N, 2\pi N)-\Delta,$$

where Δ is a set of intervals whose measure may be made arbitrarily small, and in particular may be taken to satisfy

$$\int_{\Delta}|F(\eta, t)|\,dt<\tfrac{1}{4}\epsilon. \tag{41}$$

We have
$$\gamma(\eta, x)=\int_{-\infty}^{\infty}e^{-ixt}F(\eta, t)\,dt,$$

$$\sum_{-\nu}^{\nu}\mu_j\gamma(\eta, \tfrac{1}{2}j/N)=\int_{-\infty}^{\infty}P(t)F(\eta, t)\,dt. \tag{42}$$

By (34), $$\left| \sum_{-\nu}^{\nu} \mu_j \gamma(\eta, \tfrac{1}{2}j/N) \right| \leqslant \tfrac{1}{2} \max |P(t)| \leqslant \tfrac{1}{2},$$

so that, by (42), $$\left| \int_{-\infty}^{\infty} P(t)\, F(\eta, t)\, dt \right| \leqslant \tfrac{1}{2}.$$

On the other hand,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} P(t)\, F(\eta, t)\, dt \right| &= \left| \int_{-2\pi N}^{2\pi N} + \int_{-\infty}^{-2\pi N} + \int_{2\pi N}^{\infty} \right| \\ &> \left| \int_{-2\pi N}^{2\pi N} P(t)\, F(\eta, t)\, dt \right| - \tfrac{1}{2}\epsilon \quad [\text{by (39)}] \\ &> \int_{-2\pi N}^{2\pi N} |F(\eta, t)|\, dt - \left| \int_{-2\pi N}^{2\pi N} \{P(t) - \lambda(\eta, t)\}\, F(\eta, t)\, dt \right| - \tfrac{1}{2}\epsilon; \end{aligned}$$

and

$$\begin{aligned} \left| \int_{-2\pi N}^{2\pi N} \{P(t) - \lambda(\eta, t)\}\, F(\eta, t)\, dt \right| &\leqslant \left| \int_{(-2\pi N,\, 2\pi N) - \Delta} \right| + \left| \int_{\Delta} \right| \\ &\leqslant \frac{4\pi N \epsilon}{8\pi N} + 2 \int_{\Delta} |F(\eta, t)|\, dt \\ &< \epsilon \end{aligned}$$

by (41). Thus $$\int_{-2\pi N}^{2\pi N} |F(\eta, t)|\, dt \leqslant \tfrac{1}{2} + \tfrac{3}{2}\epsilon,$$

and, since ϵ is arbitrary, and $N \to \infty$ as $\epsilon \to 0$,

$$\int_{-\infty}^{\infty} |F(\eta, t)|\, dt \leqslant \tfrac{1}{2}. \tag{43}$$

Let η take the values in a diminishing sequence η_n which tends to zero. Write

$$g_n(t) = \int_{-\infty}^{t} F(\eta_n, \xi)\, d\xi.$$

Then $$\gamma(\eta_n, x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg_n \tag{44}$$

and $$\int_{-\infty}^{\infty} |dg_n| \leqslant \tfrac{1}{2}.$$

By (35), we can apply Lemma 4. This tells us that there is a $g(t)$ satisfying

$$\int_{-\infty}^{\infty} |dg| \leqslant \tfrac{1}{2}, \quad \gamma(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg \quad (-\infty < x < \infty), \tag{45}$$

which is the required result.

LEMMA 6. *If $c(x)$ denotes the function* (25), *then* $c(\eta, x) = c(x)\, e^{-\eta x^2}$ $(\eta > 0)$ *satisfies*

$$\left| \sum_{-n}^{n} \mu_j c(\eta, x_j) \right| \leqslant \tfrac{1}{2} \max \left| \sum_{-n}^{n} \mu_j e^{-ix_j t} \right|.$$

We have

$$c(\eta, x) = \int_{-\infty}^{\infty} c(x)\, e^{ixt} e^{-t^2/(4\eta)} \frac{dt}{\sqrt{(4\pi\eta)}},$$

$$\sum_{-n}^{n} \mu_j c(\eta, x_j) = \int_{-\infty}^{\infty} \sum_{-n}^{n} \mu_j c(x_j)\, e^{ix_j t} e^{-t^2/(4\eta)} \frac{dt}{\sqrt{(4\pi\eta)}}. \tag{46}$$

Writing $\mu_j' = \mu_j e^{ix_j t}$ for a fixed t, we have

$$\begin{aligned} \left| \sum_{-n}^{n} \mu_j\, c(x_j)\, e^{ix_j t} \right| &= |\Sigma \mu_j'\, c(x_j)| \\ &\leqslant \tfrac{1}{2} \max_{(\tau)} |\Sigma \mu_j'\, e^{-ix_j \tau}| \\ &\leqslant \tfrac{1}{2} \max_{(\tau)} |\Sigma \mu_j\, e^{-ix_j(\tau - t)}| \\ &\leqslant \tfrac{1}{2} \max_{(\tau)} |\Sigma \mu_j\, e^{-ix\tau}|. \end{aligned}$$

Applying this to (46) we have the desired result, since

$$\int_{-\infty}^{\infty} e^{-t^2/(4\eta)} \frac{dt}{\sqrt{(4\pi\eta)}} = 1.$$

We can now complete the proof of the main theorem. Since $c(\eta, x)$ is continuous and absolutely integrable, it follows, by Lemma 6, that Lemma 5 is applicable. If η is given the values of a diminishing sequence η_n which tends to zero, we have, by Lemma 5,

$$c(\eta_n, x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg_n, \quad \int_{-\infty}^{\infty} |dg_n| \leqslant \tfrac{1}{2}.$$

Further, $c(\eta_n, x) \to c(x)$. Hence, by Lemma 4, there is a $g(x)$ satisfying

$$\int_{-\infty}^{\infty} |dg| \leqslant \tfrac{1}{2}, \quad c(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg.$$

This proves the theorem.

9. A consequence of the preceding theorem is

THEOREM XI. *If $c(x)$ is a continuous complex function, defined in $(-\infty, \infty)$, which satisfies $c(-x) = \bar{c}(x)$, and if (24) is satisfied for all μ_j, x_j with $\mu_{-j} = \bar{\mu}_j$, $x_{-j} = -x_j$, then, for every set of complex numbers $\Lambda_j (j = 1, \ldots, n)$ and for every set of points ξ_j,*

$$\left| \sum_1^n \Lambda_j c(\xi_j) \right| \leqslant \tfrac{1}{2} \max \left| \sum_1^n \Lambda_j e^{-i\xi_j t} \right| \quad (-\infty < t < \infty). \quad (47)$$

The condition (47) is clearly necessary in order that the conclusion of Theorem X may hold. But (24) is sufficient for this. Hence (24) implies (47).

THEOREM XII. *If the continuous function $c(x)$, defined in $(-\infty, \infty)$, satisfies $c(-x) = \bar{c}(x)$, a necessary and sufficient condition for the result of Theorem X to hold is that (24) shall be satisfied for all rational x_j.*

The condition is obviously necessary. To prove that it is sufficient, we examine the preceding proof and show how it can be modified to give the desired result. In the first place, if (34) holds for all rational x_j, then (36) holds for rational x_j. For consider the relation (38). The function

$$\mathfrak{R} \sum_{-n}^{n} \mu_j \gamma(x_j + \xi) \quad (48)$$

being a continuous function of ξ, to obtain an upper bound for its modulus we need only consider rational ξ, and then the previous argument is applicable. In the second place, in proving (43), we only use (36) for rational x_j, namely for x_j of the form $j/(2N)$. It now follows that the conclusion of Lemma 5 holds if (34) is satisfied only for rational x_j. Clearly the result of Lemma 6 is true for rational x_j under the conditions of the present theorem. Hence the result. It now appears that the condition that (24) shall hold for all x_j (satisfying $x_{-j} = -x_j$) and the condition that it shall hold only for rational x_j are equivalent.

THEOREM XIII. *The result of the preceding theorem remains true if*

$$\left| \sum_{-n}^{n}{}' \mu_j c(x_j) \right| \leqslant \tfrac{1}{2} \max \left| \sum_{-n}^{n}{}' u_j e^{-ix_j t} \right| \quad (x_{-j} = -x_j,\ \mu_{-j} = \bar{\mu}_j) \quad (49)$$

for all μ and rational x, where the dash denotes that the term $j = 0$ is omitted.

The condition (49) means that (22) holds for rational $x_j \neq 0$. If (34) holds for rational x_j with dashed sums, then (36) holds for rational x_j

with undashed sums. For, by the continuity of (48), we need only evaluate its modulus for such rational ξ as satisfy $\xi \neq 0$, $x_j \pm \xi \neq 0$, and then the previous argument is applicable. Finally, Lemma 6 holds with dashed sums for rational x_j.

The following theorem concerns the case when the function $c(x)$ of Theorem X is not assumed to be continuous.

THEOREM XIV. *Given the even function $a(x)$ and the odd function $b(x)$, both defined in $(-\infty, \infty)$, a necessary and sufficient condition for the existence of a function $g(x)$ whose total variation in $(-\infty, \infty)$ does not exceed π and which satisfies*

$$a(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos xt\, dg, \quad b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin xt\, dg$$

for almost all x in $(-\infty, \infty)$, is that, for any finite set of numbers λ_j, μ_j $(j = 1, \ldots, n)$ and for almost all points x_j, the relation (22) is satisfied.

LEMMA 7. *Let $g_n(t)$ be a sequence of functions which satisfy (26). If the sequence (27) converges for almost all x to a function $c(x)$, then there exists a function $g(x)$ which satisfies*

$$\int_{-\infty}^{\infty} |dg| \leqslant 1,$$

and such that, for almost all x,

$$c(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg.$$

The same argument as in Lemma 4 gives

$$\int_0^{\xi} c(x)\, dx = \int_0^{\xi} \gamma(x)\, dx$$

for all ξ. Here $\gamma(x)$ is continuous, and the first member has the unique derivative $c(x)$ almost everywhere. Hence the result.

LEMMA 8. *If, in Lemma 5, the functions $\alpha(x)$, $\beta(x)$ are not assumed to be continuous, and if (34) holds with dashed sums for almost all x_j, then the conclusion of Lemma 5 holds for almost all x.*

We have $$\gamma(x) = \lim_{\eta \to 0} \gamma(\eta, x)$$

for almost all x. The relation (36) holds for all x_j. For (38) holds, and we need an upper bound of the modulus of (48) only for almost all ξ. For such a set of ξ, however, the argument used in Lemma 5 applies. It is now clear that (43) holds. The proof is completed as before, using Lemma 7 instead of Lemma 4.

LEMMA 9. *The result of Lemma* 6 *with dashed sums holds for almost all* x_j.

This is clear from the proof of Lemma 6. It now follows from Lemma 8 that, if η_n is a diminishing sequence which tends to zero, then there is a $g_n(t)$ such that

$$\int_{-\infty}^{\infty} |dg_n| \leqslant \tfrac{1}{2}, \qquad c(\eta_n, x) \sim \int_{-\infty}^{\infty} e^{-ixt}\, dg_n,$$

the sign of equivalence denoting equality for almost all x. If

$$c_n(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg_n,$$

then, since $c(\eta_n, x) \to c(x)$ for all x, we have $c_n(x) \to c(x)$ for almost all x. By Lemma 7, the theorem follows.

10. THEOREM XV. *Let* $C(x)$ *be a continuous complex function, defined in* $(-\infty, \infty)$, *which satisfies* $C(-x) = -\bar{C}(x)$. *A necessary and sufficient condition that there shall exist a function* $g(x)$ *which satisfies*

$$\int_{-\infty}^{\infty} |dg| \leqslant \tfrac{1}{2}, \qquad C(x) = \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it}\, dg \quad (-\infty < x < \infty)$$

is that, given any set of $\mu_{-n}, \mu_{-(n-1)}, \ldots, \mu_0, \ldots, \mu_n$, *with* $\mu_{-j} = \bar{\mu}_j$, *and any set of rational points* $x_{-n}, \ldots, x_0, \ldots, x_n$, *with* $x_{-j} = -x_j$, *we have*

$$\left| \sum_{-n}^{n} \mu_j C(x_j) \right| \leqslant \tfrac{1}{2} \max \left| \sum_{-n}^{n} \mu_j (e^{-ix_j t} - 1)/t \right| \qquad (50)$$

$$(-\infty < t < \infty).$$

By the argument of Lemma 5, we see that the conclusion of the theorem means that

$$C(x) = \int_0^x c(\xi)\, d\xi, \quad c(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg.$$

The condition is obviously necessary. To prove the sufficiency, we observe that, if we write $C(x) = \frac{1}{2}\{A(x) - i\,B(x)\}$, the condition (50) is the same as the condition that, for any set of real numbers λ_j, μ_j $(j = 1, \ldots, n)$ and any set of rational points x_j, we have

$$\left|\sum_1^n \{\lambda_j A(x_j) + \mu_j B(x_j)\}\right| \leqslant \max \left|\sum_1^n \left(\lambda_j \frac{\sin x_j t}{t} + \mu_j \frac{1-\cos x_j t}{t}\right)\right| \qquad (51)$$

$$(-\infty < t < \infty).$$

We apply the method of Theorem I. Let the rational points in $(-\infty, \infty)$ be enumerated as $x_1, x_2, \ldots, x_n, \ldots$. Let $E_n \equiv (x_1, \ldots, x_n)$. For a fixed n let $\mathfrak{E}_n$ denote the set of points $\xi \equiv (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ in a $2n$-dimensional space such that to each ξ there corresponds a g_ξ for which

$$\int_{-\infty}^{\infty} |dg_\xi| \leqslant \pi, \qquad \alpha_j = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin x_j t}{t}\, dg_\xi, \qquad \beta_j = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos x_j t}{t}\, dg_\xi$$

$$(j = 1, \ldots, n).$$

If ξ and ξ' belong to $\mathfrak{E}_n$, then so does

$$\{\lambda\alpha_1 + (1-\lambda)\alpha_1', \ \ldots, \ \lambda\beta_1 + (1-\lambda)\beta_1', \ \ldots\}$$

for all $0 < \lambda < 1$. The set $\mathfrak{E}_n$ is closed. For let ξ^ν be a sequence of points of $\mathfrak{E}_n$ which tends to ξ, and let g_ν be the corresponding sequence of functions. We have

$$\int_{-\infty}^{\infty} |dg_\nu| \leqslant \pi, \qquad \alpha_j^\nu = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin x_j t}{t}\, dg_\nu, \qquad \beta_j^\nu = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos x_j t}{t}\, dg_\nu$$

$$(j = 1, \ldots, n).$$

We select a subsequence g_{ν_k} giving rise to a function g such that

$$\int_{-\infty}^{\infty} |dg| \leqslant \pi, \qquad (52)$$

and such that $g_{\nu_k}(x) \to g(x)$ for all rational x. Given $\epsilon > 0$, let A be a rational number such that $2\pi/A < \epsilon$. Then, as in the proof of Theorem I,

$$\int_{-A}^{A} \frac{\sin x_j t}{t}\, dg_{\nu_k} \to \int_{-A}^{A} \frac{\sin x_j t}{t}\, dg.$$

Further,
$$\left|\int_{-\infty}^{-A} + \int_{A}^{\infty} \frac{\sin x_j t}{t}\, dg_{\nu_k}\right| < \frac{1}{A}\int_{-\infty}^{\infty} |dg_{\nu_k}| \leqslant \frac{\pi}{A},$$

and similarly when g_{ν_k} is replaced by g. Hence

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin x_j t}{t}\,dg=\lim_{k\to\infty}a_j^{\nu_k}=a_j \quad (j=1,\ \ldots,\ n),$$

and similarly for β_j. This proves that $\mathfrak{E}_n$ is closed.

The condition (51) means that for the fixed E_n, given any set of λ_j, μ_j, there is a θ satisfying $\theta^2\leqslant 1$, and a $t=\tau$, such that

$$\sum_1^n\{\lambda_j A(x_j)+\mu_j B(x_j)\}=\theta\sum_1^n\left(\lambda_j\frac{\sin x_j\tau}{\tau}+\mu_j\frac{(1-\cos x_j\tau)}{\tau}\right)$$

$$=\sum_1^n(\lambda_j\gamma_j+\mu_j\delta_j),$$

where $$\gamma_j=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin x_j t}{t}\,dg, \qquad \delta_j=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos x_j t}{t}\,dg,$$

and $$g(x)=0 \quad (-\infty<x<\tau)$$

$$=\pi\theta \quad (\tau\leqslant x<\infty).$$

Thus the total variation of g does not exceed π and $(\gamma_1,\ \ldots,\ \gamma_n,\ \delta_1,\ \ldots,\ \delta_n)$ belongs to $\mathfrak{E}_n$, *i.e.* every plane

$$\sum_1^n[\lambda_j\{a_j-A(x_j)\}+\mu_j\{\beta_j-B(x_j)\}]=0$$

through $A\equiv\{A(x_1),\ \ldots,\ A(x_n),\ B(x_1),\ \ldots,\ B(x_n)\}$ has a point of $\mathfrak{E}_n$ on it. Hence, as in Theorem I, A belongs to $\mathfrak{E}_n$. Hence there is a g_n which satisfies

$$\int_{-\infty}^{\infty}|dg_n|\leqslant\pi, \qquad C(x_j)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ix_j t}-1}{-it}\,dg_n \quad (j=1,\ \ldots,\ n).$$

This is true for all n. The sequence g_n gives rise to a g satisfying (52) such that, for a subsequence n_k, $g_{n_k}(x)\to g(x)$ for all rational x. We find, by the argument just used, that

$$\int_{-\infty}^{\infty}\frac{e^{-ix_j t}-1}{-it}\,dg_{n_k}\to\int_{-\infty}^{\infty}\frac{e^{-ix_j t}-1}{-it}\,dg \quad (j=1,\ 2,\ \ldots).$$

Thus $$C(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ixt}-1}{-it}\,dg \tag{53}$$

for all rational x. Since $C(x)$ is continuous, and, as is easily seen, the second member is continuous, (53) holds for all x.

THEOREM XVI. *Let $C(x)$ be a continuous complex function, defined in $(-A, A)$, which satisfies $C(-x) = \bar{C}(x)$. A necessary and sufficient condition for the existence of a function $g(x)$ whose total variation in $(-\infty, \infty)$ does not exceed π, and such that (53) holds for $-A < x < A$, is that (50) holds for all rational x_j in $(-A, A)$.*

The proof is the same as for the preceding theorem; we merely enumerate the rational x_j in $(-A, A)$.

10. We now consider monotone functions. The following theorem is the analogue of Theorem I.

THEOREM XVII. *Given n continuous functions f_j $(j = 1, \ldots, n)$, defined in the closed interval (a, b), and n constants a_j such that, for every set of μ_j,*

$$\sum_1^n \mu_j a_j \leqslant \max \sum_1^n \mu_j f_j, \tag{54}$$

there is a non-diminishing function $g(x)$ whose variation in (a, b) does not exceed 1 and which satisfies the relations

$$a_j = \int_a^b f_j \, dg \quad (j = 1, \ldots, n). \tag{55}$$

This theorem can be deduced from Theorem I as follows. Let $f_0 = 1$, $a_0 = 1$. Then (54) implies that

$$\sum_0^n \mu_j a_j \leqslant \max \sum_0^n \mu_j f_j$$

for all μ_j. If the first member is non-negative, this gives

$$\left| \sum_0^n \mu_j a_j \right| \leqslant \max \left| \sum_0^n \mu_j f_j \right|. \tag{56}$$

If the first member is negative, then, by the hypothesis of the theorem,

$$\sum_0^n (-\mu_j) a_j \leqslant \max \sum_0^n (-\mu_j) f_j,$$

and (56) again holds. We can therefore apply Theorem I and infer the existence of a function $g(x)$ whose total variation in (a, b) does not exceed 1, which satisfies (55), and also which is such that

$$1 = a_0 = \int_a^b f_0 \, dg = \int_a^b dg.$$

Hence g is non-diminishing and its variation is equal to 1.

This theorem can also be proved independently by the same method as Theorem I. In this case, the set E is the set of points $\xi \equiv (\xi_1, \ldots, \xi_n)$ such that to each ξ there corresponds a non-diminishing function g_ξ such that

$$g_\xi(b) - g_\xi(a) \leqslant 1, \quad \xi_j = \int_a^b f_j \, dg_\xi \quad (j = 1, \ldots, n).$$

We observe that (54) means that to each set of μ_j there corresponds a θ satisfying $0 \leqslant \theta \leqslant 1$ such that

$$\sum_1^n \mu_j a_j = \theta \sum_1^n \mu_j f_j(\zeta)$$

for a suitable ζ. Hence we have

$$\sum_1^n \mu_j a_j = \sum_1^n \mu_j c_j,$$

where

$$c_j = \int_a^b f_j \, dg$$

and g is non-diminishing with $g(b) - g(a) = \theta \leqslant 1$. This is sufficient to show that the method of Theorem I is applicable.

We now prove the analogue of Theorem VII. Given n continuous functions f_j, defined in the closed interval (a, b), and n constants a_j, we denote by G the class of non-diminishing functions $g(x)$ whose variation in (a, b) does not exceed 1 and which satisfy (55).

THEOREM XVIII. *Let* 1 *be the lower bound of positive numbers* M *such that*

$$\sum_1^n \mu_j a_j \leqslant M \max \sum_1^n \mu_j f_j$$

for all μ_j. *Suppose that, for every set of* μ_j *other than the null set, the function* $F(\mu) = F(\mu, x) = \Sigma \mu_j f_j$ *does not attain its maximum at more than a finite set of points in* $a \leqslant x \leqslant b$. *Then*

(i) *G exists and the variation of every member of it is precisely equal to* 1;

(ii) *there is a finite set* Σ *such that every* $g \subset G$ *is constant in every interval which contains no point of* Σ.

The first part of the enunciation means that the condition of Theorem XVII is satisfied, but that for no ϵ satisfying $1 > \epsilon > 0$ is the

relation $$\Sigma \mu_j a_j \leqslant (1-\epsilon) \max \Sigma \mu_j f_j \tag{57}$$

satisfied for all μ_j.

The class G exists by Theorem XVII. If $g \subset G$, then the variation of g cannot be equal to $1-\epsilon$ with $1 > \epsilon > 0$, for then we should have

$$\Sigma \mu_j a_j = \int_a^b \{\Sigma \mu_j f_j\} dg \leqslant (1-\epsilon) \max \Sigma \mu_j f_j$$

for all μ_j. On grounds of continuity and the linear independence of the f_j, there exists a set of μ_j, say μ_j^0, different from the null set, such that

$$\Sigma \mu_j^0 a_j = \max \Sigma \mu_i^0 f_j(x). \tag{58}$$

Let $m = \max F(\mu^0)$, and let it be attained at the finite set $\Sigma \equiv (x_1, \ldots, x_\nu)$, where $x_1 < x_2 < \ldots < x_\nu$. Let $g \subset G$. Suppose in the first place that $m > 0$. Then, if (α, β) is completely interior to (x_i, x_{i+1}), $i = 1, \ldots, \nu-1$, there is a $\theta = \theta(\alpha, \beta)$ such that

$$\max_{(\alpha, \beta)} F(\mu^0) \leqslant \theta m \quad (0 \leqslant \theta < 1).$$

Then

$$m = \Sigma \mu_j^0 a_j = \int_a^b F(\mu^0) dg$$

$$\leqslant m\{g(\alpha)-g(a)\}+\theta m\{g(\beta)-g(\alpha)\}+m\{g(b)-g(\beta)\}. \tag{59}$$

Hence $g(\beta)-g(\alpha) = 0$. Thus $g(x)$ is constant in each open interval (x_i, x_{i+1}). If $x_1 > a$, a similar argument shows that $g(x)$ is constant in $a \leqslant x < x_1$, the interval (α, β) being of the form $(a, x_1-\epsilon)$. Similarly if $x_\nu < b$.

Suppose next that $m = 0$. Then we can write

$$\max_{(\alpha, \beta)} F(\mu^0) = -\theta \quad [\theta = \theta(\alpha, \beta) > 0].$$

We obtain $$0 = \Sigma \mu_j^0 a_j \leqslant -\theta\{g(\beta)-g(\alpha)\};$$

so that the same conclusion holds. Finally, if $m < 0$, we can write

$$\max_{(\alpha, \beta)} F(\mu^0) \leqslant \theta m \quad [\theta = \theta(\alpha, \beta) > 1],$$

and (59) holds. From this we infer that

$$1 \geqslant g(\alpha)-g(a)+\theta\{g(\beta)-g(\alpha)\}+g(b)-g(\beta),$$

and this again requires $g(\beta)-g(\alpha) = 0$.

11. We consider two cases in which the non-diminishing function $g(x)$, which satisfies (55) and whose variation is least, is unique. Given $2n$ numbers $a_1, b_1, \ldots, a_n, b_n$, we denote by F the class of functions g which are non-diminishing in $(0, 2\pi)$, whose variation in that interval does not exceed π, and which satisfy

$$g(+0) = 0, \quad a_j = \frac{1}{\pi}\int_0^{2\pi} \cos jx\, dg, \quad b_j = \frac{1}{\pi}\int_0^{2\pi} \sin jx\, dg \qquad (j = 1, \ldots, n).$$

THEOREM XIX. *Let* 1 *be the lower bound of positive numbers* M *such that*

$$\sum_1^n \lambda_j a_j + \mu_j b_j \leqslant M \max \sum_1^n (\lambda_j \cos jx + \mu_j \sin jx)$$

for all λ_j, μ_j. *Then*

(i) F *exists, and every* $g \subset F$ *has its variation equal to* π *and is a step function with not more than* n *non-equivalent points of discontinuity,* 0 *and* 2π *being equivalent points*;

(ii) *the values of* g *at its points of continuity are known*; *and, if* $h \subset F$, *then* h *can only differ from* g *at the points of discontinuity of* g, *and all the possible values of* h *at these points are determinate.*

Allowing for the difference of notation, the conditions of Theorem XVIII are satisfied. This proves (i) except for the number of discontinuities. If $g \subset F$, then the discontinuities of g are contained in the set Σ of Theorem XVIII. Let $x_1 < x_2 < \ldots < x_p$ be those points of Σ which lie in $0 \leqslant x < 2\pi$. If $x_1 > 0$, then $\Sigma \equiv (x_1, \ldots, x_p)$, and all that we need show is that $p \leqslant n$ and that the numbers $\delta_i = g(x_i + 0) - g(x_i - 0)$ are unique. If $x_1 = 0$, then, writing $x_{p+1} = 2\pi$, we have $\Sigma \equiv (x_1, \ldots, x_{p+1})$, and all that we need show is that $p \leqslant n$ and that the numbers $\delta_1 + \delta_{p+1}, \delta_2, \ldots, \delta_p$ are unique.

The set Σ is the set in $0 \leqslant x \leqslant 2\pi$ at which

$$H(x) = \sum_1^n (\lambda_j^0 \cos jx + \mu_j^0 \sin jx)$$

attains its maximum, where λ_j^0, μ_j^0, not all zero, satisfy

$$\Sigma(\lambda_j^0 a_j + \mu_j^0 b_j) = \max H(x).$$

By the argument of Theorem VIII, Σ cannot contain more than n non-equivalent points, so that $p \leqslant n$. The remainder of the proof is the same

as for Theorem VIII. In the proof of that theorem, everything depended on the fact that $p \leqslant 2n$; a relation which is satisfied in this case.

Let $a_1, \ldots, a_n$ be a set of numbers. Let $1, p_1(x), \ldots, p_n(x)$ be a set of linearly independent polynomials of degree less than or equal to n. We denote by P the class of non-diminishing functions $g(x)$ whose variation in (a, b) does not exceed 1, and which satisfy

$$g(a) = 0, \quad a_j = \int_a^b p_j \, dg \quad (j = 1, \ldots, n).$$

We need not give a proof of the following theorem. We use $[x]$ to denote the greatest integer less than x.

THEOREM XX. *Let* 1 *be the lower bound of positive numbers* M *such that*

$$\sum_1^n \lambda_j a_j \leqslant M \max \sum_1^n \lambda_j p_j \quad (a \leqslant x \leqslant b)$$

for all λ_j. *Then*

(i) P *exists, and every* $g \subset P$ *has its variation equal to* 1 *and is a step function with not more than* $[\frac{1}{2}n]+1$ *points of discontinuity in* $a \leqslant x \leqslant b$;

(ii) *the values of* g *at its points of continuity are known; and if* $h \subset P$, *then* h *can only differ from* g *at the points of discontinuity of* g, *and all the possible values of* h *at these points are determinate.*

12. Let $f(x)$ be a function defined in a neighbourhood of a point ξ. The effective maximum of f at ξ is defined to be the greatest number M such that, for all $\epsilon > 0$, the set $f(x) > M - \epsilon$ is of positive measure in every neighbourhood of ξ.

The following theorem is the analogue of Theorem XIV for monotone functions.

THEOREM XXI. *Given the even function* $a(x)$ *whose effective maximum at* $x = 0$ *is* 1, *and the odd function* $b(x)$, *both defined in* $(-\infty, \infty)$, *a necessary and sufficient condition that there shall exist a bounded non-diminishing function* g, *such that*

$$a(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos xt \, dg, \qquad b(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \sin xt \, dg, \tag{60}$$

for almost all x *in* $(-\infty, \infty)$, *is that, for any finite set of numbers*

λ_j, μ_j $(j = 1, \ldots, n)$ *and for almost all points* x_j, *we have*

$$\sum_1^n \{\lambda_j a(x_j) + \mu_j b(x_j)\} \leqslant \max \sum_1^n (\lambda_j \cos x_j t + \mu_j \sin x_j t). \tag{61}$$

If the equations (60) hold for almost all x, then, since the integrals are continuous functions of x, and since 1 is the effective maximum of $a(x)$ at $x = 0$, we have

$$1 = \lim_{x \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \cos xt \, dg = \frac{1}{\pi} \int_{-\infty}^{\infty} dg. \tag{62}$$

Hence the variation of the non-diminishing function $g(x)$ must be equal to π and then (61) is obviously satisfied.

Suppose conversely that (61) holds. The enunciation means that, given λ_j, μ_j $(j = 1, \ldots, n)$, there is a set E, $mE = 0$, depending possibly on the choice of the λ_j, μ_j, such that (61) holds for all $x_j \subset CE$. There is an enumerable set of sets of rational λ_j, μ_j $(j = 1, \ldots, n)$, where n may take all values, and a corresponding enumerable E_r. If $E = \Sigma E_r$, then $mE = 0$, and, if $x_j \subset CE$, then (61) is true for all rational λ_j, μ_j, and hence for all λ_j, μ_j.

For a fixed set of λ_j, μ_j and a fixed set of $x_j \subset CE$, if the first member of (61) is non-negative, we have

$$\left| \sum_1^n \{\lambda_j a(x_j) + \mu_j b(x_j)\} \right| \leqslant \max \left| \sum_1^n (\lambda_j \cos x_j t + \mu_j \sin x_j t) \right|. \tag{63}$$

If the first member of (61) is negative, since for the fixed set of x_j (61) holds for all λ_j, μ_j, we have

$$\sum_1^n \{(-\lambda_j) a(x_j) + (-\mu_j) b(x_j)\} \leqslant \max \sum_1^n \{(-\lambda_j) \cos x_j t + (-\mu_j) \sin x_j t\},$$

and (63) again holds. By Theorem XIV, the equations (60) hold for almost all x, where $g(x)$ has a total variation not exceeding π. But for the same reason as above, (62) holds. Hence $g(x)$ is non-diminishing.

The following analogue of Theorem XII requires no proof.

THEOREM XXII. *If* $a(x)$ *is an even function,* $b(x)$ *an odd function, both continuous and defined in* $(-\infty, \infty)$, *and* $a(0) = 1$, *then a necessary and sufficient condition that there shall be a bounded non-diminishing* $g(x)$, *such that* (60) *holds for all* x, *is that, for any finite set of* λ_j, μ_j $(j = 1, \ldots, n)$ *and for any set of rational* x_j, *the relation* (61) *is satisfied.*

Manifestly, if, for a fixed set of rational x_j, (61) holds for all λ_j, μ_j, then

$$\sum_1^n \{\lambda_j a(x_j)+\mu_j b(x_j)\} \geqslant \min \sum_1^n \{\lambda_j \cos x_j t+\mu_j \sin x_j t\}.$$

Since $a(0)=1$, the necessary and sufficient condition of Theorem XXII becomes: for each set of rational $x_j > 0$ $(j=1, \ldots, n)$ and for every set of $\lambda_0, \lambda_j, \mu_j$ $(j=1, \ldots, n)$, such that

$$\lambda_0+\sum_1^n \{\lambda_j \cos x_j t+\mu_j \sin x_j t\} \geqslant 0, \tag{64}$$

we have

$$\lambda_0 a(0)+\sum_1^n \{\lambda_j a(x_j)+\mu_j b(x_j)\} \geqslant 0. \tag{65}$$

We shall transform this condition by means of the following lemma of Fejér*.

LEMMA. *Every non-negative trigonometric polynomial of the form* (64) *can be expressed in the form*

$$\left(\sum_0^n \xi_j e^{-ijt}\right)\left(\sum_0^n \bar{\xi}_j e^{ijt}\right). \tag{66}$$

Conversely, an expression of the form (66) *is a non-negative trigonometric polynomial.*

Let the rational x_j of (64) be expressed as $x_j = p_j/N$, where p_j, N are positive integers. It is no loss of generality to suppose that the x_j are identical with a set of the form $1/N, 2/N, \ldots, M/N$, since this merely amounts to introducing some terms with zero coefficients. In this case, (65) is replaced by an expression of the form

$$\lambda_0 a(0)+\sum_1^M \{\lambda_j a(j/N)+\mu_j b(j/N)\} \geqslant 0. \tag{67}$$

By the lemma of Fejér, we can write

$$\lambda_0+\sum_0^M \{\lambda_j \cos jt+\mu_j \sin jt\} = \left(\sum_0^M \xi_k e^{-ikt}\right)\left(\sum_0^M \bar{\xi}_k e^{ikt}\right)$$

$$= \sum_{k,\,l=0}^M \xi_k \bar{\xi}_l e^{-i(k-l)t}.$$

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* Fejér, *Journal für Math.*, 146 (1916), 53–82.

Now

$$\lambda_0 a_0+\sum_1^M \lambda_j a\left(\frac{j}{N}\right)+\mu_j b\left(\frac{j}{N}\right)$$

$$=\frac{1}{\pi}\int_0^{2\pi}\left[\tfrac{1}{2}a(0)+\sum_1^M\left\{a\left(\frac{j}{N}\right)\cos jt+b\left(\frac{j}{N}\right)\sin jt\right\}\right]$$

$$\times\left(\lambda_0+\sum_1^M(\lambda_j\cos jt+\mu_j\sin jt)\right)dt. \quad (68)$$

Writing $$c(x)=\tfrac{1}{2}\{a(x)-ib(x)\},$$

we have

$$\tfrac{1}{2}a_0+\sum_1^M\left\{a\left(\frac{j}{N}\right)\cos jt+b\left(\frac{j}{N}\right)\sin jt\right\}=\sum_{-M}^M c\left(\frac{j}{N}\right)e^{ijt},$$

and the second member of (68) becomes

$$\frac{1}{\pi}\int_0^{2\pi}\left\{\sum_{-M}^M c\left(\frac{j}{N}\right)e^{ijt}\right\}\left(\sum_{k,\,l=0}^M \xi_k\bar{\xi}_l e^{-i(k-l)t}\right)dt=2\sum_{k,\,l=0}^M c\left(\frac{k}{N}-\frac{l}{N}\right)\xi_k\bar{\xi}_i.$$

We have thus proved the following theorem, which is substantially the same as a theorem of Bochner*.

THEOREM XXIII. *Let $c(x)$ be a continuous complex function defined in $(-\infty, \infty)$ and satisfying $c(0)=1$, $c(-x)=\bar{c}(x)$. A necessary and sufficient condition that there shall exist a bounded non-diminishing function $g(x)$, such that*

$$c(x)=\int_{-\infty}^{\infty}e^{-ixt}dg$$

for all x, is that, given any positive integers N and M, the Laurentian form

$$\sum_{k,\,l=0}^M c\left(\frac{k}{N}-\frac{l}{N}\right)\xi_k\bar{\xi}_l \quad (69)$$

is non-negative.

On grounds of homogeneity, the condition $c(0)=1$ may be omitted.

13. THEOREM XXIV. *Let $a(x)$, $b(x)$ be two continuous bounded functions defined in $(0, \infty)$. A necessary and sufficient condition that*

$$f(x, y)=\int_0^{\infty}e^{-yt}\{a(t)\cos xt+b(t)\sin xt\}dt \quad (70)$$

* Bochner, *loc. cit.*, 76.

shall be a positive harmonic function in $y > 0$ is, with the notation

$$c(x) = \tfrac{1}{2}\{a(x) - ib(x)\}, \quad a(-x) = a(x), \quad b(-x) = -b(x),$$

that the Laurentian form (69) *is non-negative for all positive integers* M, N.

The expression (70) is manifestly a harmonic function in $y > 0$. We may without loss of generality suppose that $a(0) = 1$. We require the following lemma, which is a special case of a theorem of Burkill*. We give a proof *ad hoc*.

LEMMA 10. *If* $F(x)$ *is continuous and of bounded variation in* $(-\infty, \infty)$ *and* $F(0) = 0$, *then*

$$F(x) = \int_0^\infty \frac{\sin xt}{t}\, dC(t) + \frac{1-\cos xt}{t}\, dD(t),$$

where

$$C(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin xt}{x}\, dF(x), \qquad D(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos xt}{x}\, dF(x).$$

We prove this for an even $F(x)$ with $F(0) = 0$, the proof for an odd $F(x)$ being the same. We must show that

$$F(x) = \int_0^\infty \frac{\sin xt}{t}\, dC(t), \qquad C(t) = \frac{2}{\pi}\int_0^\infty \frac{\sin xt}{x}\, dF(x),$$

for $x > 0$. We have, since

$$C(t) = \int_0^t c(\xi)\, d\xi, \qquad c(\xi) = \frac{2}{\pi}\int_0^\infty \cos xt\, dF(x),$$

the equations

$$\int_0^T \frac{\sin xt}{t}\, dC(t) = \int_0^T \frac{\sin xt}{t}\, c(t)\, dt = \frac{2}{\pi}\int_0^T \frac{\sin xt}{t}\, dt \int_0^\infty \cos \xi t\, dF(\xi)$$

$$= \frac{2}{\pi}\int_0^\infty dF(\xi) \int_0^T \frac{\sin xt}{t} \cos \xi t\, dt.$$

Since

$$\left| \int_0^T \frac{\sin xt}{t} \cos \xi t\, dt \right| < \pi,$$

* Burkill, *Proc. London Math. Soc.* (2), 25 (1926), 513–524.

given $\epsilon > 0$, we can choose an $A > x$ so that

$$\left| \frac{2}{\pi} \int_A^\infty dF(\xi) \int_0^T \frac{\sin xt}{t} \cos \xi t \, dt \right| < \epsilon.$$

Further,

$$\frac{2}{\pi} \int_0^A dF(\xi) \int_0^T \frac{\sin xt}{t} \cos \xi t \, dt$$

$$= \frac{2}{\pi} F(A) \int_0^T \frac{\sin xt \cos At}{t} dt + \frac{2}{\pi} \int_0^A F(\xi) d\xi \int_0^T \sin xt \sin \xi t \, dt.$$

Since $A > x$, the first term tends to zero as $T \to \infty$. The second term is

$$\frac{1}{\pi} \int_0^A F(\xi) \left[\frac{\sin(x-\xi)T}{x-\xi} - \frac{\sin(x+\xi)T}{x+\xi} \right] d\xi,$$

which tends to $F(x)$ by the theory of Dirichlet's integral.

To prove that the condition of the theorem is necessary, we observe that

$$F(\xi, y) = \int_0^\xi f(x, y) \, dx = \int_0^\infty e^{-yt} \left[a(t) \frac{\sin \xi t}{t} + b(t) \frac{1-\cos \xi t}{t} \right] dt$$

$$= \int_0^\infty \frac{\sin \xi t}{t} dA(t, y) + \frac{1-\cos \xi t}{t} dB(t, y), \quad (71)$$

where $$A(t, y) = \int_0^t e^{-y\tau} a(\tau) \, d\tau, \quad B(t, y) = \int_0^\tau e^{-y\tau} b(\tau) \, d\tau \quad (72)$$

are continuous, of bounded variation in $-\infty < t < \infty$, and vanish for $t = 0$.

Since, by hypothesis, $f(x, y) > 0$, the function $F(\xi, y)$ is a non-diminishing function of ξ. Hence

$$\lim_{\xi \to \infty} [F(\xi, y) - F(-\xi, y)] = \lim_{\xi \to \infty} 2 \int_0^\infty e^{-yt} a(t) \frac{\sin \xi t}{t} dt$$

has a unique value. This value (by the theory of Dirichlet's integral) is $\pi a_0 = \pi$. Thus $F(\xi, y)$ is bounded. Hence it is of bounded variation in ξ, and, since $F(0, y) = 0$, we have, by Lemma 10,

$$F(\xi, y) = \int_0^\infty \frac{\sin \xi t}{t} dC(t, y) + \frac{1-\cos \xi t}{t} dD(t, y), \quad (73)$$

where

$$C(t, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin t\xi}{\xi}\, dF(\xi, y), \quad D(t, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos t\xi}{\xi}\, dF(\xi, y). \quad (74)$$

Since, on writing $A(-t, y) = A(t, y)$, we obtain a function which satisfies the conditions of Lemma 10, we have, by that lemma,

$$\begin{aligned} A(t, y) &= \int_0^{\infty} \frac{\sin t\xi}{\xi}\, d\left(\frac{2}{\pi}\int_0^{\infty} \frac{\sin \xi x}{x}\, dA(x, y)\right) \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{\sin t\xi}{\xi}\, d\,[F(\xi, y) - F(-\xi, y)] \quad \text{[by (71)]} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin t\xi}{\xi}\, dF(\xi, y) \\ &= C(t, y). \end{aligned}$$

Substituting in the equation $A = C$ the expressions given by (72) and (74) and differentiating with respect to t, we get

$$e^{-yt}a(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos t\xi\, dF(\xi, y),$$

and similarly

$$e^{-yt}b(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin t\xi\, dF(\xi, y).$$

For every $y > 0$, $F(\xi, y)$ is a non-diminishing function of ξ whose variation is equal to π. Giving to y a diminishing sequence of values which tends to zero, we see by Lemma 4 that there is a non-diminishing g such that

$$c(x) = \int_{-\infty}^{\infty} e^{-ixt}\, dg. \quad (75)$$

By Theorem XXIII, the condition of the present theorem is necessary.

To prove that the condition is sufficient, we observe that, when it is satisfied, (75) holds with a non-diminishing g by Theorem XXIII. Then

$$\begin{aligned} f(x, y) &= \frac{1}{\pi}\int_0^{\infty} e^{-yt}\, dt \int_{-\infty}^{\infty} \cos(x-\xi)\, t\, dg(\xi) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} dg(\xi) \int_0^{\infty} e^{-yt} \cos(x-\xi)\, t\, dt. \end{aligned}$$

This expression is non-negative, since

$$\int_0^\infty e^{-yt}\cos(x-\xi)\,t\,dt = \frac{y}{y^2+(x-\xi)^2}$$

is non-negative. Thus $f(x, y) \geqslant 0$ for $y > 0$, and the sign of equality can hold only if $f \equiv 0$, since f is a harmonic function.

14. The condition that (64) shall imply (65) for rational x_j may be transformed in another manner by a method which Hausdorff* has applied with great effect to Fourier series.

THEOREM XXV. *Given the continuous complex function $c(x)$ which satisfies $c(-x) = \bar{c}(x)$, a necessary and sufficient condition that there shall exist a non-diminishing function g, such that*

$$c(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\,dg \tag{76}$$

for all x, is that, for every positive integer N and every positive integer p,

$$\phi(N, p, m) = \sum_{k=-p}^{p} \frac{p+1-|k|}{(p+1)^2}\, c\left(\frac{k}{N}\right) e^{ikm\pi/(p+1)} \geqslant 0 \quad (m = 0, 1, \ldots, 2p+1).$$

The method of Hausdorff depends on the following lemma.

LEMMA (Hausdorff). *Given a trigonometric polynomial of order n,*

$$f(t) = \sum_{k=-n}^{n} a_k e^{ikt} \quad (a_{-k} = \bar{a}_k),$$

and a positive integer $p \geqslant n$, then, if

$$t_m = t_{m,p} = m\pi/(p+1) \quad (m = 0, 1, \ldots, 2p+1)$$

and

$$\lambda_p(t) = \sum_{k=-p}^{p} \frac{p+1-|k|}{(p+1)^2}\, e^{-ikt},$$

we have

$$f(t) = \tfrac{1}{2}\sum_{m=0}^{2p+1} a_m \lambda_p(t-t_m),$$

where

$$a_m = \sum_{-n}^{n} \frac{p+1}{p+1-|k|}\, a_k e^{ikt_m}.$$

Thus, as $p \to \infty$, $a_m - f(t_m)$ tends to zero uniformly in m.

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* Hausdorff, *Math. Zeitschrift*, 16 (1923), 220–248.

To prove that the condition of the theorem is necessary, we observe that

$$\lambda_p(t) = \left[\frac{\sin \frac{1}{2}(p+1)t}{(p+1)\sin \frac{1}{2}t}\right]^2$$

and $\lambda_p(t/N - t_m)$ are non-negative. Hence, by the condition that (64) implies (65), we must have $\phi(N, p, m) \geqslant 0$.

To prove that the condition is sufficient, we observe that the condition that (64) implies (65) may obviously be altered to

$$\lambda_0 + \sum_1^n (\lambda_j \cos x_j t + \mu_j \sin x_j t) > 0$$

implies (65). We may suppose that, for the assigned x_j, the positive trigonometric polynomial is

$$f(t) = \lambda_0 + \sum_1^M (\lambda_j \cos jt/N + \mu_j \sin jt/N) > 0.$$

By the lemma, we can write

$$f(t) = \tfrac{1}{2} \sum_{m=0}^{2p+1} a_m \lambda_p\left(\frac{t}{N} - t_m\right)$$

for all sufficiently large p, where $a_m > 0$ for such p. Then

$$\begin{aligned}
&\lambda_0 a(0) + \sum_1^M \{\lambda_j a(j/N) + \mu_j b(j/N)\} \\
&\qquad = \frac{1}{\pi}\int_0^{2\pi} \left[\tfrac{1}{2}a_0 + \sum_1^M \left\{a\left(\frac{j}{N}\right)\cos jt + b\left(\frac{j}{N}\right)\sin jt\right\}\right] f(Nt)\,dt \\
&\qquad = \frac{1}{\pi}\int_0^{2\pi} \left\{\sum_{-M}^{M} c\left(\frac{j}{N}\right) e^{ijt}\right\}\left(\tfrac{1}{2}\sum_0^{2p+1} a_m \lambda_p(t - t_m)\right) dt \\
&\qquad = \sum_0^{2p+1} a_m \sum_{k=-p}^{p} \frac{p+1-|k|}{(p+1)^2}\, c\left(\frac{k}{N}\right) e^{ikt_m} \\
&\qquad \geqslant 0, \qquad (77)
\end{aligned}$$

since $a_m > 0$ for sufficiently large p, and the first member is independent of p. [In the expression preceding (77) we replace M by $p > M$, which is permissible on grounds of orthogonality.]

Employing this theorem, we can transform the condition of Theorem XII.

THEOREM XXVI. *Given the continuous complex function $c(x)$ which satisfies $c(-x) = \bar{c}(x)$, a necessary and sufficient condition that there shall be*

a function g whose total variation does not exceed π, and such that (76) *holds for all x, is that, for every positive integer N and every positive integer p,*

$$\sum_{m=0}^{2p+1}\left|\sum_{k=-p}^{p}\frac{p+1-|k|}{(p+1)^2}\,c\left(\frac{k}{N}\right)e^{ikm\pi/(p+1)}\right|\leqslant 1. \tag{78}$$

By Theorem XII, it is sufficient that

$$\left|\lambda_0 a(0)+\sum_1^M\left\{\lambda_j a\left(\frac{j}{N}\right)+\mu_j b\left(\frac{j}{N}\right)\right\}\right|\leqslant \max\left|\lambda_0+\sum_1^N(\lambda_j\cos jt/N+\mu_j\sin jt/N)\right| \tag{79}$$

for all N, M, λ, and μ. If we suppose, as we may on grounds of homogeneity, that the maximum in question is 1, then in the second member of (77) we have $|a_m|\leqslant 1+\epsilon$ for $p\geqslant p(\epsilon)$, and then, by (78), the first member of (79) does not exceed $1+\epsilon$ for such p. Since it is independent of ϵ, it does not exceed 1.

To prove that the condition is necessary, we observe that it must be possible to express g as g_1-g_2 where g_1, g_2 are non-diminishing, and $c=c_1-c_2$, with

$$c_1(0)+c_2(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(dg_1+dg_2)\leqslant\tfrac{1}{2}. \tag{80}$$

Then

$$\begin{aligned}\sum_{m=0}^{2p+1}|\phi(N,p,m)| &= \sum_{m=0}^{2p+1}|\phi_1(N,p,m)+\phi_2(N,p,m)|\\ &\leqslant \sum_{m=0}^{2p+1}\phi_1(N,p,m)+\phi_2(N,p,m),\end{aligned} \tag{81}$$

since, by Theorem XXV, ϕ_1, ϕ_2 are non-negative. Now

$$\begin{aligned}&\sum_{m=0}^{2p+1}\phi_1(N,p,m)\\ &\quad=\sum_{m=0}^{2p+1}\frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{-p}^{p}c_1\left(\frac{j}{N}\right)e^{ijt}\right\}\left(\sum_{-p}^{p}\frac{p+1-|k|}{(p+1)^2}\,e^{-ik\{t-m\pi/(p+1)\}}\right)dt\\ &\quad=\frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{-p}^{p}c_1\left(\frac{j}{N}\right)e^{ijt}\right\}\left(\sum_{-p}^{p}\frac{p+1-|k|}{(p+1)^2}\,e^{-ikt}\sum_{m=0}^{2p+1}e^{ikm\pi/(p+1)}\right)dt.\end{aligned}$$

But

$$\begin{aligned}\sum_{m=0}^{2p+1}e^{ikm\pi/(p+1)}&=0 \qquad [k\not\equiv 0\ (\mathrm{mod}\ p+1)]\\ &=2(p+1) \quad [k\equiv 0\ (\mathrm{mod}\ p+1)]\end{aligned}$$

Hence

$$\sum_{m=0}^{2p+1} \phi_1(N, p, m) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{-p}^{p} c_1\left(\frac{j}{N}\right) e^{ijt} \right) \frac{1}{p+1}\, 2(p+1)\, dt$$

$$= 2c_1(0).$$

Similarly for ϕ_2. By (81) and (80), we obtain the desired result.

15. We ought not to leave this subject without mentioning that various theorems of the early part of this paper hold for an infinite interval when the functions $f_j(x)$ are continuous and tend to zero as x tends to infinity in that direction, or in those directions, in which the interval is unlimited. There is no difficulty in proving this by the methods given above, but what we wish to point out is that it is sometimes possible to consider the case of an infinite interval as a specialization of the case of the finite interval. To fix the ideas, consider the analogue of Theorem I for the interval $(0, \infty)$. We are given n constants $a_1, \ldots, a_n$ and n functions $f_j(x)$, where f_j is continuous in $0 \leqslant x < \infty$ and tends to zero as $x \to \infty$. By means of the transformation

$$e^{-x} = y \tag{82}$$

there corresponds to $0 \leqslant x < \infty$ the interval $0 < y \leqslant 1$. To $f_j(x)$ there corresponds a function $F_j(y)$ which tends to zero as $y \to 0$. We define $F_j(0) = 0$, and then F_j is continuous in the closed interval $(0, 1)$. If now the condition

$$\left| \sum_1^n \mu_j a_j \right| \leqslant \max \left| \Sigma \mu_j f_j(x) \right| \quad (0 \leqslant x < \infty)$$

is satisfied for all μ_j, we have

$$\left| \sum_1^n \mu_j a_j \right| \leqslant \max \left| \Sigma \mu_j F_j(y) \right| \quad (0 < y \leqslant 1),$$

and we can replace $0 < y \leqslant 1$ by $0 \leqslant y \leqslant 1$, since $F_j(0) = 0$. By Theorem I there is a $G(y)$ such that

$$\int_0^1 |dG| \leqslant 1, \qquad a_j = \int_0^1 F_j(y)\, dG.$$

We can only transform back to the variable y if G is continuous at $x = 0$. We overcome this by writing

$$H(y) = G(y) \quad (0 < y \leqslant 1),$$

$$H(0) = G(+0).$$

Then $$\int_0^1 |dH| \leqslant \int_0^1 |dG| \leqslant 1,$$

and, since $F_j(0) = 0$,

$$\int_0^1 F_j(y)\, dG = \int_0^1 F_j(y)\, dH.$$

The transformation (82) then gives $H(y) = g(x)$ with

$$\int_0^1 |dH| = \int_0^\infty |dg|, \qquad \int_0^1 F_j(y)\, dG = \int_0^\infty f_j(x)\, dg,$$

which is the required result. Similarly for a sequence of constants a_j and of functions f_j. Thus, given a sequence c_n, the existence theorem for the solution of

$$c_n = \int_0^\infty e^{-nv}\, dg \quad (n = 0, 1, \ldots)$$

is equivalent to the same theorem for

$$c_n = \int_0^1 x^n\, dG \quad (n = 0, 1, \ldots).$$

The transformation $z = 2x - 1$ shows that this is equivalent to the existence theorem for the solution of

$$c_n = \int_{-1}^1 \left(\frac{z+1}{2}\right)^n dH \quad (n = 0, 1, \ldots).$$

It is here permissible to equate linearly independent linear combinations of the c_n to the corresponding combinations of the second members. Thus we have to consider the solution of

$$C_n = \int_{-1}^1 z^n\, dH \quad (n = 0, 1, \ldots),$$

where the C_n are known. If we again take suitable linearly independent linear combinations, we get the system

$$a_n = \int_{-1}^1 \cos(n \operatorname{arc\,cos} z)\, dH,$$

where the a_n are known. Finally, the transformation $\operatorname{arc\,cos} z = w$ gives the system

$$a_n = \int_0^\pi \cos nw\, dF. \tag{83}$$

This is equivalent to the system

$$a_n = \int_{-\pi}^{\pi} \cos nx\, dG, \qquad 0 = \int_{-\pi}^{\pi} \sin nx\, dG. \tag{84}$$

For if (83) is satisfied by an F with $F(0) = 0$, then (84) is satisfied by

$$G(x) = \tfrac{1}{2}F(x) \qquad (0 \leqslant x \leqslant \pi)$$

$$= -\tfrac{1}{2}F(x) \quad (-\pi \leqslant x \leqslant 0),$$

so that

$$\int_0^{\pi} |dF| = \int_{-\pi}^{\pi} |dG|, \tag{85}$$

and F and G are simultaneously non-diminishing. Conversely, if (84) is satisfied by a G, then (83) is satisfied by

$$F(x) = \tfrac{1}{2}\{G(x) - G(-x)\} \quad (0 \leqslant x \leqslant \pi)$$

and (85) holds, and G and F are simultaneously non-diminishing. It thus appears that the Stieltjes moment problem for a finite interval is a special case of the Fourier moment problem.

ON THE FLUCTUATION OF INTEGRAL AND MEROMORPHIC FUNCTIONS (I)

By J. M. WHITTAKER.

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1. The present work is connected with a theorem proved in a recent paper*.

If $f(z)$ is an integral function of genus 0 *or* 1 *and d a given positive number, then there is a positive number h and a sequence of points ζ, tending to infinity, such that*

$$\log|f(z)| > h \log M(|\zeta|) \quad (|z-\zeta| \leqslant d). \tag{1.1}$$

It is shown (i) that the hypothesis can be replaced by the weaker one

$f(z)$ is of order less than or equal to 2 *and satisfies the condition*

$$n(r) = o(r^2), \quad n(r) = \text{number of zeros in } |z| = r;$$

(ii) that the number h is an absolute constant, independent of the particular function considered; and (iii) that the points ζ for which (1.1) is satisfied form a set whose upper density is greater than an absolute constant. The *lower* and *upper density* of a plane set E are defined to be the lower and upper limits, as $r \to \infty$, of

$$\frac{1}{\pi r^2} \times \text{plane measure of the part of } E \text{ in } |z| = r.$$

The theorem is not, of course, valid for meromorphic functions, even if $\log M(r)$ is replaced by the Nevanlinna characteristic function $T(r)$. However, something can be proved in this direction. It is shown that, if $f(z)$ is a meromorphic function of order less than or equal to 2 such that

$$\{\text{number of poles and zeros in } |z| = r\} = o(r^2),$$

* Whittaker, **6**. This paper will be referred to as F.R. See also Whittaker, **7**.

and if $L(\zeta, d)$, $U(\zeta, d)$ denote the bounds of $|f(z)|$ in the circle $|z-\zeta| \leqslant d$, then

$$U(\zeta, d) \leqslant L(\zeta, d)\, e^{r^{1+\delta}} \quad (\delta > 0),$$

for values of ζ forming a set of unit density.

A more precise inequality holds for functions of genus 0.

As regards method, the present work differs from the former mainly in that proximate orders are not employed. To make use of them disguises the very primitive nature of the principle involved, which has no special significance for integral functions and is valid even for discontinuous functions. It is developed in three lemmas on real positive functions.

In the concluding section a modification of Pólya's method for treating functions bounded at the lattice points is employed to obtain a theorem for functions regular in a sector.

2. *Lemmas on real positive functions.*

LEMMA 1. *Let $\phi(x)$, $\omega(x)$ be real functions satisfying the conditions*

$$0 < A \leqslant \phi(x) \leqslant \omega(kx) \quad (x \geqslant a), \tag{2.1}$$

for some fixed k $(1 \leqslant k < \infty)$. Then

$$\overline{\lim_{x \to \infty}} \frac{\omega(x)}{\phi(x)} \geqslant k^{-\lambda}, \tag{2.2}$$

where

$$\lambda = \underline{\lim_{x \to \infty}} \frac{\log \omega(x)}{\log x}. \tag{2.3}$$

λ may be $-\infty$ or ∞. The result is trivial if $k = 1$ or $\lambda = \infty$, so we may suppose that $k > 1$, $\lambda < \infty$. If (2.2) is false, there is a number $l > k^{\lambda}$, such that

$$l\omega(x) \leqslant \phi(x) \qquad (x \geqslant b).$$

Then
$$l\omega(x) \leqslant \phi(x) \leqslant \omega(kx) \quad (x \geqslant c),$$

so that
$$l^n \omega(x) \leqslant \omega(k^n x) \qquad (x \geqslant c),$$

and thus, if $1 \leqslant \theta \leqslant k$,

$$\frac{\log \omega(k^n c\theta)}{\log (k^n c\theta)} \geqslant \frac{n \log l + \log \omega(c\theta)}{n \log k + \log c\theta} \quad (n \geqslant 0).$$

$$> \frac{\log l}{\log k} - \frac{K}{n}.$$

On making $n \to \infty$ this gives

$$\underline{\lim}_{x \to \infty} \frac{\log \omega(x)}{\log x} \geqslant \frac{\log l}{\log k} > \lambda,$$

which contradicts (2.3).

(2.1) cannot be replaced by

$$0 < \phi(x) \leqslant \omega(kx) \quad (x \geqslant a). \tag{2.1}'$$

For example, let

$$\phi(x) = \omega(x) = \frac{1}{q} \quad (x = p/q, \text{ a rational number in its lowest terms}),$$

$$= 1 \quad (x \text{ irrational}).$$

Then $$\phi(x) \leqslant \omega(2x), \quad \lambda = -\infty,$$

and (2.2) is false.

If $\omega(x)$ is assumed to be an increasing function, the conclusion can be amplified.

LEMMA 2. *If* (2.1)′ *is satisfied and*

$$0 < \omega(x) \leqslant \omega(x') \quad (x' > x \geqslant d), \tag{2.4}$$

there is a sequence $x_1, x_2, \ldots$, *tending to infinity, such that*

$$\frac{\log \omega(x_n)}{\log x_n} \to \kappa, \tag{2.5}$$

and

$$\underline{\lim}_{n \to \infty} \frac{\omega(x_n)}{\phi(x_n)} \geqslant k^{-\kappa}, \tag{2.6}$$

where

$$\kappa = \overline{\lim}_{x \to \infty} \frac{\log \omega(x)}{\log x}. \tag{2.7}$$

We may suppose that $k > 1$, $\kappa < \infty$. It is sufficient to prove that, corresponding to each $\epsilon > 0$, there is a sequence $\xi_1, \xi_2, \ldots$, tending to infinity, such that

$$\omega(\xi_n) > \max\{(k^{-\kappa} - \epsilon)\phi(\xi_n), \ \xi_n^{\kappa-\epsilon}\}.$$

If this is false, then, for some ϵ,

$$\omega(x) \leqslant \max\{(k^{-\kappa} - \epsilon)\phi(x), \ x^{\kappa-\epsilon}\} \quad (x \geqslant f). \tag{2.8}$$

Take δ so that $0 < \delta < \epsilon$ and

$$l = (k^{-\kappa} - \epsilon)^{-1} > k^{\eta} \quad [\eta = \epsilon(\kappa - \delta)(\epsilon - \delta)^{-1}].$$

By (2.7) there is a sequence of points tending to infinity for which

$$\omega(x) \geqslant x^{\kappa - \delta}.$$

Let $X > \max(a, d, f)$ be a point of such a sequence, and let

$$Y = X^{(\kappa - \delta)/(\kappa - \epsilon)},$$

so that

$$\omega(x) \geqslant \omega(X) \geqslant X^{\kappa - \delta} = Y^{\kappa - \epsilon} \geqslant x^{\kappa - \epsilon} \quad (X \leqslant x \leqslant Y). \tag{2.9}$$

Then (2.8), (2.9) give

$$l\omega(x) \leqslant \phi(x) \quad (X \leqslant x \leqslant Y),$$

so that

$$l\omega(x) \leqslant \omega(kx) \quad (X \leqslant x \leqslant Y),$$

and

$$l^n \omega(X) \leqslant \omega(k^n X) \quad (k^{n-1} X \leqslant Y). \tag{2.10}$$

Let ν be the smallest integer such that

$$k^{\nu} > \frac{Y}{X} = X^{(\epsilon - \delta)/(\kappa - \epsilon)},$$

so that

$$\nu \log k > \frac{\epsilon - \delta}{\kappa - \epsilon} \log X \geqslant (\nu - 1) \log k,$$

and

$$\nu \log k = \frac{\epsilon - \delta}{\kappa - \epsilon} \log X \left\{1 + O\left(\frac{1}{\nu}\right)\right\}.$$

Now, by (2.10),

$$\omega(k^{\nu} X) \geqslant l^{\nu} \omega(X) \geqslant l^{\nu} X^{\kappa - \delta},$$

so that

$$\log \omega(k^{\nu} X) \geqslant \nu \log l + (\kappa - \delta) \log X$$

$$= \left\{\frac{\epsilon - \delta}{\kappa - \epsilon} \frac{\log l}{\log k} + O\left(\frac{1}{\nu}\right) + \kappa - \delta\right\} \log X,$$

while

$$\log k^{\nu} X = \left\{\frac{\epsilon - \delta}{\kappa - \epsilon} + O\left(\frac{1}{\nu}\right) + 1\right\} \log X,$$

On dividing these and making $X \to \infty$, so that $\nu \to \infty$, we get

$$\overline{\lim_{x\to\infty}} \frac{\log \omega(x)}{\log x} \geqslant \left\{\frac{\epsilon-\delta}{\kappa-\epsilon}\frac{\log l}{\log k}+\kappa-\delta\right\}\left\{\frac{\epsilon-\delta}{\kappa-\epsilon}+1\right\}^{-1}$$

$$= \frac{\epsilon-\delta}{\kappa-\delta}\frac{\log l}{\log k}+\kappa-\epsilon$$

$$> \frac{(\epsilon-\delta)}{(\kappa-\delta)}\frac{\epsilon(\kappa-\delta)}{(\epsilon-\delta)}+\kappa-\epsilon = \kappa,$$

a contradiction.

LEMMA 3. *If* (2.4) *is satisfied,* κ *is defined by* (2.7), *and*

$$\psi(x) = \frac{\alpha+1}{x^{\alpha+1}}\int_0^x t^\alpha \omega(t)\,dt \quad (\alpha \geqslant 0), \tag{2.11}$$

then there is a sequence $x_1, x_2, \ldots$, *tending to infinity, such that* (2.5) *is satisfied and*

$$\lim_{n\to\infty} \frac{\psi(x_n)}{\omega(x_n)} \geqslant \frac{\alpha+1}{\kappa+\alpha+1}. \tag{2.12}$$

If the conclusion is false, then, for some ϵ,

$$\psi(x) \leqslant \max\left\{\left(\frac{\alpha+1}{\kappa+\alpha+1}-\epsilon\right)\omega(x),\ x^{\kappa-\epsilon}\right\} \quad (x \geqslant x_0).$$

Write

$$\beta = \frac{\{\kappa+\epsilon(\kappa+\alpha+1)\}(\alpha+1)}{\alpha+1-\epsilon(\kappa+\alpha+1)} > \kappa+\epsilon,$$

and take δ so that $0 < \delta < \min(\epsilon, \epsilon^2/\kappa)$. Now

$$\omega(2x) \geqslant \psi(2x) \geqslant \frac{\alpha+1}{(2x)^{\alpha+1}}\int_x^{2x} t^\alpha \omega(t)\,dt \geqslant \frac{\alpha+1}{2^{\alpha+1}}\omega(x),$$

so that

$$\overline{\lim_{x\to\infty}} \frac{\log \psi(x)}{\log x} = \kappa, \tag{2.13}$$

and there is a sequence of points, tending to infinity, such that

$$\psi(x) \geqslant x^{\kappa-\delta}.$$

X, Y are chosen as in the last lemma, and

$$\psi(x) \leqslant \left(\frac{\alpha+1}{\kappa+\alpha+1}-\epsilon\right)\omega(x) \quad (X \leqslant x \leqslant Y)$$

$$= \left(\frac{\alpha+1}{\kappa+\alpha+1}-\epsilon\right)\left\{\psi(x)+\frac{x}{\alpha+1}\psi'(x)\right\},$$

so that

$$\frac{\beta}{x} \leqslant \frac{\psi'(x)}{\psi(x)}.$$

By integration,

$$\beta \log x - \beta \log X \leqslant \log \psi(x) - \log \psi(X) \quad (X \leqslant x \leqslant Y),$$

and so

$$\begin{aligned} \log \psi(Y) &\geqslant \beta \log Y - \beta \log X + \log \psi(X) \\ &\geqslant \beta \log Y - (\beta - \kappa + \delta) \log X \\ &= \left\{\beta - (\beta - \kappa + \delta)\frac{\kappa - \epsilon}{\kappa - \delta}\right\} \log Y. \end{aligned}$$

Since the coefficient of $\log Y$ is greater than κ and there is a sequence of Y tending to infinity, a contradiction to (2.13) has been reached.

Consideration of the function x^κ shows that the constants occurring in these lemmas are the best possible constants.

3. *Integral functions of order 2.*

LEMMA 4*. *Divide the z-plane into squares of side λ by drawing lines parallel to the axes and mark points $c_1, c_2, \ldots$ on it. Shade every square containing a c and the eight neighbouring squares. Of the remaining squares shade every one whose twenty-four neighbours contain eight or more c's, and, generally, every square whose $4q(q+1)$ neighbours contain $4q(q-1)$ or more c's. Let $c(r)$ denote the number of c's for which $|c_s| \leqslant r$. Then, if*

$$\frac{c(r)}{r^2} \to 0 \quad as \quad r \to \infty,$$

almost every square is unshaded.

The conclusion of the lemma means that

$$\frac{N_s}{N} \to 0 \quad \text{as} \quad N \to \infty,$$

where N_s is equal to the number of shaded squares among the N squares nearest the origin.

LEMMA 5. *Let $f(z)$ be an integral function of order less than or equal to 2 satisfying the condition*

$$n(r) = o(r^2), \tag{3.1}$$

and let λ, δ be given positive numbers. Then, for almost every square of

* F.R., Lemma 5.

side λ, drawn as above,

$$|\log f(z_1)-\log f(z_2)|<r^{1+\delta}, \tag{3.2}$$

where z_1, z_2 are any points of the square, and r is the distance of the centre of the square from the origin.

$n(r)$ denotes the number of zeros a_s of $f(z)$ for which $r_s=|a_s|\leqslant r$. Multiple zeros are counted multiply.

There is no loss of generality in taking $f(0)=1$, so that $f(z)$ is of the form

$$f(z)=\exp(kz+lz^2)\prod_{s=1}^{\infty}\left\{\left(1-\frac{z}{a_s}\right)\exp\left(\frac{z}{a_s}+\frac{z^2}{2a_s^2}\right)\right\}. \tag{3.3}$$

Mark the points a_1, a_2, ... on the z-plane and shade as in the last lemma. Then almost every square is unshaded. Take points z_1, z_2 in an unshaded square and denote by τ_s the smaller of $|a_s-z_1|$, $|a_s-z_2|$. Then

$$\begin{aligned}
&|\log f(z_1)-\log f(z_2)|\\
&\quad\leqslant|k||z_1-z_2|+|l||z_1^2-z_2^2|+\sum_{s=1}^{\infty}\left|\log\left(1-\frac{z_1-z_2}{a_s-z_2}\right)+\frac{z_1-z_2}{a_s}+\frac{z_1^2-z_2^2}{2a_s^2}\right|\\
&\quad<\kappa r|z_1-z_2|+\sum_{s=1}^{\infty}\left|-\frac{z_1-z_2}{a_s-z_2}-\frac{(z_1-z_2)^2}{2(a_s-z_2)^2}+\frac{z_1-z_2}{a_s}+\frac{z_1^2-z_2^2}{2a_s^2}\right|\\
&\quad<\kappa r|z_1-z_2|+|z_1-z_2|\sum_{s=1}^{\infty}\left|-\frac{z_1z_2}{a_s(a_s-z_2)^2}+\frac{z_2^2(z_1+z_2)}{2a_s^2(a_s-z_2)^2}\right|\\
&\quad<\kappa\lambda r+2\lambda r^2\sum_{s=1}^{\infty}\frac{1}{r_s\tau_s^2}+2\lambda r^3\sum_{s=1}^{\infty}\frac{1}{r_s^2\tau_s^2}.
\end{aligned} \tag{3.4}$$

Take α so that $1-\frac{1}{2}\delta<\alpha<1$, and consider the square P of side $(2p+1)\lambda$, $p=[r^{\alpha}]$, symmetrically disposed about the square containing z_1, z_2. For points a_s inside P, $r_s>\frac{1}{2}r$, so that

$$\sum_P\frac{1}{r_s\tau_s^2}<\frac{2}{r}\sum_P\frac{1}{\tau_s^2},\quad \sum_P\frac{1}{r_s^2\tau_s^2}<\frac{4}{r^2}\sum_P\frac{1}{\tau_s^2}.$$

Now, since the square containing z_1, z_2 is unshaded, there are no points a_s in the first ring of squares round it, nor more than seven in the second ring, nor more than twenty-three in the second and third ring, etc. Thus

$$\begin{aligned}
\sum_P\frac{1}{\tau_s^2}&\leqslant\frac{7}{\lambda^2}+\frac{23-7}{(2\lambda)^2}+\ldots+\frac{4p(p-1)-4(p-1)(p-2)}{\{(p-1)\lambda\}^2}\\
&<\frac{8}{\lambda^2}\sum_{q=1}^{p-1}\frac{1}{q}<\frac{8}{\lambda^2}\log r.
\end{aligned}$$

Again, applying Hölder's inequality to the sum of the remaining terms of the first series, we have

$$\sum_{CP} \frac{1}{r_s \tau_s^2} \leqslant \left(\sum_{CP} r_s^{\alpha-3} \right)^{1/(3-\alpha)} \left(\sum_{CP} \tau_s^{2(\alpha-3)/(2-\alpha)} \right)^{(2-\alpha)/(3-\alpha)}$$

$$< \kappa \left(\int_{r^\alpha}^{\infty} \int_0^{2\pi} u^{2(\alpha-3)/(2-\alpha)} u \, du \, d\theta \right)^{(2-\alpha)/(3-\alpha)}$$

$$< \kappa r^{-2\alpha/(3-\alpha)}.$$

Moreover, if $\beta > 2(\delta+2\alpha-2)^{-1}$,

$$\sum_{CP} \frac{1}{r_s^2 \tau_s^2} \leqslant \left(\sum_{CP} r_s^{2\beta/(1-\beta)} \right)^{(\beta-1)/\beta} \left(\sum_{CP} \tau_s^{-2\beta} \right)^{1/\beta}$$

$$< \kappa \left(\int_{r^\alpha}^{\infty} \int_0^{2\pi} u^{-2\beta} u \, du \, d\theta \right)^{1/\beta}$$

$$< \kappa r^{(2-2\beta)\alpha/\beta}.$$

Since $2-2\alpha/(3-\alpha) < 1+\delta$, $3+(2-2\beta)\alpha/\beta < 1+\delta$,

these results establish the lemma.

4. *Upper and lower density.* Let E be a plane measurable set of points, let $m(E, r)$ be the (plane) measure of the parts of E contained in the circle of centre the origin, radius r. We call

$$(4.1) \qquad \underline{D}E = \varliminf_{r\to\infty} \frac{m(E, r)}{\pi r^2}, \quad \overline{D}E = \varlimsup_{r\to\infty} \frac{m(E, r)}{\pi r^2},$$

the *lower* and *upper densities of* E. If they are equal, the common value will be called the *density of* E.

LEMMA 6. *If E_1 is any set and E_2 a set of density unity, then*

$$\underline{D}(E_1 E_2) = \underline{D}E_1, \quad \overline{D}(E_1 E_2) = \overline{D}E_1.$$

The proof is immediate*.

5. *Meromorphic functions of order* 2. Let $L(\zeta, d)$, $U(\zeta, d)$ denote the bounds of an integral or meromorphic function in the circle $|z-\zeta| \leqslant d$,

* The density of a linear set has been defined by Besicovitch, 1.

and let $n(r, 0)$, $n(r, \infty)$ denote, as usual, the number of zeros and the number of poles of the function in the circle $|z| \leqslant r$.

THEOREM 1. *Let $f(z)$ be an integral or meromorphic function of order less than or equal to 2 satisfying the condition*

$$n(r, 0)+n(r, \infty) = o(r^2), \tag{5.1}$$

and let d, δ be given positive numbers. Then the values of ζ for which

$$U(\zeta, d) \leqslant L(\zeta, d)\, e^{r^{1+\delta}} \quad (|\zeta| = r) \tag{5.2}$$

form a set of unit density.

$f(z)$ can be expressed in the form* $f_1(z)/f_2(z)$, where $f_1(z)$, $f_2(z)$ are integral functions satisfying the conditions of Lemma 5. Given ϵ, take λ so that

$$\frac{\lambda^2-(\lambda-2d)^2}{\lambda^2} < \epsilon, \tag{5.3}$$

and divide the z-plane into squares of side λ. If ζ is in the square of side $\lambda-2d$ concentric with a square of side λ, the inequalities

$$|\log f_1(z)-\log f_1(\zeta)| < \tfrac{1}{4}r^{1+\delta}, \quad |\log f_2(z)-\log f_2(\zeta)| < \tfrac{1}{4}r^{1+\delta},$$

where $|z-\zeta| \leqslant d$, are satisfied for almost every such square. Thus, for almost every square,

$$\begin{aligned}
&\log U(\zeta, d)-\log L(\zeta, d)\\
&\quad = \log\left|\frac{f_1(z_1)}{f_2(z_1)}\right| - \log\left|\frac{f_1(z_2)}{f_2(z_2)}\right| \quad (z_1, z_2 \text{ points in } |z-\zeta| \leqslant d)\\
&\quad \leqslant \big|\log|f_1(z_1)|-\log|f_1(z_2)|\big| + \big|\log|f_2(z_1)|-\log|f_2(z_2)|\big|\\
&\quad \leqslant |\log f_1(z_1)-\log f_1(z_2)| + |\log f_2(z_1)-\log f_2(z_2)|\\
&\quad < r^{1+\delta}.
\end{aligned}$$

If r is sufficiently large, the squares (of side λ) for which this holds cover a fraction greater than $1-\epsilon$ of the circle of radius r, and, by (5.3), the measure of the set of ζ for which the inequality holds is greater than $(1-\epsilon)^2\pi r^2$. This is the result stated.

If $f(z)$ is of order $\rho < 2$, more precise information, which will be useful later, can be obtained.

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* Nevanlinna, **2**, 40.

LEMMA 7. *If $f(z)$ is an integral or meromorphic function of order ρ $(1 \leqslant \rho < 2)$, and $\sigma > \frac{1}{2}\rho$, the values of ζ for which*

$$U(\zeta, d) \leqslant L(\zeta, d)\, e^{r^\sigma} \tag{5.4}$$

form a set of unit density.

$f(z)$ is now of genus 0 or 1, and (3.4) is replaced by

$$|\log f(z_1) - \log f(z_2)| < \kappa\lambda + 2\lambda r \sum_{s=1}^{\infty} \frac{1}{r_s \tau_s}.$$

The number α defining the size of the large square P is now chosen so that $\sigma > \alpha > \frac{1}{2}\rho$, and

$$\sum_P \frac{1}{r_s \tau_s} < \frac{2}{r} \sum_P \frac{1}{\tau_s} < \frac{\kappa}{r} \int_0^{r^\alpha} \frac{u\, du}{u} < \kappa r^{\alpha - 1},$$

while

$$\begin{aligned} \sum_{CP} \frac{1}{r_s \tau_s} &< \left(\sum_{CP} r_s^{-2\alpha} \right)^{1/(2\alpha)} \left(\sum_{CP} \tau_s^{2\alpha/(1-2\alpha)} \right)^{(2\alpha-1)/(2\alpha)} \\ &< \kappa \left(\int_{r^\alpha}^{\infty} u^{1+2\alpha/(1-2\alpha)}\, du \right)^{(2\alpha-1)/(2\alpha)} \\ &< \kappa r^{\alpha - 1}. \end{aligned}$$

These inequalities contain a proof of the result.

If $f(z)$ is of genus zero, a one-dimensional shading process is appropriate.

LEMMA 8*. *Let $r_1, r_2, \ldots$ be an increasing sequence of positive numbers. Divide the real positive axis into segments of given length λ and mark the points $r_1, r_2, \ldots$ on it. Shade every segment containing an r and the two neighbouring segments. Of the remaining segments, shade every one whose q neighbours on the right contain q or more r's. Perform the same process for segments on the left. Let $n(r)$ denote the number of r's for which $r_s \leqslant r$. Then, if*

$$n(r) = o(r), \tag{5.5}$$

almost every segment is unshaded.

LEMMA 9. *If $f(z)$ is an integral or meromorphic function of genus 0, and d, δ are given positive numbers, the values of ζ for which*

$$U(\zeta, d) \leqslant L(\zeta, d)\, r^\delta \tag{5.6}$$

form a set of unit density.

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* F.R., Lemma 1.

Write $f(z)$ in the form $f_1(z)/f_2(z)$, where $f_1(z)$, $f_2(z)$ are integral functions of genus 0, mark on the real positive axis the moduli $r_1, r_2, \ldots$ of the zeros $a_1, a_2, \ldots$ of $f_1(z), f_2(z)$, and shade as above. (5.5) is satisfied, so that almost every segment is unshaded. Let $\{p\lambda, (p+1)\lambda\}$ be an unshaded segment, and let

$$r_n < p\lambda \leqslant |z_1|, \quad |z_2| \leqslant (p+1)\lambda < r_{n+1}.$$

Then
$$|a_{n-s+1}-z_1|, \quad |a_{n-s+1}-z_2| \geqslant s\lambda \quad (s = 1, 2, \ldots, n),$$

$$|a_{n+s}-z_1|, \quad |a_{n+s}-z_2| \geqslant s\lambda \quad (s = 1, 2, \ldots, \infty).$$

In particular,

$$|z_1|, \quad |z_2| \leqslant (p+1)\lambda, \quad |a_{n+p+1}-z_2| \geqslant (p+1)\lambda,$$

so that

$$|a_{n+p+t}| \leqslant |a_{n+p+t}-z_2|+|z_2| \leqslant 2|a_{n+p+t}-z_2| \quad (t = 1, 2, \ldots).$$

Thus, if
$$|z_1-\zeta| \leqslant d, \quad |z_2-\zeta| \leqslant d,$$

$$|\log f(z_1)-\log f(z_2)|$$

$$\leqslant \sum_{k=1}^{\infty} \left|\log\left(1+\frac{z_2-z_1}{a_k-z_2}\right)\right| \leqslant |z_1-z_2| \sum_{k=1}^{\infty} \frac{1}{|a_k-z_2|}$$

$$\leqslant |z_1-z_2| \left\{ \sum_{s=1}^{n} \frac{1}{s\lambda} + \sum_{s=1}^{p} \frac{1}{s\lambda} + 2 \sum_{t=1}^{\infty} \frac{1}{r_{n+p+t}} \right\}$$

$$< \frac{6d}{\lambda} \log p < \delta \log|\zeta|,$$

by suitable choice of λ.

Thus the set of ζ for which (5.6) is satisfied includes all points lying in the annulus

$$p\lambda \leqslant |z| \leqslant (p+1)\lambda,$$

provided that the corresponding segment is unshaded. It follows that the set of ζ has unit density.

6. *Relations between $M(r)$, $T(r)$, $S(r)$.* Let $M(r)$ denote the maximum modulus and $T(r)$ the Nevanlinna characteristic function

(6.1)
$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} {}^{+} \log|f(re^{i\theta})|\, d\theta,$$

of an integral function $f(z)$, of order ρ and lower order* λ, so that

$$(6.2) \qquad \overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \rho, \quad \underline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \lambda.$$

It is known that†

$$(6.3) \qquad \log M(r) \leqslant \frac{k+1}{k-1} T(kr) \quad (k>1),$$

so, by Lemma 1,

$$(6.4) \qquad \overline{\lim_{r\to\infty}} \frac{T(r)}{\log M(r)} \geqslant \frac{k-1}{k+1} k^{-\lambda} \quad (k>1).$$

Taking $k = 1+\lambda^{-1}$, we get

$$(6.5) \qquad \left.\begin{aligned} \overline{\lim_{r\to\infty}} \frac{T(r)}{\log M(r)} &\geqslant \frac{1}{2\lambda+1}\left(1+\frac{1}{\lambda}\right)^{-\lambda} \quad (\lambda>0) \\ &= 1 \qquad\qquad\qquad (\lambda=0). \end{aligned}\right\}$$

With ρ in place of λ, this result was proved by Miss Cartwright‡.

Paley shows, in the paper referred to, that functions of all orders exist for which

$$(6.6) \qquad \underline{\lim_{r\to\infty}} \frac{T(r)}{\log M(r)} = 0.$$

Consider next the function

$$(6.7) \qquad S(r) = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \log^+ |f(ue^{i\theta})|\, u\, du\, d\theta$$

$$= \frac{2}{r^2} \int_0^r T(u)\, u\, du.$$

If $k>1$,

$$S(kr) \geqslant \frac{2}{k^2 r^2} \int_{\sqrt{k}r}^{kr} T(u)\, u\, du$$

$$\geqslant \frac{2}{k^2 r^2} T(\sqrt{k}r) \int_{\sqrt{k}r}^{kr} u\, du$$

$$\geqslant \frac{k-1}{k} \frac{\sqrt{k}-1}{\sqrt{k}+1} \log M(r) \quad \text{by (6.3)}.$$

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* *Cf.* Whittaker, **8**.

† Nevanlinna, **2**, 24.

‡ See Paley, **3**. Miss Cartwright kindly showed me her proof, which depends on the properties of proximate orders.

Use Lemma 2 and take $k=(1+1/\rho)^2$. Then follows

LEMMA 10. *If $f(z)$ is an integral function of finite order ρ, there is a sequence $r_1, r_2, \ldots$, tending to infinity, such that*

$$\frac{\log S(r_n)}{\log r_n} \to \rho, \tag{6.8}$$

and

$$\lim_{n\to\infty} \frac{S(r_n)}{\log M(r_n)} \geqslant \frac{\rho^{2\rho}}{(\rho+1)^{2\rho+2}}. \tag{6.9}$$

The relation between $S(r)$ and $T(r)$ follows from Lemma 3, namely that there is a sequence for which (6.8) is satisfied and

$$\lim_{n\to\infty} \frac{S(r_n)}{T(r_n)} \geqslant \frac{2}{2+\rho}. \tag{6.10}$$

The constant on the right of (6.9) is not "best possible". That is to say, it can be replaced by larger numbers (except when $\rho=0$, when it takes the value 1) without invalidating the result. Let $R(\rho)$ denote the upper bound of such numbers. $R(\rho)$ cannot be larger than

$$\frac{2}{\pi\rho(\rho+2)} \quad (\rho > \tfrac{1}{2}),$$

since, for Mittag-Leffler's function*,

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)} \quad (0<\alpha=\rho^{-1}<2), \tag{6.11}$$

$$\frac{S(r)}{\log M(r)} \to \frac{2}{\pi\rho(\rho+2)}.$$

It is not unlikely that this is the true value of $R(\rho)$.

On the other hand, the constant on the right of (6.10) is "best possible", since, for Mittag-Leffler's function,

$$\frac{S(r)}{T(r)} \to \frac{\rho}{2+\rho}.$$

7. *"Flat" regions of integral functions of order* 2. The main theorem is proved by combining the results of the last three sections.

THEOREM 2. *There is an absolute constant H, not less than $\frac{16}{729}$, with the following property. If $f(z)$ is an integral function of order $\rho \leqslant 2$,*

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* *Cf.* Paley, **3**, for the necessary properties of the function.

satisfying the condition

$$n(r) = o(r^2), \tag{6.1}$$

and $h\,(<H)$, $\eta\,(<\rho)$, *and* d *are given positive numbers, the values of* ζ *for which the inequality*

$$\log|f(z)| > h \log M(|\zeta|) > |\zeta|^{\eta} \quad (|z-\zeta| \leqslant d), \tag{7.2}$$

is satisfied form a set of upper density greater than or equal to $(H-h)/(1-h)$.

$H = R(2) \geqslant \frac{1.6}{7.2.9}$. Given $\epsilon > 0$, let $r_1, r_2, \ldots$ be a sequence for which (in accordance with Lemma 10)

$$S(r_n) > (H-\epsilon) \log M(r_n), \tag{7.3}$$

and

$$\log M(r_n) > \frac{1}{h} r_n^{\eta}. \tag{7.4}$$

Take $h' > h$ and write $t\pi r_n^2 =$ measure of set of ζ in $|z| \leqslant r_n$ for which

$$\log|f(\zeta)| > h' \log M(r_n), \tag{7.5}$$

so that

$$\pi r_n^2 S(r_n) \leqslant t\pi r_n^2 \log M(r_n) + (1-t)\pi r_n^2 h' \log M(r_n).$$

Comparing this with (7.3), we see that

$$t + (1-t)h' > H - \epsilon,$$

so that

$$t > \frac{H-h'-\epsilon}{1-h'}.$$

As ϵ is arbitrary, it follows from (7.4), (7.5) that the set of ζ for which

$$\log|f(\zeta)| > h' \log M(|\zeta|) > \frac{h'}{h}|\zeta|^{\eta} \tag{7.6}$$

has upper density greater than or equal to $(H-h')(/1-h')$.

There are now three cases to be considered, according as $1 < \rho \leqslant 2$, $\rho = 1$, or $\rho < 1$. In the first case, η may be supposed greater than 1 and δ so chosen that $0 < \delta < \eta - 1$.

By Lemma 6, (5.2), (7.6) are simultaneously true for a set of upper density greater than or equal to $(H-h')/(1-h')$, and for this set

$$\log|f(z)| > h' \log M(|\zeta|) - |\zeta|^{1+\delta} \quad (|z-\zeta| \leqslant d),$$

$$> h \log M(|\zeta|), \tag{7.7}$$

and also

$$h \log M(|\zeta|) > |\zeta|^{\eta}. \tag{7.8}$$

(7.7), (7.8) do not involve h'. Hence the upper density of the set of ζ for which they are true is greater than or equal to $(H-h)/(1-h)$.

The other cases can be treated in a similar manner with the aid of Lemmas 7, 9.

8. *Lattice point interpolation.* A particular case of Theorem 2 is as follows.

An integral function $f(z)$ is a constant if

(8.1) *$f(z)$ is of order less than or equal to 2 and satisfies the condition*

$$n(r) = o(r^2),$$

$$|f(m+ni)| < \kappa \quad (m, n = 0, \pm 1, \pm 2, \ldots). \tag{8.2}$$

With the hypothesis

$$\log M(r) = o(r^2), \tag{8.3}$$

in place of (8.1), this result has been proved in a very elegant manner by Pólya*. The method can be developed and indicates directions in which it may be possible to extend Theorem 2.

The starting point is the interpolation formula†

$$f(z) = \phi(z) \sum_{m,\, n=-\infty}^{\infty} \frac{f(m+ni)}{c_{mn}(z-m-ni)}, \tag{8.4}$$

where

$$\phi(z) = e^{\pi z^2/2}\, \vartheta_1(\pi z/i),$$

$$c_{mn} = \phi'(m+ni) = (-)^{m+n+mn} \exp\{\pi(m^2+n^2)/2\}\, \phi'(0).$$

It was shown that this is valid for an integral function $f(z)$ satisfying the condition

$$\overline{\lim_{r \to \infty}} \frac{\log M(r)}{r^2} < \tfrac{1}{2}\pi. \tag{8.5}$$

---

* Pólya, 5. Littlewood conjectured that a function of order less than 2 cannot be bounded at the lattice points unless it is a constant, and it was proved in F.R. that this is the case for any function of genus 0 or 1. Pólya's hypothesis is weaker than this, but stronger than (8.1).

† F.R., (2.5).

As is often the case with theorems of "minimum modulus type", the upper limit can be replaced by the corresponding lower limit*.

LEMMA 11. *If $f(z)$ satisfies the condition*

$$\lim_{r\to\infty} \frac{\log M(r)}{r^2} < \tfrac{1}{4}\pi, \tag{8.6}$$

(8.4) *is valid provided that the terms of the series are bracketed together in a certain way.*

(8.6) implies that there is a number $\alpha < \frac{1}{4}\pi$ and a sequence $r_1, r_2, \ldots$, tending to infinity, such that

$$M(r_k) < e^{\alpha r_k^2}.$$

Write $p(k) = [(r_k/\sqrt{2}) - \frac{1}{2}]$, so that on the square Q_k, with corners $(1\pm i)\{p(k)+\frac{1}{2}\}$,

$$|f(z)| < e^{\alpha r_k^2}. \tag{8.7}$$

Thus, if z is a point inside Q_k other than one of the points $m+ni$,

$$\frac{f(z)}{\phi(z)} + \sum_{Q_k} \frac{f(m+ni)}{c_{mn}(m+ni-z)} = \frac{1}{2\pi i}\int_{Q_k} \frac{f(\zeta)}{\phi(\zeta)(\zeta-z)}\, d\zeta. \tag{8.8}$$

Now

$$\phi(z+m+ni) = (-)^{m+n+mn}\phi(z)\exp\{\tfrac{1}{2}\pi(m^2+n^2)+\pi(m-ni)z\}, \tag{8.9}$$

so that, on Q_k,

$$|\phi(z)| > Ce^{\frac{1}{2}\pi p^2} > Ce^{\frac{1}{2}\pi(r_k-1)^2}. \tag{8.10}$$

It follows from (8.7), (8.10) that the integral on the right of (8.8) tends to zero as $k\to\infty$, and this is the result stated. It should be noticed that the system of brackets depends on the function but not on the particular value of z.

Pólya's argument may now be used to prove the following result.

THEOREM 3. *An integral function $f(z)$ is a constant if it satisfies* (8.2) *and*

$$\lim_{r\to\infty} \frac{\log M(r)}{r^2} = 0. \tag{8.11}$$

* *E.g.*, Hadamard's theorem on the minimum modulus. See Whittaker, **8.**

For, if k is any integer, $f(kz)$ satisfies (8.6) and so

$$f(z) = k\phi\left(\frac{z}{k}\right) \sum_{m,\,n=-\infty}^{\infty} \frac{f(km+kni)}{c_{mn}\{z-k(m+ni)\}}, \tag{8.12}$$

the terms of the series being bracketed together in a certain way.

The brackets may, however, be removed, since, by (8.2), the resulting series is absolutely convergent. When we make $k \to \infty$, (8.12) becomes

$$f(z) = z\phi'(0)\frac{f(0)}{c_{00}z} = f(0).$$

A sector theorem may be proved by a similar device.

THEOREM 4. *Let $f(z)$ be regular in the domain T defined by*

$$r > R, \quad -\alpha < \theta < \alpha \quad (0 < \alpha \leqslant \pi),$$

where $z = re^{i\theta}$, and, given $\epsilon > 0$, let

$$|f(z)| < e^{\epsilon r^2} \quad (r \geqslant r_\epsilon), \quad z \text{ in } T, \tag{8.13}$$

$$|f(z)| \leqslant \kappa, \quad \text{on boundary of } T, \tag{8.14}$$

$$|f(m+ni)| \leqslant \kappa, \quad \text{for points } m+ni \text{ in } T. \tag{8.15}$$

Then

$$|f(z)| \leqslant \kappa \quad \text{in } T. \tag{8.16}$$

Draw lines parallel to $\theta = \pm\alpha$ at a distance $\frac{1}{4}$ on either side, and consider the canonical product $\chi(z)$ with zeros at the lattice points lying inside the strips so formed. It will be shown that the function

$$\omega(z) = \frac{\phi(z)}{\chi(z)}$$

satisfies the condition

$$\frac{1}{r^2}\log|\omega(re^{\pm i\alpha})| \to \tfrac{1}{2}\pi, \quad \text{as } r \to \infty. \tag{8.17}$$

$\chi(z)$ is evidently an integral function of order 1. By (8.9), in the neighbourhood of a lattice point $M+Ni$ lying inside the strips,

$$|\phi(z)| = |\phi(z-M-Ni)| \exp\{\tfrac{1}{2}\pi(M^2+N^2)+\pi Mx'+\pi Ny'\},$$

where $z = M+Ni+x'+iy'$. Moreover,

$$\chi(z)-\chi(M+Ni) = (z-M-Ni)\chi'\{z+\gamma(z-M-Ni)\} \quad (|\gamma| < 1),$$

so that

$$|\chi(z)| < Ae^{r^{1+\delta}}|z-M-Ni|.$$

Thus

$$|\omega(z)| > \exp\{\tfrac{1}{2}\pi(M^2+N^2)+O(r^{1+\delta})\}\,\frac{|\phi(z-M-Ni)|}{|z-M-Ni|}$$

$$> B\exp\{\tfrac{1}{2}\pi r^2+O(r^{1+\delta})\},$$

which gives (8.17), the contrary inequality being readily established.

Now apply Cauchy's theorem to the function $f(z)/\omega(z)$ in the domain T_p, the part of T enclosed by lines $x=p+\frac{1}{2}$, $y=\pm i(p+\frac{1}{2})$, where p is an integer. On these lines,

$$|\omega(z)| > Ce^{\frac{1}{2}\pi r^2}. \tag{8.18}$$

The resulting equation may be written

$$f(z)=\omega(z)\,\Sigma_{\omega,T_p}\frac{f(m+ni)}{d_{mn}(z-m-ni)}+\frac{\omega(z)}{2\pi i}\int_{C_p}\frac{f(\zeta)}{\omega(\zeta)(\zeta-z)}\,d\zeta,$$

where C_p is the contour bounding T_p, the summation is taken over the zeros of $\omega(z)$ inside T_p, and

$$d_{mn}=\omega'(m+ni).$$

Making use of (8.13), (8.17), (8.18), we get, as $p\to\infty$,

$$f(z)=\omega(z)\,\Sigma_{\omega,T}\frac{f(m+ni)}{d_{mn}(z-m-ni)}+\frac{\omega(z)}{2\pi i}\int_{C}\frac{f(\zeta)}{\omega(\zeta)(\zeta-z)}\,d\zeta,$$

where C is the boundary of T.

The same formula is valid for the function $\{f(z)\}^k$, where k is any positive integer. Thus, making use of (8.14), (8.15), we have

$$|f(z)|^k \leqslant |\omega(z)|\,\Sigma_{\omega,T}\frac{\kappa^k}{|d_{mn}(z-m-ni)|}+\frac{|\omega(z)|}{2\pi}\int_C\frac{\kappa^k|d\zeta|}{|\omega(\zeta)(\zeta-z)|}$$

$$< \kappa^k\Phi(z), \quad \text{with } \Phi(z) \text{ independent of } k.$$

Hence

$$|f(z)| < \kappa\{\Phi(z)\}^{1/k},$$

and, on making $k\to\infty$, we have the result stated*.

Theorem 4 should be compared with the corresponding result of Phragmén and Lindelöf†. This asserts that (8.16) follows from (8.13), (8.14), and

$$\alpha < \tfrac{1}{4}\pi. \tag{8.19}$$

* The same device has been used by Landau to prove the maximum modulus principle.

† Phragmén and Lindelöf, **4**. The result is the case $\alpha > 2$ of the theorem on p. 385.

It seems possible that in Theorem 4 the hypothesis that $f(z)$ is bounded on the lines $\theta = \pm\alpha$ may be replaced by the hypothesis that α is greater than a certain value. Theorem 2 yields a result of this type. For, if $\gamma = \pi/(1-H)$ and $f(z)$ satisfies the conditions of Theorem 2, every sector $-\alpha < \theta < \alpha$, $(\gamma < \alpha \leqslant \pi)$, must contain an infinite sequence of circles $|z-\zeta_s| \leqslant d$ in which the function is of the same order as $\log M(|\zeta_s|)$. Hence, if $f(z)$ is bounded at the lattice points in such a sector, it must be a constant.

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ON THE TOTAL VARIATION OF $\{f(t+\tau)-f(t)\}$

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Section 1 contains a proof of an elementary proposition to the effect that, if two real functions of bounded variation approximate one another in the sense that their difference has a small total variation, then the positive variations of the two functions approximate one another in the same sense. This result, which applies just as well to functions of a rational variable as to functions of a real variable, is in the nature of a comment on the work which follows. The proof depends essentially on the inequalities (1.6) and (1.7), which show that the whole range can be divided into small intervals on each of which the functions are effectively monotonic.

The two main results of the paper are contained in § 4. The first one, the corollary to the theorem of § 4, states that, if $f(t)$ is a finite measurable function defined in the interval (0, 1), and if, for each particular value of τ in (0, 1), the function $f(t+\tau)-f(t)$ is of bounded variation on the interval $(0, 1-\tau)$ in which it is defined, then $f(t)$ is itself of bounded variation on (0, 1). An example given in § 3 shows (on the basis of the axiom of choice) that the hypothesis that $f(t)$ is measurable is, in fact, necessary. The second main result, § 4, Lemma 1, Corollary 2, states that, if $f(t)$, $f'(t)$ are finite measurable functions defined in (0, 1), then the total variation of $f(t+\tau)-f'(t)$ on the interval $(0, 1-\tau)$ in which it is defined is a measurable function of τ. An example (in which f' coincides with f) shows that the hypothesis of measurability is again needed.

It is well known that, if $f(t)$ is absolutely continuous, then the total variation of $f(t+\tau)-f(t)$ tends to zero with τ. The converse has also been

proved* on the assumption that $f(t)$ is of bounded variation: § 4 shows that it is sufficient to assume that $f(t)$ is measurable.

Section 2 contains a discussion of two analogous problems for functions of a rational variable. In this case we assume that the total variation of $\{f(t+\tau)-f(t)\}$ is not only finite but bounded in order to reach the result that $f(t)$ is itself of bounded variation: and another example given in § 3 shows that a more restrictive hypothesis is, in fact, required. The results of § 2 I applied in an earlier proof of my first main result: they are not needed in the present version, but they seem to me to have some interest of their own.

I use the abbreviations b.v. and a.c. for the adjectives " of bounded variation " and " absolutely continuous ", and I use p.p. (= *presque partout*) for the phrase " almost everywhere in the sense of Lebesgue ". I use the letters Δ and δ, with or without a suffix, to denote intervals and in every such case I denote the length of the interval by $|\Delta|$ or $|\delta|$. If a and b are the origin and end point respectively of an interval Δ, then I also write

$$\Delta f(t) = f(b)-f(a).$$

Further, t is the current variable *par excellence*, *e.g.*

$$\Delta f\left(t+\frac{k}{M}\right) = f\left(b+\frac{k}{M}\right)-f\left(a+\frac{k}{M}\right).$$

1. If $f(t)$ is real and b.v. in a certain range of values of t, we shall denote by $T(f, \Delta)$ its total variation in an interval Δ of the range. Thus we may write

$$T(f, \Delta) = \int_\Delta |df(t)| = \Delta p(t)+\Delta n(t), \qquad (1.1)$$

where $p(t)$ and $n(t)$ are the positive and negative variations of $f(t)$ and

$$f(t) = p(t)-n(t). \qquad (1.2)$$

Suppose that we have two real b.v. functions $f(t)$ and $f'(t)$ and write

$$T(f', \Delta) = \int_\Delta |df'(t)| = \Delta p'(t)+\Delta n'(t),$$

$$f'(t) = p'(t)-n'(t).$$

* Wiener and Young, *Trans. American Math. Soc.*, 35 (1933), 327–340. These authors prove that, if $f(t)$ is a singular function of bounded variation, then

$$\int_0^{1-\tau} |df(t+\tau)-df(t)| = \int_0^{1-\tau} |df(t)|+\int_\tau^1 |df(t)|$$

for almost all values of τ. § 4 of the present paper may be regarded as an extension of this result to measurable functions of unbounded variation.

We shall prove that

$$T(f-f', \Delta) = T(p-p', \Delta)+T(n-n', \Delta). \tag{1.3}$$

Given $\epsilon > 0$, we can subdivide Δ into intervals δ_k such that

$$\Delta p+\Delta n = T(f, \Delta) \leqslant \Sigma|\delta_k f|+\epsilon, \tag{1.4}$$

$$\Delta p'+\Delta n' = T(f', \Delta) \leqslant \Sigma|\delta_k f'|+\epsilon. \tag{1.5}$$

Let Σ_1 denote summation over those values of k for which $\delta_k f$ is positive, and let Σ_2 denote summation over the rest. Then

$$\Sigma_1|\delta_k f| = \Sigma_1 \delta_k f = \Sigma_1 \delta_k p-\Sigma_1 \delta_k n,$$

$$\Sigma_2|\delta_k f| = -\Sigma_2 \delta_k f = \Sigma_2 \delta_k n-\Sigma_2 \delta_k p,$$

and hence

$$\Sigma|\delta_k f| = \Sigma\delta_k p+\Sigma\delta_k n-2(\Sigma_1 \delta_k n+\Sigma_2 \delta_k p).$$

By (1.4), $$\Sigma_1 \delta_k n+\Sigma_2 \delta_k p \leqslant \tfrac{1}{2}\epsilon. \tag{1.6}$$

Similarly, if Σ_1' denotes summation over those values of k for which $\delta_k f'$ is positive, and if Σ_2' denotes summation over the rest, we find that

$$\Sigma_1' \delta_k n'+\Sigma_2' \delta_k p' \leqslant \tfrac{1}{2}\epsilon. \tag{1.7}$$

Now let $\Sigma_{\alpha\beta}$ denote summation over those values of k covered both by Σ_α and by Σ_β'. Then

$$\Sigma_{11} T(p-p', \delta_k)+\Sigma_{11} T(n-n', \delta_k) \leqslant \Sigma_{11} T(f-f', \delta_k)+2\Sigma_{11} T(n-n', \delta_k)$$

$$\leqslant \Sigma_{11} T(f-f', \delta_k)+2\Sigma_{11}(\delta_k n+\delta_k n'),$$

and similarly

$$\Sigma_{22} T(p-p', \delta_k)+\Sigma_{22} T(n-n', \delta_k) \leqslant \Sigma_{22} T(f-f', \delta_k)+2\Sigma_{22}(\delta_k p+\delta_k p').$$

Again,

$$\Sigma_{12} T(p-p', \delta_k)+\Sigma_{12} T(n-n', \delta_k) \leqslant \Sigma_{12}\delta_k (p+p'+n+n')$$

$$\leqslant \Sigma_{12} T(f-f', \delta_k)+2\Sigma_{12}(\delta_k p'+\delta_k n),$$

and similarly

$$\Sigma_{21} T(p-p', \delta_k)+\Sigma_{21} T(n-n', \delta_k) \leqslant \Sigma_{21} T(f-f', \delta_k)+2\Sigma_{21}(\delta_k p+\delta_k n').$$

Adding and using (1.6) and (1.7), we find that

$$T(p-p', \Delta)+T(n-n', \Delta) \leqslant T(f-f', \Delta)+2\epsilon. \tag{1.8}$$

Since ϵ is arbitrarily small, it follows that

$$T(p-p', \Delta)+T(n-n', \Delta) \leqslant T(f-f', \Delta).$$

But it is clear that

$$T(f-f', \Delta) \leqslant T(p-p', \Delta)+T(n-n', \Delta),$$

and, combining these two inequalities, we get (1.3).

2. Let $f(t)$ be defined for rational values (only) of t in (0, 1) and let us define $v(\tau;\ df)$ for rational values of τ in (0, 1) as the total variation of $f(t+\tau)-f(t)$ in the interval $(0,\ 1-\tau)$ in which it is defined, *i.e.* as the upper bound of the sums

$$\Sigma|\delta_j\{f(t+\tau)-f(t)\}|$$

for all rational subdivisions of $(0,\ 1-\tau)$ into non-overlapping intervals δ_j. If we assume that $f(t)$ is real and b.v., then we get

$$v(\tau;\ df) = v(\tau;\ dp)+v(\tau;\ dn), \tag{2.1}$$

but we shall not make this assumption.

LEMMA 1. *If $v(\tau;\ df)$ is bounded, then $f(t)$ is of bounded variation.*

Divide the interval (0, 1) into M equal parts δ_j. Then

$$\sum_j \sum_k |\delta_j f-\delta_k f| \leqslant 2\sum_{h=1}^{M-1} v\left(\frac{h}{M}\right),$$

and hence

$$M\sum_j|\delta_j f| \leqslant \sum_j\left|M\delta_j f-\sum_k \delta_k f\right|+\sum_j\left|\sum_k \delta_k f\right|$$

$$\leqslant 2\sum_{h=1}^{M-1} v\left(\frac{h}{M}\right)+M|f(1)-f(0)|.$$

If then K is the upper bound of $v(\tau)$, we have

$$\sum_j|\delta_j f| \leqslant 2K+|f(1)-f(0)|, \tag{2.2}$$

which is independent of M.

The converse of the lemma is immediate: if T is the total variation of $f(t)$ in the interval, then

$$v(\tau;\ df) \leqslant 2T.$$

THEOREM 1. *If $v(\tau;\ df)\to 0$ with τ, then $f(t)$ is absolutely continuous.*

Since
$$v(\tau_1+\tau_2)\leqslant v(\tau_1)+v(\tau_2), \tag{2.3}$$
we see at once that $v(\tau)$ is bounded. By the lemma, $f(t)$ is b.v. in the interval; let T be its total variation. Divide the range into M equal parts and denote them in their natural order by

$$\delta_1,\ \delta_2,\ \ldots,\ \delta_M.$$

If
$$0<\nu\leqslant \tfrac{1}{2}(M+1), \tag{2.4}$$

and if also
$$0<j\leqslant \tfrac{1}{2}(M+1),$$

then
$$\sum_{k=0}^{\nu-1}|\delta_{j+k}f|\leqslant T.$$

Let
$$\Delta_1,\ \Delta_2,\ \ldots,\ \Delta_n$$

be any selection of those intervals δ_j for which $j\leqslant \frac{1}{2}(M+1)$. Then

$$\sum_{k=0}^{\nu-1}\left|\Delta_i f\left(t+\frac{k}{M}\right)\right|\leqslant T$$

for each interval Δ_i of the set. But

$$\sum_{i=1}^{n}\left|\Delta_i f\left(t+\frac{k}{M}\right)-\Delta_i f(t)\right|\leqslant v\left(\frac{k}{M}\right)$$

for all $k<\nu$. Hence

$$\nu\sum_{i=1}^{n}|\Delta_i f(t)|\leqslant\sum_{k=1}^{\nu-1}v\left(\frac{k}{M}\right)+nT. \tag{2.5}$$

There is a like result for selections of those intervals δ_j for which $j\geqslant\frac{1}{2}(M+1)$: it may be obtained by applying (2.5) to the function $f(1-t)$ in place of $f(t)$. Combining the two results, we get

$$\nu\sum_{i=1}^{n}|\Delta_i f(t)|\leqslant 2\sum_{k=1}^{\nu-1}v\left(\frac{k}{M}\right)+nT$$

for any selection of intervals δ_j and any value of ν satisfying (2.4). Hence

$$\sum_{i=1}^{n}|\Delta_i f(t)|\leqslant\frac{2}{\nu}\sum_{k=1}^{\nu-1}v\left(\frac{k}{M}\right)+\frac{MT}{\nu}\sum_{i=1}^{n}|\Delta_i|. \tag{2.6}$$

In this form the inequality holds for any set of non-overlapping intervals Δ_i provided that M is a common denominator for their end points.

Given $\epsilon > 0$, we can by hypothesis choose δ so that

$$0 < \delta < \tfrac{1}{2}, \quad v(\tau) < \tfrac{1}{3}\epsilon \text{ for all } \tau < \delta.$$

Then, if $\{\Delta_i\}$ is a set of non-overlapping intervals for which

$$\Sigma|\Delta_i| < \frac{\epsilon\delta}{3T}, \tag{2.7$_1$}$$

we get, from (2, 6),

$$\Sigma|\Delta_i f(t)| < \epsilon, \tag{2.7$_2$}$$

which proves the theorem.

The converse of the theorem can be deduced from known results in the theory of Lebesgue integration. For if $f(t)$ is an a.c. function of a rational variable t in the interval (0 . 1), then there is an a.c. function $F(t)$ of a real variable t in the interval (0, 1) which coincides with $f(t)$ for rational values of t. $F(t)$ is the indefinite integral of some summable function $\phi(t)$ and

$$v(\tau;\ df) = \int_0^{1-\tau} |\phi(x+\tau)-\phi(x)|\,dx.$$

And it is known that the right-hand side tends to zero with τ.

Let $f(t), f'(t)$ be defined for rational values (only) of t in (0, 1), and let us define $v(\tau;\ df,\ df')$ for rational values of τ in (0, 1) as the total variation of $f(t+\tau)-f'(t)$ in the interval $(0,\ 1-\tau)$ in which it is defined. If we assume that both functions are real and b.v., then we get

$$v(\tau;\ df,\ df') = v(\tau;\ dp,\ dp')+v(\tau;\ dn,\ dn'),$$

but we shall not make this assumption.

LEMMA 2. *If $v(\tau;\ df,\ df')$ is bounded in $0 \leqslant \tau < 1$, then both $f(t)$ and $f'(t)$ are of bounded variation.*

For every τ,

$$v(\tau;\ df) \leqslant v(\tau;\ df,\ df')+v(0;\ df,\ df'), \tag{2.8}$$

which is bounded: by Lemma 1, $f(t)$ is b.v. Similarly for $f'(t)$.

COROLLARY. *If $v(\tau;\ df,\ df')$ is bounded in an interval $a < \tau < b$, then $f(t)$ is b.v. in the interval $(a,\ 1)$ and $f'(t)$ is b.v. in the interval $(0,\ 1-a)$.*

We assume that $0 \leqslant a < b \leqslant 1$, but we do not assume that a is rational. If α, β, γ are rational and

$$a < \alpha < b, \quad 0 < \beta < b-\alpha, \quad 0 \leqslant \gamma \leqslant 1-\alpha-\beta,$$

then, by applying Lemma 2 to the functions

$$f(\alpha+\gamma+\beta t), \quad f'(\gamma+\beta t)$$

in place of $f(t)$ and $f'(t)$, we see that $f'(t)$ is b.v. in the interval $(\gamma, \gamma+\beta)$. The interval $(0, 1-\alpha)$ can be covered by a finite number of such intervals and therefore $f'(t)$ is b.v. in $(0, 1-\alpha)$. Similarly $f(t)$ is b.v. in the interval $(\alpha, 1)$.

Choose a rational number α_0 such that

$$a < \alpha_0 < b, \quad \alpha_0 < \tfrac{1}{2}(1+a),$$

and let T_0 be the total variation of $f(t)$ in $(\alpha_0, 1)$, T_0' that of $f'(t)$ in $(0, 1-\alpha_0)$. Let K be the upper bound of $v(\tau)$ in $a < \tau < b$. If

$$a < \alpha < \alpha_0,$$

then

$$1-\alpha_0+\alpha > \alpha_0.$$

and hence the total variation of $f(t)$ in the interval (α, α_0) does not exceed

$$T_0' + v(\alpha;\ df, df').$$

The total variation in $(\alpha, 1)$ therefore does not exceed

$$T_0 + T_0' + K.$$

Since this is independent of α, it follows that $f(t)$ is b.v. in the interval $(a, 1)$. Similarly $f'(t)$ is b.v. in the interval $(0, 1-a)$.

The converse of the lemma and its corollary is immediate: if the total variation of $f(t)$ in the interval $(a, 1)$ is T, and if the total variation of $f'(t)$ in the interval $(0, 1-a)$ is T', then

$$v(\tau;\ df, df') \leqslant T + T' \text{ for } \tau \geqslant a.$$

THEOREM 2. *If $v(\tau;\ df, df') \to 0$ with τ, then $f(t)$ is absolutely continuous in $0 < t \leqslant 1$, $f'(t)$ is absolutely continuous in $0 \leqslant t < 1$, and $f(t)-f'(t)$ is constant in $0 < t < 1$.*

If $0 < \delta < 1-\tau$, then the total variation of $f(t+\tau)-f(t)$ in the interval $(\delta, 1-\tau)$ does not exceed

$$v(\tau+\delta;\ df, df') + v(\delta;\ df, df')$$

which is arbitrarily small with τ and δ. It follows by Theorem 1, applied to the function $f(\frac{1}{2}+\frac{1}{2}t)$ in place of $f(t)$, that $f(t)$ is a.c. in $(\frac{1}{2}, 1)$. Similarly, $f'(t)$ is a.c. in $(0, \frac{1}{2})$.

If $0<\delta<\delta_0<\frac{1}{2}$, then the total variation of $f(t)-f'(t)$ in the interval $(\frac{1}{2}, 1-\delta_0)$ does not exceed

$$\int_{\frac{1}{2}}^{1-\delta_0}|df(t+\delta)-df(t)|+\int_{\frac{1}{2}}^{1-\delta_0}|df(t+\delta)-df'(t)|.$$

Since each term is arbitrarily small with δ, it follows that $f(t)-f'(t)$ is constant in $(\frac{1}{2}, 1-\delta_0)$. And since δ_0 is arbitrarily small, $f(t)-f'(t)$ is constant in $\frac{1}{2}\leqslant t<1$.

Similarly, it may be shown that $f(t)-f'(t)$ is constant in $0<t\leqslant\frac{1}{2}$. This completes the proof of the theorem.

The converse of Theorem 2, like that of Theorem 1, can be deduced from known results in the theory of Lebesgue integration.

3. Let $f(t)$ be defined for all real values of t in $(0, 1)$, and let us define $v(\tau;\ df)$ for $0<\tau<1$ as the total variation of $f(t+\tau)-f(t)$ in the interval $(0, 1-\tau)$ in which it is defined. If we assume that $f(t)$ is real and b.v., then we get

$$v(\tau;\ df)=v(\tau;\ dp)+v(\tau;\ dn),$$

but we shall not make this assumption. Our main result will be that if $f(t)$ is measurable and if, for every τ,

$$v(\tau;\ df)<\infty, \tag{3.1}$$

in other words, if $f(t+\tau)-f(t)$ is b.v. for every τ, then $f(t)$ is itself b.v.

In Lemma 1 we obtained a similar result for functions of a rational variable, but we assumed more about $v(\tau)$, namely that it is bounded. So long as we keep to the rational points we need more information about $v(\tau)$ than is given by (3.1). In fact let $u(x)$ be the function of period 1 which is equal to 1 when the fractional part of x lies between $\frac{1}{4}$ and $\frac{3}{4}$ and which is equal to 0 for all other values of x. Then the series

$$\sum_{n=1}^{\infty}\frac{u(n!\,t)}{2^n}$$

is uniformly convergent: denote its sum by $f_1(t)$. If $n!\,\tau$ is an integer then

$$f_1(t+\tau)-f_1(t)=\sum_{\nu<n}2^{-\nu}[u\{\nu!\,(t+\tau)\}-u(\nu!\,t)],$$

showing that $f_1(t+\tau)-f_1(t)$ is b.v. But the total variation of $f_1(t)$ over the points t of $(0, 1)$, for which $n!\,t$ is an integer, is equal to the total variation of

$$\sum_{\nu<n}\frac{u(\nu!\,t)}{2^\nu}$$

over the same set of points. Now, if $n > 1$, the total variation of the last term over these points is $(n-1)!\,2^{2-n}$, and hence that of $f_1(t)$ is at least

$$(n-1)!\,2^{2-n} - \sum_{\nu=1}^{n-2} \nu!\,2^{1-\nu}$$

which tends to infinity with n. Hence $f_1(t)$ is not of bounded variation over the rational points although, for each rational value of τ, $f_1(t+\tau)-f_1(t)$ is b.v. even when t is allowed to take irrational values.

The postulate that $f(t)$ is measurable is essential to our purpose: without this, no amount of information about $v(\tau)$ can tell us that $f(t)$ is b.v. In fact, if the continuum can be well-ordered, there is a real function $f_2(t)$ such that $v(\tau;\, df_2)$ vanishes identically, but which is not measurable, and *a fortiori* not b.v. For let $\{\theta_\alpha\}$ be an additive rational base (Hamel base) so that every real number can be expressed in one and only one way as a finite (possibly null) sum

$$t = \Sigma r_\alpha \theta_\alpha$$

of rational multiples of members of the base. If we divide the base into two parts $\{\theta_\beta'\}$ and $\{\theta_\gamma''\}$ and write, accordingly,

$$t = t' + t'',$$

where $$t' = \Sigma r_\beta' \theta_\beta', \quad t'' = \Sigma r_\gamma'' \theta_\gamma'',$$

and if we define $$f_2(t) = A' t' + A'' t'',$$

where $$A' \neq A'',$$

then, for each value of τ, we find that

$$f_2(t+\tau) - f_2(t) = A'\tau' + A''\tau'' = f_2(\tau),$$

and hence $$v(\tau;\, df_2) = 0. \tag{3.2}$$

On the other hand f_2 is not measurable. For, if E is a measurable set of mean density $1-\eta$ on an interval δ and if a is any real number, then the set H of points y which can be expressed in the form

$$y = x + ra, \tag{3.3}$$

where $x \subset E$ and r is rational, is of mean density at least $1-2\eta$ on every interval of length greater than $|\delta|$, and consequently, if E is a set of positive measure, then almost all points can be expressed in this form. Now, if $f_2(t)$ was measurable, we could choose a set E of positive measure so that the

oscillation of f_2 in E and the diameter of E itself would both be arbitrarily small. Taking $a = a'$, we should get

$$f_2(y)-A'y=f_2(x)-A'x \tag{3.4}$$

for all values of y of the form (3.3) and therefore almost everywhere. The oscillation of the right-hand side can be made as small as we please by choice of E and hence

$$f_2(y) = C'+A'y \quad \text{p.p.}$$

Similarly, by taking $a = a''$, we should get

$$f_2(y) = C''+A''y \quad \text{p.p.}$$

And these are incompatible since $A' \neq A''$.

4. Let $f(t)$, $f'(t)$ be finite measurable functions defined for all values of t in (0, 1), and let us define $v(\tau;\ df,\ df')$ for all values of τ in (0, 1) as the total variation of $f(t+\tau)-f'(t)$ in the interval $(0,\ 1-\tau)$ in which it is defined.

LEMMA 1. *If $f'(t)$ has a more than simple discontinuity* at t_0 where $0 \leqslant t_0 < 1$, then $v(\tau) = \infty$ almost everywhere in $(0,\ 1-t_0)$.*

To fix the ideas, suppose that $f'(t)$ is more than simply discontinuous to the right at t_0. If the oscillation of $f'(t)$ to the right at t_0 is finite and equal to $2k$, then we can find a sequence

$$b_1,\ a_1,\ b_2,\ a_2,\ \ldots \tag{4.1}$$

decreasing to zero and such that

$$t_0+b_1 < 1,$$

$$|f'(b_n+t_0)-f'(a_n+t_0)| > \tfrac{3}{2}k \quad \text{for all } n. \tag{4.2}$$

If the oscillation is infinite, we can choose such a sequence for any assigned value of k: in this case we take $k = 1$ and choose a sequence accordingly.

Let N be a positive integer and ϵ a positive number such that

$$\epsilon < \frac{k}{6}, \quad \epsilon < \frac{1-t_0}{N(2N+1)}. \tag{4.3}$$

* We say that $f(t)$ has a more than simple discontinuity to the right at t_0 and that the oscillation there is infinite whenever $f(t)$ is unbounded to the right at t_0, and, in particular, when $f(t) \to \pm\infty$ as $t \to t_0$ from the right.

Take a continuous function $\phi(t)$ such that

$$|f(t)-\phi(t)|<\epsilon$$

except in a set of measure at most ϵ; and let δ be a positive number such that

$$|\phi(t_1)-\phi(t_2)|<\epsilon$$

for every pair of numbers t_1, t_2 in the interval (0, 1) for which

$$|t_1-t_2|<\delta.$$

We can choose δ arbitrarily small and we shall suppose it so chosen that

$$\delta<\epsilon.$$

And we shall choose h so that

$$b_{h+1}<\delta.$$

Then, for every positive integer n,

$$|f(b_{h+n}+t)-f(a_{h+n}+t)|<3\epsilon$$

for every value of t in $(0,\ 1-\epsilon)$ except those of a set of measure at most 2ϵ. Whence

$$|f(b_{h+n}+t_0+\tau)-f(a_{h+n}+t_0+\tau)|<3\epsilon$$

for every value of τ in $(0,\ 1-t_0-\epsilon)$ except those of a set E_n of measure at most 2ϵ. Whence

$$|f(b_{h+n}+t_0+\tau)-f(a_{h+n}+t_0+\tau)|<3\epsilon \text{ for all } n\leqslant N \qquad (4.4)$$

for every value of τ in $(0,\ 1-t_0-\epsilon)$ except those of a set ΣE_n of measure at most $2N\epsilon$.

Now for every value of τ for which (4.4) holds, and therefore for a set of measure at least

$$(1-t_0)\left(1-\frac{1}{N}\right)$$

in the interval $(0,\ 1-t_0)$, we have

$$v(\tau)\geqslant\sum_{n=1}^{N}|\{f(b_{h+n}+t_0+\tau)-f'(b_{h+n}+t_0)\}-\{f(a_{h+n}+t_0+\tau)-f'(a_{h+n}+t_0)\}|$$

$$>N(\tfrac{3}{2}k-3\epsilon)>Nk.$$

Making $N\to\infty$, we get the desired result.

COROLLARY 1. *If $f(t)$ has a more than simple discontinuity at the point t_0, where $0<t_0\leqslant 1$, then $v(\tau)=\infty$ almost everywhere in $(0, t_0)$.*

COROLLARY 2. *$v(\tau)$ is measurable.*

If $f(t)$ has more than simple discontinuities arbitrarily near the point 1, then $v(\tau)=\infty$ p.p. If $f'(t)$ has more than simple discontinuities arbitrarily near the point 0, then $v(\tau)=\infty$ p.p. If $f(t)$, $f'(t)$ have only simple discontinuities in $0<t\leqslant 1$ and in $0\leqslant t<1$ respectively, then the discontinuities, if any, are countable and $v(\tau)$ is easily expressed as the limit of an increasing sequence of measurable functions.

In the remaining cases there is a maximum value of a such that $f(t)$ has only simple discontinuities in $1-a<t\leqslant 1$ and $f'(t)$ has only simple discontinuities in $0\leqslant t<a$. The discontinuities in these intervals are countable and hence $v(\tau)$ is measurable in $\tau>1-a$. And by the lemma $v(\tau)=\infty$ p.p. in $\tau<1-a$.

The corollary depends on the hypothesis that $f(t)$ and $f'(t)$ are measurable. If $\{\theta_\alpha\}$ is an additive rational base, and if we write

$$t=\Sigma r_\alpha\theta_\alpha=r_0\theta_0+t',$$

$$f_3(t)=r_0^2,$$

then we find
$$f_3(t+\tau)-f_3(t)=2\rho_0 r_0+\rho_0^2,$$

where
$$\tau=\Sigma\rho_\alpha\theta_\alpha=\rho_0\theta_0+\tau'.$$

Hence, if $\rho_0\neq 0$, the function $f_3(t+\tau)-f_3(t)$ is not measurable and therefore $v(\tau)=\infty$. But, if $\rho_0=0$, we get $v(\tau)=0$. Hence $v(\tau;\ df_3)$ is not measurable.

LEMMA 2. *If $f(t)$ has at most simple discontinuities in $t>0$ and, if $f'(t)$ is of unbounded variation in the neighbourhood of a point $t_0<1$, then $v(\tau)$ is totally unbounded on the interval $(0,\ 1-t_0)$, i.e., it is unbounded on every subinterval.*

It is sufficient to prove $v(\tau)$ unbounded on an interval (l, m) strictly interior to $(0,\ 1-t_0)$. To fix the ideas, suppose that $f'(t)$ is of unbounded variation to the right at t_0 and choose a sequence

$$a_1, a_2, \ldots$$

decreasing to zero and such that

$$a_1<1-t_0-m,$$

$$\sum_{n=1}^{\infty}|f'(a_n+t_0)-f'(a_{n+1}+t_0)|=\infty.$$

Then

$$\int_l^m v(\tau)\,d\tau \geqslant \sum_{n=1}^{\infty} \int_l^m |\{f(a_n+t_0+\tau)-f(a_{n+1}+t_0+\tau)\} - \{f'(a_n+t_0)-f'(a_{n+1}+t_0)\}|\,d\tau.$$

The n-th term is at least equal to

$$(m-l)\,|f'(a_n+t_0)-f'(a_{n+1}+t_0)|-\left|\int_l^m \{f(a_n+t_0+\tau)-f(a_{n+1}+t_0+\tau)\}\,d\tau\right|$$

and

$$\sum_{n=1}^{\infty}\left|\int_l^m \{f(a_n+t_0+\tau)-f(a_{n+1}+t_0+\tau)\}\,d\tau\right| \leqslant \sum_{n=1}^{\infty}\int_{l+t_0+a_{n+1}}^{l+t_0+a_n}|f(t)|\,dt+\int_{m+t_0+a_{n+1}}^{m+t_0+a_n}|f(t)|\,dt = \int_{l+t_0}^{l+t_0+a_1}|f(t)|\,dt+\int_{m+t_0}^{m+t_0+a_1}|f(t)|\,dt < \infty$$

since $f(t)$ is bounded in $t > l$. It follows that

$$\int_l^m v(\tau)\,d\tau = \infty$$

and hence that $v(\tau)$ is unbounded in (l, m).

COROLLARY. *If $f'(t)$ has at most simple discontinuities in $t < 1$ and if $f(t)$ is of unbounded variation in the neighbourhood of a point $t_0 > 0$, then $v(\tau)$ is totally unbounded on the interval $(0, t_0)$.*

THEOREM. *If $v(\tau)$ is finite on a set E of positive measure on the interval $(0, t_0)$, then $f(t)$ is of bounded variation on the interval $(t_0, 1)$ and $f'(t)$ is of bounded variation on the interval $(0, 1-t_0)$. Whence also $v(\tau)$ is bounded in $\tau \geqslant t_0$.*

Choose K so great that the set E_K of points τ at which $v(\tau) < K$ is of positive measure on $(0, t_0)$. Let t_1 be a point of density of the set E_K strictly interior to $(0, t_0)$, and let δ_0 be a positive number less than t_1 and such that the mean density of E_K on $(t_1-\delta_0, t_1)$ is at least $\frac{2}{3}$. If $\delta < \frac{1}{3}\delta_0$, then the points τ, $\tau+\delta$ belong simultaneously to E_K for all points τ of the interval $(t_1-\delta_0, t_1-\delta)$ save a set of measure at most $\frac{2}{3}\delta_0$. Hence there is at least one value of τ in this interval such that both τ and $\tau+\delta$ belong to E_K.

The total variation of $f(t+\delta)-f(t)$ on the interval $(t_1, 1-\delta)$ does not exceed

$$\int_{t_1}^{1-\delta}|df(t+\delta)-df'(t-\tau)|+\int_{t_1}^{1-\delta}|df(t)-df'(t-\tau)|$$

which is less than $2K$. If we write

$$\phi(t) = f(t_1+t-t_1t) \quad (0 \leqslant t \leqslant 1),$$

then $$v(\tau;\ d\phi) < 2K \text{ for } 0 < \tau < \frac{\frac{1}{3}\delta_0}{1-t_1},$$

and it follows easily that $v(\tau;\ d\phi)$ is bounded, in fact that

$$v(\tau;\ d\phi) < 2K\left(1+\frac{1-t_1}{\frac{1}{3}\delta_0}\right).$$

By Lemma 1, applied with $\phi(t)$ in place of both $f(t)$ and $f'(t)$, it follows that $\phi(t)$ has at most simple discontinuities; and from Lemma 2 it then follows that there is no point at which $\phi(t)$ is of unbounded variation. Hence $\phi(t)$ is b.v., *i.e.*, $f(t)$ is b.v. on $(t_1, 1)$, and *a fortiori* on $(t_0, 1)$.

And similarly $f'(t)$ is b.v. on $(0, 1-t_0)$.

COROLLARY. *If $f(t+\tau)-f(t)$ is of bounded variation for all τ, then $f(t)$ is itself b.v.*

INFINITESIMAL DEFORMATIONS OF SUB-SPACES IN A GENERAL METRICAL SPACE

By H. A. HAYDEN.

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In an earlier paper* I investigated certain infinitesimal deformations of a curve in a Riemannian space, and especially parallel deformations, *i.e.* deformations which displace the tangent parallelly at each point. In the present paper I consider infinitesimal deformations of a sub-space in a general metrical space (space with torsion).

The variations of the fundamental tensors associated with a sub-space (*e.g.* the metric tensor g'_{ij}, the Eulerian curvature tensor $\Omega_{ij}^{\ a}$, the absolute curvature tensor $R'^{i}_{\ jkl}$) have been given by Schouten for the case of Riemannian spaces†. The present methods of obtaining the variations of those tensors are however different. For example, the variation of the tensor $\Omega_{ij}^{\ a}$ is obtained here as the simplest application of the general formula (7).

1. Let V_n be a metrical space‡ with connection $\Gamma^{\gamma}_{\alpha\beta}$, metric tensor $g_{\alpha\beta}$, and torsion tensor $S_{\alpha\beta}^{\ \ \gamma}$, and let V_m be a sub-space of V_n, with connection Γ'^{k}_{ij}, metric tensor g'_{ij}, and torsion tensor $S'^{\ k}_{ij}$.

Let the points of V_m receive an infinitesimal displacement in V_n, given by

$$\delta x^{\alpha} = \epsilon v^{\alpha}, \tag{1}$$

where ϵ is an infinitesimal constant and v^{α} is a vector-field defined over V_m. We shall denote the deformed sub-space by $\bar{V}_m$, and the point of $\bar{V}_m$, obtained by displacing the point x^{α} of V_m, by $\bar{x}^{\alpha}$; so that

$$\bar{x}^{\alpha} = x^{\alpha} + \epsilon v^{\alpha}. \tag{2}$$

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* "Deformations of a curve in a Riemannian n-space . . .", *Proc. London Math. Soc.* (2), 32 (1931), 321–336.

† J. A. Schouten, "On infinitesimal deformations of a V_m in a V_n", *Proc. K. Akad. v. Wet., Amsterdam*, 31 (1928), 208–218.

‡ The notation employed throughout is that of my previous paper, "Sub-spaces of a space with torsion", *Proc. London Math. Soc.* (2), 34 (1932), 27–50.

We shall choose as parameter curves in $\bar{V}_m$ the curves which correspond, as a result of the deformation, to the parameter curves of V_m. We can then obtain the components of any tensor, located in $\bar{V}_m$, in other coordinates by the ordinary laws of tensor transformation in $\bar{V}_m$.

In the chosen coordinates,

$$\bar{\theta}_{i}^{\ \alpha} \equiv \frac{\partial \bar{x}^{\alpha}}{\partial x'^{i}} = \theta_{i}^{\ \alpha} + \epsilon \frac{\partial v^{\alpha}}{\partial x'^{i}}$$

$$= \theta_{i}^{\ \alpha} + \epsilon(\nabla_i v^{\alpha} - \Gamma^{\alpha}_{\beta\gamma} v^{\beta} \theta_{i}^{\ \gamma}). \tag{3}$$

$\bar{\theta}_{i}^{\ \alpha} - \theta_{i}^{\ \alpha}$ is not a tensor, of course; but if, for fixed i, $\bar{\theta}_{i}^{\ \alpha}$ is the vector at the point x^{α} parallel to the vector $\bar{\theta}_{i}^{\ \alpha}$ at $\bar{x}^{\alpha}$, then $\bar{\bar{\theta}}_{i}^{\ \alpha} - \theta_{i}^{\ \alpha}$ is a vector at the point x^{α}; while, on varying i, we obtain a tensor $\bar{\bar{\theta}}_{i}^{\ \alpha} - \theta_{i}^{\ \alpha}$. This is the "variation" of the tensor $\theta_{i}^{\ \alpha}$ due to the deformation ϵv^{α}. Since

$$\bar{\bar{\theta}}_{i}^{\ \alpha} = \bar{\theta}_{i}^{\ \alpha} + \epsilon \bar{\Gamma}^{\alpha}_{\beta\gamma} \bar{\theta}_{i}^{\ \beta} v^{\gamma} = \theta_{i}^{\ \alpha} + \epsilon(\nabla_i v^{\alpha} + 2 S_{\beta\gamma}^{\ \ \alpha} \theta_{i}^{\ \beta} v^{\gamma}) + O(\epsilon^2),$$

the first variation of $\theta_{i}^{\ \alpha}$ is given by

$$\boxed{\delta\theta_{i}^{\ \alpha} = \epsilon(\nabla_i v^{\alpha} + 2 S_{\beta\gamma}^{\ \ \alpha} \theta_{i}^{\ \beta} v^{\gamma}).} \tag{4}$$

2. The quantities $g_{\alpha\beta}$ and $S_{\alpha\beta}^{\ \ \gamma}$ behave like constants under the operator δ, since, although they vary as we pass from the point x^{α} to the point $\bar{x}^{\alpha}$, that variation is not a consequence of the deformation.

Thus the first variation of the metric tensor of V_m is

$$\delta g'_{ij} = \delta(g_{\alpha\beta} \theta_{i}^{\ \alpha} \theta_{j}^{\ \beta}) = 2 g_{\alpha\beta} \theta_{(i}^{\ \ \alpha} \delta\theta_{j)}^{\ \ \beta},$$

i.e.

$$\boxed{\delta g'_{ij} = 2\epsilon\{\theta_{(i}^{\ \ \alpha} \nabla_{j)} v_{\alpha} + 2\theta_{(ij)}^{\ \ \alpha\beta} S_{\beta\gamma\alpha} v^{\gamma}\}.} \tag{5}$$

The first variation of the torsion tensor of V_m is given by

$$\delta S'_{ijk} = \delta(S_{\alpha\beta\gamma} \theta_{ijk}^{\ \ \alpha\beta\gamma}) = S_{\alpha\beta\gamma}(\theta_{jk}^{\ \ \beta\gamma} \delta\theta_{i}^{\ \alpha} + \theta_{ki}^{\ \ \gamma\alpha} \delta\theta_{j}^{\ \beta} + \theta_{ij}^{\ \ \alpha\beta} \delta\theta_{k}^{\ \gamma}).$$

Since $S_{\alpha\beta\gamma}$ is skew-symmetric in α, β,

$$\delta S'_{ijk} = 2 S_{\alpha\beta\gamma}\{2\theta_{k[i}^{\ \ \gamma\alpha} \delta\theta_{j]}^{\ \ \beta} + \theta_{[ij]}^{\ \ \alpha\beta} \delta\theta_{k}^{\ \gamma}\}$$

$$= 2\epsilon S_{\alpha\beta\gamma}[2\theta_{k[i}^{\ \ \gamma\alpha} \nabla_{j]} v^{\beta} + \theta_{[ij]}^{\ \ \alpha\beta} \nabla_k v^{\gamma} + 2 v^{\mu}\{2\theta_{[ij]k}^{\ \ \ \ \alpha\lambda\gamma} S_{\lambda\mu}^{\ \ \beta} + \theta_{[ij]k}^{\ \ \ \ \alpha\beta\lambda} S_{\lambda\mu}^{\ \ \gamma}\}],$$

i.e.

$$\boxed{\delta S'_{ijk} = 2\epsilon[S_{\alpha\beta\gamma}\{2\theta_{k[i}^{\ \ \gamma\alpha} \nabla_{j]} v^{\beta} + \theta_{[ij]}^{\ \ \alpha\beta} \nabla_k v^{\gamma}\} + 2\theta_{[ij]k}^{\ \ \ \ \alpha\beta\gamma} v^{\mu}\{2 S_{\beta\mu}^{\ \ \lambda} S_{\alpha\lambda\gamma} + S_{\gamma\mu}^{\ \ \lambda} S_{\alpha\beta\lambda}\}].}$$

(6)

3. We now proceed to find a formula for

$$(\delta\nabla_k - \nabla_k\delta)\, T^{a i_1 \cdots i_r}_{j_1 \cdots j_s},$$

where $T^{a i_1 \cdots i_r}_{j_1 \cdots j_s}$ $(a = 1, \ldots, n;\ i_1, \ldots, j_s = 1, \ldots, m)$ is an arbitrary tensor of the type indicated by its indices.

By means of such a formula we shall be able to write down immediately, by successive applications, the variations of the tensors $\Omega_{ij}{}^a$, $\Omega_{ijk}{}^a$, etc., since we already know $\delta\theta_i{}^a$.

When there is no ambiguity, we shall write $T^{a i_1 \cdots i_r}_{j_1 \cdots j_s}$ as $T^{a \cdots}_{\cdots}$. If $\bar{T}^{a \cdots}_{\cdots}$ is the tensor correspondingly defined at points of $\bar{V}_m$, then, so far as first order terms are concerned,

$$\bar{T}^{a \cdots}_{\cdots} = T^{a \cdots}_{\cdots} - \epsilon\Gamma^a_{\beta\gamma}\, T^{\beta \cdots}_{\cdots}\, v^\gamma + \delta T^{a \cdots}_{\cdots}.$$

(This is really our definition of $\delta T^{a \cdots}_{\cdots}$.)

Thus

$$\delta\nabla_k\, T^{a \cdots}_{\cdots} = \bar{\nabla}_k\, \bar{T}^{a \cdots}_{\cdots} - \nabla_k\, T^{a \cdots}_{\cdots} + \epsilon\Gamma^a_{\beta\gamma}\nabla_k\, T^{\beta \cdots}_{\cdots}\, v^\gamma,$$

where $\bar{\nabla}_k$ denotes covariant differentiation in $\bar{V}_m$. If $\bar{\Gamma}'^k_{ij}$ is the connection of $\bar{V}_m$, Γ'^k_{ij} and $\bar{\Gamma}'^k_{ij}$ obey the same law of transformation, for coordinate-transformations in V_m, and hence it follows that $\delta\Gamma'^k_{ij}$ is a tensor. We shall indicate it by $\epsilon\Pi^k_{ij}$.

For brevity, we shall denote by $\binom{l}{i_\rho}$ the operation of replacing i_ρ by l, so that

$$\binom{l}{i_\rho} T^{a \cdots}_{\cdots} = T^{a i_1 \cdots i_{\rho-1} l i_{\rho+1} \cdots i_r}_{j_1 \cdots j_s};$$

and similarly by $\binom{j_\rho}{l}$ we shall denote the operation of replacing j_ρ by l.

Then, on substituting the expression for $\bar{T}^{a \cdots}_{\cdots}$ in the expression for $\delta\nabla_k\, T^{a \cdots}_{\cdots}$, we find that

$$\begin{aligned}
\delta\nabla_k\, T^{a \cdots}_{\cdots} = {} & \frac{\partial}{\partial x'^k}\{\delta T^{a \cdots}_{\cdots} - \epsilon\Gamma^a_{\beta\gamma}\, T^{\beta \cdots}_{\cdots} v^\gamma\} + \epsilon\Gamma^a_{\beta\gamma}\left(\nabla_k\, T^{\beta \cdots}_{\cdots} v^\gamma + T^{\beta \cdots}_{\cdots}\frac{\partial v^\gamma}{\partial x'^k}\right) \\
& + \Gamma^a_{\beta\gamma}\theta_k{}^\gamma\{\delta T^{\beta \cdots}_{\cdots} - \epsilon\Gamma^\beta_{\lambda\mu}\, T^{\lambda \cdots}_{\cdots} v^\mu\} + \epsilon T^{\beta \cdots}_{\cdots}\theta_k{}^\gamma\frac{\partial\Gamma^a_{\beta\gamma}}{\partial x^\lambda}\, v^\lambda \\
& + \sum_{p=1}^{r}\Gamma'^{i_p}_{lk}\binom{l}{i_\rho}\{\delta T^{a \cdots}_{\cdots} - \epsilon\Gamma^a_{\beta\gamma}\, T^{\beta \cdots}_{\cdots} v^\gamma\} + \sum_{p=1}^{s}\Gamma'^{l}_{j_\rho k}\binom{j_\rho}{l}\{\delta T^{a \cdots}_{\cdots} - \epsilon\Gamma^a_{\beta\gamma}\, T^{\beta \cdots}_{\cdots} v^\gamma, \\
& + \epsilon\sum_{p=1}^{r}\Pi^{i_\rho}_{lk}\binom{l}{i_\rho}\, T^{a \cdots}_{\cdots} - \epsilon\sum_{p=1}^{s}\Pi^{l}_{j_\rho k}\binom{j_\rho}{l}\, T^{a \cdots}_{\cdots}.
\end{aligned}$$

Also

$$\nabla_k \delta T^{\alpha\cdots}_{\cdots} = \frac{\partial}{\partial x'^k}\delta T^{\alpha\cdots}_{\cdots} + \Gamma^{\alpha}_{\beta\gamma}\delta T^{\beta\cdots}_{\cdots}\theta_k^{\ \gamma} + \sum_{p=1}^{r}\Gamma'^{i_p}_{lk}\binom{l}{i_p}\delta T^{\alpha\cdots}_{\cdots} - \sum_{p=1}^{s}\Gamma'^{l}_{j_p k}\binom{j_p}{l}\delta T^{\alpha\cdots}_{\cdots}.$$

On reduction we find the desired formula, viz.:

$$(\delta\nabla_k - \nabla_k\delta)\,T^{\alpha\cdots}_{\cdots} = \epsilon\Big\{R^{\alpha}_{\cdot\beta\gamma\delta}T^{\beta\cdots}_{\cdots}v^{\gamma}\theta_k^{\ \delta} + \sum_{p=1}^{r}\Pi_{lk}^{\ i_p}\binom{l}{i_p}T^{\alpha\cdots}_{\cdots} - \sum_{p=1}^{s}\Pi_{j_p k}^{\ l}\binom{j_p}{l}T^{\alpha\cdots}_{\cdots}\Big\}. \tag{7}$$

4. The next step, obviously, is to find $\Pi_{ij}^{\ k}$, the variation of the connection of V_m.

In the Riemannian case ($\overset{0}{V}_m$ in $\overset{0}{V}_n$) this can easily be calculated directly, since the coefficients of connection are then the Christoffel symbols. It is easily shown*, in fact, that, for a $\overset{0}{V}_m$ in $\overset{0}{V}_n$,

$$\begin{aligned}\Pi_{ij}^{\ k} &= g'^{kl}\{2\nabla_{(i}\theta_{(j)}^{\ \alpha}\nabla_{l)}v_{\alpha} - \nabla_l\theta_{(i}^{\ \alpha}\nabla_{j)}v_{\alpha}\}\\ &= \theta^k_{\cdot\alpha}\nabla_j\nabla_i v^{\alpha} + g'^{kl}\Omega_{ji\alpha}\nabla_l v^{\alpha} + g'^{kl}\theta_{jil}^{\ \ \ \alpha\beta\gamma}R_{\gamma\beta\delta\alpha}v^{\delta}.\end{aligned}$$

In the general (non-Riemannian) case direct calculation is very tedious, but it can be avoided by the following method.

Putting $$T^{\alpha\cdots}_{\cdots} \equiv \theta_j^{\ \alpha}$$

in (7), we have immediately

$$\delta\Omega_{ij}^{\ \ \alpha} = \epsilon(P_{ij}^{\ \ \alpha} - \theta_k^{\ \alpha}\Pi_{ji}^{\ \ k}), \tag{8}$$

where $$P_{ij}^{\ \ \alpha} \equiv \nabla_i\nabla_j v^{\alpha} + R^{\alpha}_{\cdot\beta\gamma\delta}\theta_{ij}^{\ \ \delta\beta}v^{\gamma} + 2\nabla_i(S_{\beta\gamma}^{\ \ \alpha}\theta_j^{\ \beta}v^{\gamma}). \tag{9}$$

From (8), $$\theta^k_{\cdot\alpha}\delta\Omega_{ij}^{\ \ \alpha} = \epsilon(\theta^k_{\cdot\alpha}P_{ij}^{\ \ \alpha} - \Pi_{ij}^{\ \ k}).$$

But, since $$\theta^k_{\cdot\alpha}\Omega_{ij}^{\ \ \alpha} = 0,$$

$$\begin{aligned}\theta^k_{\cdot\alpha}\delta\Omega_{ij}^{\ \ \alpha} &= -\Omega_{ij}^{\ \ \alpha}\delta\theta^k_{\cdot\alpha} = -\Omega_{ij}^{\ \ \alpha}g_{\alpha\beta}\delta(g'^{kl}\theta_l^{\ \beta})\\ &= -\Omega_{ij\beta}(g'^{kl}\delta\theta_l^{\ \beta} + \theta_l^{\ \beta}\delta g'^{kl})\\ &= -g'^{kl}\Omega_{ij\beta}\delta\theta_l^{\ \beta}\\ &= -\epsilon g'^{kl}\Omega_{ij\alpha}(\nabla_l v^{\alpha} + 2S_{\beta\gamma}^{\ \ \alpha}\theta_l^{\ \beta}v^{\gamma}),\end{aligned}$$

* E. Bortolotti, "Scostamento geodetico e sue generallizzazioni", *Giornale di Matem.*, 66 (1928), 153–191; 176, eqn. (94).

whence $$\Pi_{ji}^{\cdot\cdot k} = \theta^k_{\cdot\alpha} P_{ij}^{\cdot\cdot\alpha} + g'^{kl}\Omega_{ij\alpha}(\nabla_l v^\alpha + 2S_{\beta\gamma}^{\cdot\cdot\alpha}\theta_l^{\cdot\beta} v^\gamma), \tag{10}$$

i.e.

$$\boxed{\begin{aligned}\Pi_{ji}^{\cdot\cdot k} = \theta^k_{\cdot\alpha}\nabla_i\nabla_j v^\alpha + g'^{kl}\Omega_{ij\alpha}\nabla_l v^\alpha + g'^{kl}\theta_{ijl}^{\cdot\cdot\cdot\alpha\beta\gamma} R_{\gamma\beta\delta\alpha} v^\delta \\ + 2\Omega_{ij\alpha}\theta^{k\beta} S_{\beta\gamma}^{\cdot\cdot\alpha} v^\gamma + 2\theta^k_{\cdot\alpha}\nabla_i(S_{\beta\gamma}^{\cdot\cdot\alpha}\theta_j^{\cdot\beta} v^\gamma).\end{aligned}} \tag{10a}$$

Also, since $$\theta_k^{\cdot\alpha}\theta^k_{\cdot\beta} = g_\beta^{\cdot\alpha} - C_\beta^{\cdot\alpha},$$

we have*, from (8) and (10),

$$\boxed{\delta\Omega_{ij}^{\cdot\cdot\alpha} = \epsilon\{C_\beta^{\cdot\alpha} P_{ij}^{\cdot\cdot\beta} - \theta^{k\alpha}\Omega_{ij\beta}(\nabla_k v^\beta + 2S_{\lambda\mu}^{\cdot\cdot\beta}\theta_k^{\cdot\lambda} v^\mu)\}.} \tag{11}$$

The term $\epsilon C_\beta^{\cdot\alpha} P_{ij}^{\cdot\cdot\beta}$ is the component of $\delta\Omega_{ij}^{\cdot\cdot\alpha}$ normal to V_m, while the other term is the component tangential to V_m.

5. We can express v^α in the form

$$v^\alpha = u^i\theta_i^{\cdot\alpha} + \sum_{r=1}^{q}\psi_r\chi_r^\alpha, \tag{12}$$

where u^i is the V_m component of v^α, and $\underset{r}{\chi}{}^\alpha$ $(r = 1, \ldots, q;\ q \equiv n-m)$ are q mutually orthogonal unit vector fields at points of V_m, each normal to V_m, and the ψ's are scalar functions of the x'^i. Then, since

$$\theta_i^{\cdot\alpha} v_\alpha = u_i,$$

$$\theta_i^{\cdot\alpha}\nabla_j v_\alpha = \nabla_j u_i - \Omega_{ji}^{\cdot\cdot\alpha} v_\alpha,$$

whence

$$\boxed{\delta g'_{ij} = 2\epsilon\{\nabla_{(i}u_{j)} - \Omega_{(ij)}^{\cdot\cdot\alpha} v_\alpha + 2\theta_{(ij)}^{\cdot\cdot\alpha\beta} S_{\beta\gamma\alpha} v^\gamma\}.} \tag{13}$$

We shall write, for brevity,

$$\delta g'_{ij} = 2\epsilon\phi_{ij},$$

where $$\phi_{ij} \equiv \nabla_{(i}u_{j)} - \Omega_{(ij)}^{\cdot\cdot\alpha} v_\alpha + 2S_{\beta\gamma\alpha}\theta_{(ij)}^{\cdot\cdot\alpha\beta} v^\gamma. \tag{14}$$

6. If ds is the length of an infinitesimal element dx'^i in V_m,

$$ds^2 = g'_{ij}\,dx'^i\,dx'^j,$$

and $$\frac{\delta ds}{ds} = \tfrac{1}{2}\frac{\delta ds^2}{ds^2} = \epsilon\phi_{ij}\frac{dx'^i}{ds}\frac{dx'^j}{ds} = \epsilon\phi_{ij}\xi'^i\xi'^j,$$

* For the case of $\overset{0}{V}_m$ in $\overset{0}{V}_n$, *cf.* Schouten, *loc. cit.*, 211.

where ξ'^i is the unit vector in the direction dx'^i. This is the "extension" of V_m in the direction ξ'^i at the point considered, due to the deformation. If dv is an infinitesimal volume-element of V_m,

$$dv = \sqrt{g'}\, dx'^1 dx'^2 \dots dx'^m,$$

and

$$\delta dv = \frac{1}{2\sqrt{g'}}\, \delta g' \,.\, dx'^1 \dots dx'^m,$$

whence

$$\frac{\delta dv}{dv} = \frac{1}{2g'}\, \delta g' = \frac{1}{2g'}\, \frac{\partial g'}{\partial g'_{ij}}\, \delta g'_{ij} = \epsilon\phi,$$

where

$$\phi \equiv g'^{ij}\, \phi_{ij}.$$

This is the "dilatation" of V_m at the point.

If $\underset{h}{\xi'^i}$ $(h = 1, \dots, m)$ are any m mutually orthogonal directions in V_m, at a point of V_m, the sum of the extensions in those directions is equal to $\epsilon\phi_{ij} \sum_{h=1}^{m} \underset{h}{\xi'^i}\, \underset{h}{\xi'^j} = \epsilon\phi_{ij}\, g'^{ij} = \epsilon\phi$, which is independent of the particular directions and is equal to the dilatation at the point*.

The directions of maximum or minimum extension at a point are the principal directions of the tensor ϕ_{ij}. They are mutually orthogonal (if distinct), being given by the equations

$$(\phi_{ij} - e g'_{ij})\, \xi'^i = 0,$$

where e is a root of the equation

$$|\phi_{ij} - e g'_{ij}| = 0.$$

The directions of zero extension at a point are given by

$$\phi_{ij}\, \xi'^i\, \xi'^j = 0,$$

and are the generators of an $(m-1)$-dimensional quadratic cone with vertex at the point.

The torsion tensor $S_{\alpha\beta\gamma}$ is always skew-symmetric in α, β. If it is also skew-symmetric in β, γ (and hence also in α, γ), the connection sometimes is called† "Euclidean". A V_n with a Euclidean connection will be denoted

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* For a $\overset{0}{V}_2$ in E_3, *cf.* Weatherburn, "On small deformations of surfaces and of thin elastic shells", *Quart. J. of Math.*, 50 (1925), 272–296. *Added in proof:* Since this paper was written a paper has appeared by A. G. Walker, "On small deformations of sub-spaces of a flat space" (*i.e.* $\overset{0}{V}_m$ in E_n), *Proc. Edin. Math. Soc.* (2), 3 (1932), 77–86, *q.v.*

† E. Bortolotti, "Su una classe di connessioni euclidee in V_3", *Rend. di Palermo*, 51 (1927), 98–105. Such a space is not to be confused, of course, with a "Euclidean space" *i.e.* one whose *metric* is Euclidean.

by U_n. A characteristic feature of a U_n is that its geodesics (*i.e.* auto-parallel curves, or "paths") are minimal curves, and *vice versa*. (A Riemannian space is a particular case of a U_n.) It is clear from the definition of S'_{ijh} that every sub-space of a U_n is a U_m.

For a deformation of a U_m in a U_n,

$$\phi_{ij} = \nabla_{(i} u_{j)} - \Omega_{(ij)}{}^{a} v_a.$$

It follows that the extension of a U_m in a U_n under any normal deformation is zero in every asymptotic direction.

The extension is zero in every direction at every point of V_m if ϕ_{ij} vanishes; in that case the deformation is what is usually called a "motion". Every deformation of a U_m in a U_n which is perpendicular to U_m and to its first normal space at each point is a motion. If $m_1 < q$ (where m_1 is the dimensionality of the first normal space), U_m admits of an infinite number of normal motions; also, since m_1 is essentially less than or equal to $\frac{1}{2}m(m+1)$, it follows that U_m admits of an infinite number of normal motions if $n-m > \frac{1}{2}m(m+1)$, *i.e.* if $n > \frac{1}{2}m(m+3)$. Hence *a U_m admits of an infinite number of normal motions in a U_n if $n > \frac{1}{2}m(m+3)$.*

It is clear, of course, that a U_{n-1} in a U_n does not admit of a normal motion; also, every normal deformation of a geodesic sub-space of a U_n is a motion.

The extension at a point of V_m is the same in all directions, though possibly varying from point to point, if

$$\phi_{ij} = \sigma g'_{ij},$$

where σ is a scalar function of the x'^i. In that case the deformation is conformal.

The equations for a conformal normal deformation of a U_m in U_n are

$$\sum_{r=1}^{m_1} \omega_{(ij)} \overset{r}{\psi} = \sigma g'_{ij},$$

where $\overset{r}{\chi}{}^a$ $(r = 1, \ldots, m_1)$ are m_1 mutually orthogonal normals determining the first normal space. Since

$$\sigma = \frac{\phi}{m} = \frac{1}{m} \sum_{r=1}^{m_1} \omega \overset{r}{\psi},$$

we can write these equations as

$$\sum_{r=1}^{m_1} \left\{ \omega_{(ij)} - \frac{1}{m} \omega g'_{ij} \right\} \overset{r}{\psi} = 0 \quad (i, j = 1, \ldots, m). \quad (15)$$

On account of symmetry in i, j, these are $\frac{1}{2}m(m+1)$ equations for the m_1

quantities $\underset{r}{\psi}$ $(r=1, \ldots, m_1)$, and, since $m_1 \leqslant \frac{1}{2}m(m+1)$, we must have, in general,

$$\underset{r}{\psi} = 0 \quad (r=1, \ldots, m_1),$$

whence $$\sigma = 0.$$

Thus, *in general, the only conformal normal deformations of a U_m in a U_n are motions.*

Equations (15) are satisfied identically if, and only if,

$$\overset{r}{\omega}_{(ij)} = \overset{r}{\sigma} g'_{ij} \quad (r=1, \ldots, m_1),$$

the $\overset{r}{\sigma}$ being scalar functions over U_m; *i.e.* if, and only if, every point of U_m is an umbilic. Hence *every normal deformation of a U_m in U_n is conformal if, and only if, U_m is totally umbilical.*

7. The tangent m-space to V_m is determined by the m vectors θ_i^α $(i=1, \ldots, m)$, and that to $\bar{V}_m$ by the m vectors $\bar{\theta}_i^\alpha$ $(i=1, \ldots, m)$. The m vectors $\bar{\bar{\theta}}_i^\alpha$ $(i=1, \ldots, m)$ determine a space at the point x^α which we may speak of as parallel to the space at $\bar{x}^\alpha$ defined by the vectors $\bar{\theta}_i^\alpha$. We are justified by the fact that, if any vector at $\bar{x}^\alpha$ tangential to V_m is displaced parallelly to the point x^α, the displaced vector lies in the space defined by the vectors $\bar{\bar{\theta}}_i^\alpha$, as is easily verified. We shall say, then, that V_m is displaced parallelly, if the space determined by the vectors $\bar{\bar{\theta}}_i^\alpha$ is identical with the tangent m-space to V_m, at the same point, and such a deformation we shall call a "parallel deformation" of V_m in V_n. This requires that the $\bar{\bar{\theta}}_i^\alpha$, and hence also the $\delta\theta_i^\alpha$, are homogeneous linear functions of the θ_i^α; that is, that the vectors $\delta\theta_i^\alpha$ $(i=1, \ldots, m)$ are all tangential to V_m.

Necessary and sufficient conditions for a parallel deformation of V_m in V_n are, therefore,

$$\underset{r}{\chi^\alpha}(\nabla_i v_\alpha + 2S_{\beta\gamma\alpha}\theta_i^\beta v^\gamma) = 0 \qquad (16)$$

$$(r=1, \ldots, q;\ i=1, \ldots, m).$$

Since $$\underset{r}{\chi^\alpha} v_\alpha = \underset{r}{\psi},$$

we have

$$\underset{r}{\chi^\alpha}\nabla_i v_\alpha = \nabla_i \underset{r}{\psi} - v_\alpha \nabla_i \underset{r}{\chi^\alpha}$$

$$= \nabla_i \underset{r}{\psi} - v_\alpha \left(\overset{r}{\omega}_{ij}\theta^{j\alpha} + \sum_{s=1}^{q} \overset{rs}{v}_i \underset{s}{\chi^\alpha} \right)$$

$$= \nabla_i \underset{r}{\psi} - \overset{r}{\omega}_{ij} u^j - \sum_{s=1}^{q} \overset{rs}{v}_i \underset{s}{\psi};$$

and the conditions (16) are

$$\nabla_i \underset{r}{\psi} = \left(\overset{r}{\omega}_{ij} - 2S_{\beta\gamma\alpha}\theta_{ij}^{\ \ \beta\gamma}\underset{r}{\chi^\alpha}\right)w^j + \sum_{s=1}^{q}\left(\overset{rs}{v}_i - 2S_{\beta\gamma\alpha}\theta_i^{\ \beta}\underset{r}{\chi^\alpha}\underset{s}{\chi^\gamma}\right)\underset{s}{\psi}.$$

Also
$$S_{\beta\gamma}^{\ \ \ \alpha}\theta_{ij}^{\ \ \beta\gamma} = S'_{ij}^{\ \ k}\theta_k^{\ \alpha} - \Omega_{[ij]}^{\ \ \ \alpha}, \tag{17}$$

whence
$$S_{\beta\gamma\alpha}\theta_{ij}^{\ \ \beta\gamma}\underset{r}{\chi^\alpha} = -\Omega_{[ij]\alpha}\underset{r}{\chi^\alpha} = \overset{r}{\omega}_{[ij]}.$$

Thus necessary and sufficient conditions for a parallel deformation are

$$\boxed{\nabla_i \underset{r}{\psi} = \overset{r}{\omega}_{ji}w^j + \sum_{s=1}^{q}\left(\overset{rs}{v}_i - 2S_{\beta\gamma\alpha}\theta_i^{\ \beta}\underset{r}{\chi^\alpha}\underset{s}{\chi^\gamma}\right)\underset{s}{\psi}.} \tag{18}$$

The most interesting case is that in which the deformation is normal to V_m at each point*.

The equations for a normal parallel deformation are

$$\nabla_i \underset{r}{\psi} = \sum_{s=1}^{q}\left(\overset{rs}{v}_i - 2S_{\beta\gamma\alpha}\theta_i^{\ \beta}\underset{r}{\chi^\alpha}\underset{s}{\chi^\gamma}\right)\underset{s}{\psi}. \tag{19}$$

Since
$$\nabla_{[i}\nabla_{j]}\underset{r}{\psi} = 2S'_{ij}^{\ \ k}\nabla_k\underset{r}{\psi},$$

the integrability conditions of (19) are

$$\sum_{t=1}^{q}\left\{\nabla_{[i}\overset{rt}{f}_{j]} - \sum_s \overset{rs}{f}_{[i}\overset{st}{f}_{j]} - 2S'_{ij}^{\ \ k}\overset{rt}{f}_k\right\}\underset{t}{\psi} = 0,$$

where
$$\overset{rs}{f}_i \equiv \overset{rs}{v}_i - 2S_{\beta\gamma\alpha}\theta_i^{\ \beta}\underset{r}{\chi^\alpha}\underset{s}{\chi^\gamma}.$$

The conditions for complete integrability are

$$\boxed{\nabla_{[i}\overset{rt}{f}_{j]} - \sum_s \overset{rs}{f}_{[i}\overset{st}{f}_{j]} - 2S'_{ij}^{\ \ k}\overset{rt}{f}_k = 0} \tag{20}$$

$$(r, t = 1, \ldots, q;\ i, j = 1, \ldots, m).$$

If these are satisfied, V_m admits of normal parallel deformations arbitrarily assigned at any one point.

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* It is seen, from (18), that there does not exist, *in general*, a tangential parallel deformation. For example, for a hypersurface ($m = n-1$) such a deformation exists only if the tensor ω_{ij} is of rank less than or equal to $n-2$.

For the Riemannian case ($\overset{0}{V}_m$ in $\overset{0}{V}_n$) they can be expressed in a form free from derivatives; for, in virtue of Kühne's equation, they become

$$(\tfrac{1}{2} R_{\alpha\beta\gamma}^{\quad\cdot\cdot\delta}\, \theta_{ij}^{\quad\cdot\cdot\alpha\beta} - \Omega_{[i}^{\quad\cdot\cdot k\delta}\, \Omega_{j]k\gamma})\, \underset{r}{\chi^{\gamma}}\, \underset{t}{\chi_{\delta}} = 0. \tag{21}$$

For a hypersurface all the $\overset{rs}{v}_i$ vanish and there is a single vector f_i, viz.

$$f_i = -2 S_{\beta\gamma\alpha}\, \theta_i^{\quad\cdot\beta}\, \chi^{\alpha}\, \chi^{\gamma}.$$

This vanishes for every hypersurface if V_n has a Euclidean connection, and in that case (20) are satisfied identically. Thus *every U_{n-1} in a U_n admits of a normal parallel deformation, arbitrarily assigned at any one point.*

A normal deformation of constant length* of a $\overset{0}{V}_{n-1}$ in a $\overset{0}{V}_n$ is a parallel deformation, as is evident from (19). This is not so for a V_{n-1} in a V_n unless

$$S_{\beta\gamma\alpha}\, \theta_i^{\quad\cdot\beta}\, \chi^{\alpha}\, \chi^{\gamma} = 0,$$

i.e.

$$S_{\beta\gamma\alpha}\, \theta_i^{\quad\cdot\beta} (g^{\alpha\gamma} - \theta_{jk}^{\quad\cdot\cdot\alpha\gamma}\, g'^{jk}) = 0,$$

i.e.

$$S'^{\quad\cdot\cdot j}_{ij} = \theta_i^{\quad\cdot\alpha}\, S_{\alpha\beta}^{\quad\cdot\cdot\beta}.$$

Hence *a constant normal deformation of a V_{n-1} in V_n is a parallel deformation if, and only if, the vector $S'^{\quad\cdot\cdot j}_{ij}$ is the V_{n-1}-component of the vector $S_{\alpha\beta}^{\quad\cdot\cdot\beta}$; in that case, the only normal parallel deformations are those of constant length.* This condition is satisfied by any U_{n-1} in a U_n, since in that case the vectors $S_{\alpha\beta}^{\quad\cdot\cdot\beta}$ and $S'^{\quad\cdot\cdot j}_{ij}$ vanish.

Returning to general deformations of a V_{n-1} in V_n, we see that equations (18) are only $n-1$ in number, viz.

$$\nabla_i \psi = \omega_{ij}\, u^j - 2 S_{\beta\gamma\alpha}\, \theta_i^{\quad\cdot\beta}\, C^{\alpha\gamma}\, \psi,$$

so that, if ψ is given arbitrarily over V_{n-1}, we can solve for the u^i. Thus *a V_{n-1} in V_n admits of an infinite number of parallel deformations, but the deformation is, in general, completely determined by its normal component.*

In the case of a U_{n-1} in U_n, if the normal component is of constant length the deformation must be wholly normal, unless the rank of ω_{ij} is less than or equal to $n-2$.

8. Individual directions in V_m are not, in general, displaced parallelly under a parallel deformation of V_m. But there are, in general, m directions

* For brevity we shall refer to such a deformation, somewhat loosely, as a "constant normal deformation".

at each point which are displaced parallelly. For we can write the equations of a parallel deformation of V_m in the form

$$\delta\theta_i^{\ \alpha} \equiv \epsilon(\nabla_i v^\alpha + 2S_{\beta\gamma}^{\ \ \alpha}\,\theta_i^{\ \beta} v^\gamma) = \epsilon b_i^{\ j}\,\theta_j^{\ \alpha}, \tag{22}$$

where $b_i^{\ j}$ is some tensor.

The direction dx'^i, at a point x'^i of V_m, is displaced parallelly if δdx^α is along dx^α. But, under the deformation (22),

$$\delta dx^\alpha = \delta(\theta_i^{\ \alpha}\,dx'^i) = dx'^i\,\delta\theta_i^{\ \alpha} = b_i^{\ j}\,\theta_j^{\ \alpha}\,dx'^i,$$

and this is along dx^α if, and only if,

$$b_i^{\ j}\,\theta_j^{\ \alpha}\,dx'^i = \sigma dx^\alpha = \sigma\theta_j^{\ \alpha}\,dx'^i,$$

where σ is a scalar; that is, if

$$(b_{ij} - \sigma g'_{ij})\,dx'^i = 0. \tag{23}$$

This gives, in general, m directions at the point, corresponding to the m roots of the equation

$$|b_{ij} - \sigma g'_{ij}| = 0. \tag{24}$$

(If two roots are equal there corresponds a plane of directions, and so on, in the usual way.)

For the deformation (22),

$$\phi_{ij} = \theta_{(i}^{\ \ \alpha}\,b_{j)}^{\ \ k}\,\theta_{k\alpha} = b_{(ij)}, \tag{25}$$

and the directions of principal extension at a point are given by

$$(b_{(ij)} - e g'_{ij})\,dx'^i = 0.$$

Thus the directions which are displaced parallelly, under a parallel deformation of V_m, coincide with the directions of principal extension provided the tensor b_{ij} is symmetric.

For a $\overset{0}{V}_m$ in $\overset{0}{V}_n$ this condition is equivalent to

$$\nabla_{[i} u_{j]} = 0;$$

that is, the vector u_i must be a gradient.

9. Directions other than those given by (23) undergo a "rotation" when V_m is displaced parallelly, and we shall call the deformation "*irrotational*" if *every* direction in V_m is displaced parallelly. (An irrotational deformation is of necessity, of course, a parallel deformation.)

A necessary and sufficient condition for an irrotational deformation is

$$b_{ij} = \sigma g'_{ij},$$

where σ is a scalar function of the x'^i.

This is equivalent to the two conditions

$$b_{(ij)} = \sigma g'_{ij}, \quad b_{[ij]} = 0, \tag{26}$$

and, from (25), we see that *every irrotational deformation is conformal.*

The first of the conditions (26) is satisfied by any conformal parallel deformation, while the second is equivalent to

$$\nabla_{[i} u_{j]} - \Omega_{[ij]}{}^{\alpha} v_{\alpha} - 2S_{\beta\gamma\alpha}\theta_{[ij]}{}^{\alpha\beta} v^{\gamma} = 0,$$

which for a $\overset{0}{V}_m$ in $\overset{0}{V}_n$ is simply

$$\operatorname{curl} u_i = 0.$$

In particular, *every normal conformal parallel deformation of a* $\overset{0}{V}_m$ *in* $\overset{0}{V}_n$ *is irrotational.*

The variation of the connection of V_m under a parallel deformation (22) is, from (10) and (22),

$$\Pi_{ji}{}^{k} = \theta^{k}_{\cdot\alpha} P_{ij}{}^{\alpha},$$

and we find that $P_{ij}{}^{\alpha}$, in this case, is equal to

$$b_j{}^{k}\,\Omega_{ik}{}^{\alpha} + \theta^{k\alpha}\nabla_i b_{jk} + R^{\alpha}_{\cdot\beta\gamma\delta}\theta_{ij}{}^{\delta\beta} v^{\gamma},$$

whence

$$\Pi_{ji}{}^{k} = \nabla_i b_j{}^{k} + R_{\alpha\beta\gamma\delta}\theta_{ij}{}^{\delta\beta}\theta^{k\alpha} v^{\gamma}.$$

If the deformation is irrotational,

$$b_j{}^{k} = \sigma g'_j{}^{k},$$

and

$$\Pi_{ji}{}^{k} = g'_i{}^{k}\nabla_j \sigma + R_{\alpha\gamma\gamma\delta}\theta_{ij}{}^{\delta\beta}\theta^{k\alpha} v^{\gamma}.$$

But a necessary and sufficient condition that the deformation shall preserve geodesics (*i.e.* that geodesics of V_m deform into geodesics of $\bar{V}_m$) is that*

$$\Pi_{(ij)}{}^{k} = g'_{(i}{}^{k}\varpi_{j)}, \tag{27}$$

where ϖ_j is an arbitrary covariant vector. Thus *an irrotational deformation* (v^{α}) *preserves geodesics if, and only if,*

$$R_{\alpha\beta\gamma\delta}\theta_{(ij)k}{}^{\delta\beta\alpha} v^{\gamma} = g'_{k(i}\lambda_{j)},$$

where λ_i *is some covariant vector.*

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* Eisenhart, *Non-Riemannian geometry* (New York, 1927), 31.

The parallel deformation (22) is a motion if

$$b_{(ij)} = 0;$$

a parallel motion is not irrotational unless, in addition,

$$b_{[ij]} = 0,$$

in which case $$\delta\theta_i^{\ \alpha} = 0.$$

The equations for an irrotational motion are, therefore,

$$\nabla_i v^\alpha + 2S_{\beta\gamma}^{\ \ \alpha}\theta_i^{\ \beta} v^\gamma = 0.$$

In the case of a $\overset{0}{V}_m$ in $\overset{0}{V}_n$, these become simply

$$\nabla_i v^\alpha = 0,$$

so that v^α is a parallel vector-field of constant length over $\overset{0}{V}_m$ (parallel in $\overset{0}{V}_n$), and the deformation is a "translation".

A constant normal deformation of a $\overset{0}{V}_{n-1}$ in $\overset{0}{V}_n$ is a parallel deformation, but it is not irrotational unless $\overset{0}{V}_{n-1}$ is totally-umbilical. For, if the deformation is given by

$$v^\alpha = \psi\chi^\alpha,$$

where ψ is constant,

$$b_{ij} = \theta_{j\alpha}\nabla_i v^\alpha = \theta_{j\alpha}\omega_{ik}\theta^{k\alpha}\psi = \omega_{ij}\psi,$$

and the result follows.

If it is not irrotational, the directions of principal extension are the directions of principal curvature, and, since ω_{ij} is symmetric in the case of Riemannian spaces, it follows that *the directions of principal curvature (and hence the lines of curvature) of a* $\overset{0}{V}_{n-1}$ *in* $\overset{0}{V}_n$ *are displaced parallelly under a constant normal deformation.*

This is well known, of course, for a $\overset{0}{V}_2$ in an E_3.

10. We may inquire whether any parallel deformation of a V_m in V_n also displaces the first normal space parallelly*.

We must first observe, however, that the dimensionality of the first normal space may alter as a result of a deformation. We also remark that the

* For a $\overset{0}{V}_1$ in $\overset{0}{V}_n$, *cf.* Hayden, *Proc. London Math. Soc.* (2), 32 (1931), 321-336.

dimensionality of a vector-space is unaltered by a parallel displacement, since, if there is a linear relation between any number of vectors, there is a linear relation of the same type between the displaced vectors, as is immediately verified.

Let the first normal space to V_m be of dimensions m_1, and let the first normal space to $\bar{V}_m$ at the corresponding point be of dimensions $\bar{m}_1$. The space determined by the vectors $\bar{\Omega}_{ij}{}^{\alpha}$ is, therefore (from the remark above), also of dimensions $\bar{m}_1$. A necessary condition that the first normal space be displaced parallelly is that the $\delta\Omega_{ij}{}^{\alpha}$ are homogeneous linear functions of the $\Omega_{ij}{}^{\alpha}$; that is, that $\delta\Omega_{ij}{}^{\alpha}$ are of the form

$$\delta\Omega_{ij}{}^{\alpha} = \epsilon a_{ij}{}^{kl}\,\Omega_{kl}{}^{\alpha}, \tag{28}$$

where $a_{ij}{}^{kl}$ is a tensor.

It is not immediately evident that this condition is sufficient. If, however, the condition (28) is satisfied, the space determined by the vectors $\bar{\Omega}_{ij}{}^{\alpha}$ is certainly contained in the first normal space to V_m, so that $\bar{m}_1 \leqslant m_1$. But we can regard V_m as obtained by deforming $\bar{V}_m$, and, if $\bar{\delta}$ refers to this deformation, it is easily seen, on writing out the expressions concerned (retaining only first order terms), that

$$\bar{\delta}\bar{\Omega}_{ij}{}^{\alpha} = -\epsilon a_{ij}{}^{kl}\,\bar{\Omega}_{kl}{}^{\alpha},$$

which implies $m_1 \leqslant \bar{m}_1$. Hence $\bar{m}_1 = m_1$, and the space determined by the $\bar{\Omega}_{ij}{}^{\alpha}$ is identical with the first normal space to V_m; that is, the first normal space is displaced parallelly. Thus the condition is also sufficient.

Now let us consider a parallel deformation, given by (22). If we substitute for $\nabla_i v^{\alpha}$ from (22) in (9), we find that

$$P_{ij}{}^{\alpha} = \theta_k{}^{\alpha}\nabla_i b_j{}^k + b_j{}^k\,\Omega_{ik}{}^{\alpha} + R^{\alpha}{}_{.\beta\gamma\delta}\,\theta_{ij}{}^{\delta\beta}\,v^{\gamma},$$

and hence, from (11),

$$\delta\Omega_{ij}{}^{\alpha} = \epsilon(b_j{}^k\,\Omega_{ik}{}^{\alpha} + C_{\beta}{}^{\alpha}\,R^{\beta}{}_{.\lambda\mu\nu}\,\theta_{ij}{}^{\nu\lambda}\,v^{\mu}). \tag{29}$$

If the deformation is irrotational,

$$b_j{}^k = \sigma g'_j{}^k,$$

and

$$\delta\Omega_{ij}{}^{\alpha} = \epsilon(\sigma\Omega_{ij}{}^{\alpha} + C_{\beta}{}^{\alpha}\,R^{\beta}{}_{.\lambda\mu\nu}\,\theta_{ij}{}^{\nu\lambda}\,v^{\mu}).$$

Thus, *if*

$$C_{\beta}{}^{\alpha}\,R^{\beta}{}_{.\lambda\mu\nu}\,\theta_{ij}{}^{\nu\lambda} = 0,$$

the first normal space to V_m is displaced parallelly under any parallel deformation of V_m; moreover, if the deformation of V_m is irrotational, the first normal space is also displaced irrotationally.

Furthermore, if the curvature tensor of V_n vanishes*, every sub-space V_m has the property just stated†.

11. We can easily prove further that, *if the curvature tensor of V_n vanishes, all the normal spaces to any sub-space V_m are displaced parallelly under any parallel deformation of V_m.*

Let us denote by O_p the "p-th osculating space"‡ of V_m in V_n, that is, the vector-space determined by the aggregate of the vectors

$$\theta_i^{\alpha}, \quad \Omega_{ij}^{\ \ \alpha}, \quad \ldots, \quad \Omega_{i_1 i_2 \ldots i_p}^{\ \ \ \ \ \ \alpha}.$$

The p-th normal space is the vector-space of greatest dimensions contained in O_{p+1} and perpendicular to O_p. It is clear that, if O_p and O_{p+1} are each displaced parallelly, so also is the p-th normal space.

By the argument of § 10, necessary and sufficient conditions that O_p be displaced parallelly are that

$$\delta\Omega_{i_1 i_2 \ldots i_r}^{\ \ \ \ \ \ \alpha} = \epsilon(a_{i_1}^{\ j_1}\theta_{j_1}^{\ \alpha} + b_{i_1 i_2}^{\ \ j_1 j_2}\Omega_{j_1 j_2}^{\ \ \alpha} + \ldots + k_{i_1 i_2 \ldots i_r}^{\ \ \ \ \ \ j_1 \ldots j_r}\Omega_{j_1 \ldots j_r}^{\ \ \ \ \ \alpha}) \quad (r = 1, \ldots, p). \quad (30)$$

Suppose that these are satisfied. Then it follows from (7), on putting

$$T_{\ldots}^{\alpha \ldots} = \Omega_{i_1 \ldots i_p}^{\ \ \ \ \ \alpha}$$

and using (30), assuming the curvature tensor of V_n to vanish, that $\delta\Omega_{i_1 \ldots i_{p+1}}^{\ \ \ \ \ \ \ \alpha}$ is of the form

$$\delta\Omega_{i_1 \ldots i_p i_{p+1}}^{\ \ \ \ \ \ \ \alpha} = \epsilon(A_{i_1}^{\ j_1}\theta_{j_1}^{\ \alpha} + B_{i_1 i_2}^{\ \ j_1 j_2}\Omega_{j_1 j_2}^{\ \ \alpha} + \ldots + L_{i_1 \ldots i_{p+1}}^{\ \ \ \ \ \ \ j_1 \ldots j_{p+1}}\Omega_{j_1 \ldots j_{p+1}}^{\ \ \ \ \ \ \ \alpha}). \quad (31)$$

But (30) and (31) together are necessary and sufficient conditions that O_{p+1} is displaced parallelly. Hence, if O_p is displaced parallelly, so also is O_{p+1} and, therefore, also the p-th normal space. The result follows immediately.

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* It is to be noted that $R^{\alpha}_{.\beta\gamma\delta}$ is the curvature tensor, *not* the Riemann-Christoffel tensor (except for a $\overset{0}{V}_n$); *i.e.* it is formed with respect to the connection, not with respect to the metric.

† For a $\overset{0}{V}_m$ in E_n the latter part of the above theorem also follows from the results of my paper quoted in the first foot-note and the fact that the first normal space at any point of V_m is the locus of the first normals to all possible curves in V_m through the point.

‡ Duschek und Mayer, *Lehrbuch der Differentialgeometrie*, 2 (Leipzig, 1930), 202, ("p-ter Schmiegraum"); Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie* (Berlin, 1922), 110 ["Krümmungsgebiet $(p-1)$-ter Ordnung"]; Hayden, *Proc. London Math. Soc.* (2), 34 (1932), 27–50 (48).

It is seen, from (7), that, *for a particular sub-space V_m, all the normal spaces are displaced parallelly under any parallel deformation of V_m if*

$$R^{\alpha}_{\cdot\,\beta\gamma\delta}\,\theta_{i}^{\cdot\,\delta} = 0.$$

12. A sub-space V_m is a Riemannian space if its torsion tensor S'_{ijk} vanishes, and, in order that it may remain Riemannian under deformation, we require that

$$\delta S'_{ijk} = 0.$$

For a $\overset{0}{V}_m$ in V_n we have, from (17),

$$S_{\beta\gamma}^{\cdot\cdot\,\alpha}\,\theta_{ij}^{\cdot\cdot\,\beta\gamma} = -\Omega_{[ij]}^{\cdot\cdot\,\alpha},$$

and hence, from (6),

$$\delta S'_{ijk} = 2\epsilon\{2S_{\alpha\lambda\gamma}(2S_{\beta\mu}^{\cdot\cdot\,\lambda}\,\theta_{[ij]k}^{\cdot\cdot\,\alpha\beta\gamma}\,v^{\mu} + \theta_{k[i}^{\cdot\cdot\,\gamma\alpha}\,\nabla_{j]}\,v^{\lambda}) - \Omega_{[ij]\alpha}(2S_{\beta\gamma}^{\cdot\cdot\,\alpha}\,\theta_{k}^{\cdot\,\beta}\,v^{\gamma} + \nabla_k v^{\alpha})\}.$$

In a parallel deformation of $\overset{0}{V}_m$ the second part of this expression vanishes, since $\nabla_k v^{\alpha} + 2S_{\beta\gamma}^{\cdot\cdot\,\alpha}\,\theta_{k}^{\cdot\,\beta}\,v^{\gamma}$ is tangential to $\overset{0}{V}_m$ (for every k), and, if we substitute for $\nabla_j v^{\lambda}$ in the first part the value obtained from (22), we find that the above expression reduces to

$$\delta S'_{ijk} = 4\epsilon S_{\alpha\lambda\gamma}\,\theta_{kl[i}^{\cdot\cdot\,\gamma\lambda\alpha}\,b_{j]}^{\cdot\,l} = 4\epsilon\, b_{[j}^{\cdot\,l}\,S'_{i]lk} = 0.$$

Thus *a $\overset{0}{V}_m$ in a V_n remains Riemannian under any infinitesimal parallel deformation.*

It follows from the second theorem of § 7 that *a $\overset{0}{V}_{n-1}$ in a U_n remains Riemannian under a constant infinitesimal normal deformation.*

Let us now consider general normal deformations of a $\overset{0}{V}_{n-1}$ in a U_n. (From the second theorem of § 7 this is *not* a parallel deformation unless it is of constant length.)

If
$$v^{\alpha} = \psi\chi^{\alpha},$$

$$\nabla_i v^{\alpha} = \psi_i\chi^{\alpha} + \omega_{il}\,\theta^{l\alpha}\psi,$$

where we have written ψ_i for $\nabla_i\psi$, *i.e.* for $\dfrac{\partial\psi}{\partial x^i}$. For a $\overset{0}{V}_{n-1}$ in V_n,

$$S_{\beta\gamma}^{\cdot\cdot\,\alpha}\,\theta_{ij}^{\cdot\cdot\,\beta\gamma} = \omega_{[ij]}\,\chi^{\alpha}.$$

Using this, and remembering that, for a U_n, $S_{\alpha\beta\gamma}$ is skew-symmetric in β, γ, and in α, γ, we find that, in the present case (of a $\overset{0}{V}_{n-1}$ in U_n),

$$\delta S'_{ijk} = 2\epsilon\{\omega_{[jk]}\,\psi_i + \omega_{[ki]}\,\psi_j + \omega_{[ij]}\,\psi_k\}.$$

Necessary and sufficient conditions that $\overset{0}{V}_{n-1}$ remain Riemannian under deformation are, therefore,

$$\omega_{[jk]}\psi_i+\omega_{[ki]}\psi_j+\omega_{[ij]}\psi_k=0. \tag{32}$$

The left-hand side of (32) vanishes identically if two or more of the indices i, j, k are equal, and either remains unchanged or merely changes sign on interchanging the indices i, j, k in any way. Hence there are, at most, $\frac{1}{6}(n-1)(n-2)(n-3)$ independent equations (32).

It follows that, for $n=3$, $\delta S'_{ijk}$ vanishes for arbitrary ψ. Thus *a* $\overset{0}{V}_2$ *in a* U_3 *remains Riemannian under any infinitesimal normal deformation.*

13. V_m is a geodesic sub-space of V_n if, and only if,

$$\Omega_{(ij)}{}^{\alpha}=0.$$

It follows, from (11), that a necessary and sufficient condition that a geodesic sub-space remains geodesic under deformation is that

$$C_\beta{}^\mu[\nabla_{(i}\nabla_{j)}v^\beta+2S_{\lambda\mu}{}^{\cdot\cdot\beta}\theta_{(j}{}^{\lambda}\nabla_{i)}v^\mu+\{R^\beta{}_{\cdot\lambda\mu\nu}\theta_{(ij)}{}^{\nu\lambda}+2\theta_{(j}{}^{\lambda}\nabla_{i)}S_{\lambda\mu}{}^{\cdot\cdot\beta}\}v^\mu]=0. \tag{33}$$

For $m=1$, this reduces to

$$C_\beta{}^\alpha\left[\frac{D^2v^\beta}{Ds^2}+2S_{\lambda\mu}{}^{\cdot\cdot\beta}\xi^\lambda\frac{Dv^\mu}{Ds}+\left(R^\beta{}_{\cdot\lambda\mu\nu}\xi^\nu+2\frac{D}{Ds}S_{\lambda\mu}{}^{\cdot\cdot\beta}\right)\xi^\lambda v^\mu\right]=0, \tag{34}$$

where $\dfrac{D}{Ds}$ denotes absolute differentiation along the curve (geodesic) and ξ^α is the unit tangent to the curve.

We can obtain this in a different form, by using (8) instead of (11). For $m=1$, there is a single component of the torsion tensor $S'_{ij}{}^{\cdot\cdot k}$ and it vanishes identically on account of skew-symmetry; there is a single component of ϕ_{ij}, which is ϕ; and the single component of $\Pi_{(ij)}{}^{k}$ is equal to $\epsilon\dfrac{d\phi}{ds}$ (if we take s as the single coordinate in V_1). We get therefore, from (8), as the equation of "geodesic deviation",

$$\frac{D^2v^\alpha}{Ds^2}+2S_{\lambda\mu}{}^{\cdot\cdot\alpha}\xi^\lambda\frac{Dv^\mu}{Ds}+\left(R^\alpha{}_{\cdot\lambda\mu\nu}\xi^\nu+2\frac{D}{Ds}S_{\lambda\mu}{}^{\cdot\cdot\alpha}\right)\xi^\lambda v^\mu-\xi^\alpha\frac{d\phi}{ds}=0. \tag{34'}$$

This is the equation obtained by Hlavatý*. If V_n is Riemannian it reduces to Levi-Civita's equation†.

* Hlavatý, "Sur la déformation infinitésimale d'une courbe dans la variété métrique avec torsion", *Bull. de la Soc. Math. de France*, 61 (1928), 18–25.

† Levi-Civita, "Sur l'écart géodésique", *Math. Annalen*, 97 (1926), 291–320.

In the case when $m = 1$, it is well known that we can get a solution which shall satisfy one other condition, and the usual condition† to impose is that the first variation of the arc-length shall vanish, *i.e.* that the deformation shall be a motion.

If we impose on (33) the analogous condition, that the deformation of V_m shall be a motion, we have

$$\delta g'_{ij} = 0, \quad \delta \left\{ {k \atop ij} \right\} = 0,$$

and hence, since

$$\Gamma'^{k}_{ij} = \left\{ {k \atop ij} \right\} + S'^{\cdot\cdot k}_{ij} - 2S'^{k}_{\cdot(ij)}, \tag{35}$$

$$\Pi^{\cdot\cdot k}_{(ij)} = -2\delta S'^{k}_{\cdot(ij)} = -2g'^{kl}\delta S'_{l(ij)}.$$

Using (8), we have, therefore, as necessary and sufficient conditions that a geodesic-V_m remains geodesic under a *motion*,

$$\nabla_{(i}\nabla_{j)}v^{\alpha} + 2S^{\cdot\cdot\alpha}_{\beta\gamma}\theta^{\cdot\beta}_{(j}\nabla_{i)}v^{\gamma} + \{R^{\alpha}_{\cdot\beta\gamma\delta}\theta^{\cdot\cdot\delta\beta}_{(ij)} + 2\theta^{\cdot\beta}_{(j}\nabla_{i)}S^{\cdot\cdot\alpha}_{\beta\gamma}\}v^{\gamma} - \theta^{k\alpha}\delta S'_{k(ij)} = 0, \tag{36}$$

where $\delta S'_{k(ij)}$ is obtained from (6).

For a U_m in U_n this takes the more simple form

$$\nabla_{(i}\nabla_{j)}v^{\alpha} + 2S^{\cdot\cdot\alpha}_{\beta\gamma}\theta^{\cdot\beta}_{(j}\nabla_{i)}v^{\gamma} + \{R^{\alpha}_{\cdot\beta\gamma\delta}\theta^{\cdot\cdot\delta\beta}_{(ij)} + 2\theta^{\cdot\beta}_{(j}\nabla_{i)}S^{\cdot\cdot\alpha}_{\beta\gamma}\}v^{\gamma} = 0. \tag{37}$$

Equation (37) has a geometrical interpretation for a general connection. From (8), we see that it is the equation obtained by imposing on (33) the conditions

$$\Pi^{\cdot\cdot k}_{(ij)} = 0,$$

which are sufficient (though not necessary) conditions that geodesics of V_m deform into geodesics of $\bar{V}_m$. Hence, for a general connection, (37) are the equations of a deformation of a geodesic V_m such that V_m remains geodesic and its geodesic lines are preserved.

Equation (34), or (34′), also has a geometrical significance for a V_m in V_n. It is the equation for the deformation along an asymptotic line of V_m in order that that curve shall deform into an asymptotic line of $\bar{V}_m$.

For normal deformations of a geodesic hypersurface of a Riemannian space of constant curvature K, (37) becomes*

$$\nabla_i\nabla_j\psi + Kg'_{ij}\psi = 0. \tag{38}$$

Since the hypersurface is geodesic, it is also of constant curvature K, and

* We have seen in § 6 that a normal deformation of a geodesic U_m in U_n is essentially a motion.

hence these equations are completely integrable*, a first integral being

$$\Delta_1 \psi + K\psi^2 = \text{const.},$$

where Δ_1 is Beltrami's differential parameter of the first order.

14. The mean normal curvature vector at a point of V_m is the vector

$$N^\alpha \equiv \frac{1}{m} g'^{ij} \Omega_{ij}^{\ \ \alpha},$$

and the mean normal curvature at the point, which we shall denote by J, is the length of the vector N^α.

The variations of N^α and J are readily found. We have

$$\delta N^\alpha = \frac{1}{m} (g'^{ij} \delta\Omega_{ij}^{\ \ \alpha} + \Omega_{ij}^{\ \ \alpha} \delta g'^{ij}).$$

Since $$g'^{ij} g'_{jk} = \delta_k^{\ i},$$

$$g'_{jk} \delta g'^{ij} = -g'^{ij} \delta g'_{jk} = -2\epsilon\phi^i_{\ .k};$$

whence $$\delta g'^{ij} = -2\epsilon\phi^{ij}.$$

Thus

$$\delta N^\alpha = \frac{\epsilon}{m} [C_\beta^{\ \alpha} P_i^{\ i\beta} - 2\phi^{ij} \Omega_{ij}^{\ \ \alpha} - m\theta^{k\alpha} N_\beta (\nabla_k v^\beta + 2S_{\lambda\mu}^{\ \ \beta} \theta_k^{\ \lambda} v^\mu)]. \qquad (39)$$

Also $$J^2 = g_{\alpha\beta} N^\alpha N^\beta,$$

$$2J\delta J = 2g_{\alpha\beta} N^\alpha \delta N^\beta,$$

$$\delta J = \frac{1}{J} N_\alpha \delta N^\alpha,$$

i.e.

$$\boxed{\delta J = \frac{\epsilon}{mJ} N_\alpha (P_i^{\ i\alpha} - 2\phi^{ij} \Omega_{ij}^{\ \ \alpha}).} \qquad (40)$$

For $m = n-1$, and a normal deformation

$$v^\alpha = \psi\chi^\alpha,$$

* Bianchi, "Sui sistemi di equazioni lineari ai differenziali totali", *Rend. Accad. Lincei* (4), 5 (1889), 312–323.

we have

$$N^\alpha = J\chi^\alpha, \quad \omega = -(n-1)J, \quad \Omega_{ij}^{\ \ \alpha} = -\omega_{ij}\chi^\alpha,$$

$$\phi_{ij} = \psi\{\omega_{(ij)} + 2\theta_{(ij)}^{\ \ \alpha\beta} S_{\beta\gamma\alpha}\chi^\gamma\},$$

and we find* that

$$\delta J = \frac{\epsilon}{n-1}\,[\Delta_2\psi + 2S_{\alpha\beta\gamma}\theta^{i\alpha}\chi^\beta\chi^\gamma\nabla_i\psi$$

$$+\{\omega_{ij}\omega^{ij} - R_{\alpha\beta}\chi^\alpha\chi^\beta + 4\omega^{ij}\theta_{(ij)}^{\ \ \alpha\beta}\chi^\gamma S_{\beta\gamma\alpha} + 2\theta^{i\alpha}\chi^\beta\chi^\gamma\nabla_i S_{\alpha\beta\gamma}\}\psi], \quad (41)$$

where Δ_2 is Beltrami's differential parameter of the second order.

For a U_{n-1} in U_n this becomes

$$\delta J = \frac{\epsilon}{n-1}\,[\Delta_2\psi + (\omega_{ij}\omega^{ij} - R_{\alpha\beta}\chi^\alpha\chi^\beta)\psi]. \quad (42)$$

Further, from Gauss's equation,

$$\omega_{ij}\omega^{ij} = \omega^2 - (n-1)(n-2)\bar{K} = (n-1)^2 J^2 - (n-1)(n-2)\bar{K},$$

where $\bar{K}$ is the relative curvature of U_{n-1} in U_n. Also $R_{\alpha\beta}\chi^\alpha\chi^\beta$ is the mean curvature of U_n in the direction of the normal to the hypersurface. Denoting this by M, we have, therefore, *for normal deformations of a* U_{n-1} *in* U_n,

$$\boxed{\delta J = \epsilon\left[\frac{1}{n-1}\,\Delta_2\psi + \{(n-1)(J^2+M) - (n-2)\bar{K}\}\psi\right].} \quad (43)$$

If U_n is of constant curvature K, this takes the form

$$\delta J = \epsilon\left[\frac{1}{n-1}\,\Delta_2\psi + \{(n-1)J^2 + (2n-3)K - (n-2)K'\}\psi\right], \quad (44)$$

where K' is the curvature of U_{n-1}; and, for a U_2 in a U_3 of zero curvature,

$$\delta J = \epsilon\,[\tfrac{1}{2}\Delta_2\psi + (2J^2 - K')\psi], \quad (45)$$

which is identical with Schwarz's formula† for a $\overset{0}{V}_2$ in E_3.

* The calculation of P_i^{ia} and the steps in the reduction are omitted, but the reader will not find it difficult to verify the result.

† Blaschke, *Vorlesungen über Differentialgeometrie*, 1 (Berlin, 1924), 187.

15. From the definition of the curvature tensor $R'^{i}_{.jkl}$, we have, to the first order,

$$\delta R'^{i}_{.jkl} = \epsilon\left(\frac{\partial \Pi_{jl}^{\cdot\cdot i}}{\partial x'^{k}} - \frac{\partial \Pi_{jk}^{\cdot\cdot i}}{\partial x'^{l}} + \Gamma'^{i}_{hk}\,\Pi_{jl}^{\cdot\cdot h} - \Gamma'^{i}_{hl}\,\Pi_{jk}^{\cdot\cdot h} + \Gamma'^{h}_{jl}\,\Pi_{hk}^{\cdot\cdot i} - \Gamma'^{h}_{jk}\,\Pi_{hl}^{\cdot\cdot i}\right),$$

giving

$$\boxed{\delta R'^{i}_{.jkl} = \epsilon(\nabla_k \Pi_{jl}^{\cdot\cdot i} - \nabla_l \Pi_{jk}^{\cdot\cdot i} - 2S'^{\cdot\cdot h}_{kl}\,\Pi_{jh}^{\cdot\cdot i}).} \tag{46}$$

Hence

$$\delta R'_{jk} \equiv \delta R'^{i}_{.jki} = \epsilon(\nabla_k \Pi_{ji}^{\cdot\cdot i} - \nabla_i \Pi_{jk}^{\cdot\cdot i} - 2S'^{\cdot\cdot h}_{ki}\,\Pi_{jh}^{\cdot\cdot i}),$$

and

$$\delta K' \equiv -\frac{1}{m(m-1)}\,\delta(g'^{jk} R'_{jk})$$

$$= \frac{\epsilon}{m(m-1)}\,[\nabla_i \Pi_{k}^{\cdot\cdot ki} - \nabla_k \Pi_{\cdot\cdot i}^{ki} + 2S'_{ijk}\,\Pi^{ijk} + 2\phi^{ij} R'_{ij}]. \tag{47}$$

In the case of a $\overset{0}{V}_m$ *in* $\overset{0}{V}_n$, this reduces to the simple form

$$\boxed{\delta K' = \frac{2\epsilon}{m(m-1)}\,[(R'_{ij} + \nabla_i \nabla_j)\,\phi^{ij} - \Delta_2 \phi].} \tag{48}$$

For conformal deformations of a $\overset{0}{V}_m$ in $\overset{0}{V}_n$, (48) takes the specially simple form*

$$\delta K' = -\frac{2\epsilon}{m}\left(K'\,\phi + \frac{1}{m}\,\Delta_2 \phi\right). \tag{49}$$

16. The second variations of the metric tensor and of the volume are of interest, in view of the classical results for an ordinary surface in Euclidean 3-space. The process of calculating the second variations becomes extremely laborious in the general case of spaces with torsion, but it can be very much simplified for the case of Riemannian spaces, and we shall, therefore, confine ourselves to the latter case (of a $\overset{0}{V}_m$ in $\overset{0}{V}_n$).

If $\bar{g}'_{ij}$ is the metric tensor of $\overset{0}{\bar{V}}_m$,

$$\bar{g}'_{ij} = \bar{g}_{\alpha\beta}\,\bar{\theta}_{i}^{\cdot\cdot\alpha}\,\bar{\theta}_{j}^{\cdot\cdot\beta},$$

* For a $\overset{0}{V}_2$ in E_3, *cf.* E. Cesàro, "Teoria intrinseca della deformazioni infinitesime", *Rend. Accad. Napoli* (2), 8 (1894), 149–154.

where $\bar{g}_{\alpha\beta}$ is the value of $g_{\alpha\beta}$ at the point $\bar{x}^\alpha$. Since $\bar{g}'_{ij}$ is invariant for coordinate transformations in $\overset{0}{V}_n$, we can evaluate it in any coordinate system we please. By Fermi's theorem, we can find a coordinate-system in $\overset{0}{V}_n$ such that the $g_{\alpha\beta}$ are constant all along any arbitrary curve. Choosing as this curve the (infinitesimal) curve of displacement through the point $\bar{x}^\alpha$, at which we are evaluating, we have, in the special coordinates,

$$\bar{\theta}_i^{\ \alpha} = \theta_i^{\ \alpha} + \epsilon \nabla_i v^\alpha, \quad \bar{g}_{\alpha\beta} = g_{\alpha\beta},$$

and hence

$$\bar{g}'_{ij} = g_{\alpha\beta}(\theta_i^{\ \alpha} + \epsilon \nabla_i v^\alpha)(\theta_j^{\ \beta} + \epsilon \nabla_j v^\beta).$$

Thus the second variation of g'_{ij} is

$$\boxed{\delta^2 g'_{ij} = \epsilon^2 \nabla_i v^\alpha \nabla_j v^\alpha.} \tag{50}$$

We shall write, for brevity,

$$\delta^2 g'_{ij} = \epsilon^2 \Phi_{ij},$$

where
$$\Phi_{ij} = \nabla_i v^\alpha \nabla_j v_\alpha. \tag{51}$$

If g', $\bar{g}'$ denote respectively the determinants $|g'_{ij}|$, $|\bar{g}'_{ij}|$,

$$\bar{g}' = |g'_{ij} + F_{ij}|,$$

where
$$F_{ij} \equiv 2\epsilon\phi_{ij} + \epsilon^2 \Phi_{ij}.$$

Expanding the determinant, we have

$$\bar{g}' = g' + F_{ij}\frac{\partial g'}{\partial g'_{ij}} + \frac{1}{(2!)^2}(F_{ik}F_{jl} - F_{jk}F_{il})\frac{\partial^2 g'}{\partial g'_{ik}\partial g'_{jl}} + \dots$$

$$= g' + g' g'^{ij} F_{ij} + \frac{1}{(2!)^2}(F_{ik}F_{jl} - F_{jk}F_{il}) g' (g'^{ik} g'^{jl} - g'^{jk} g'^{il}) + \dots$$

$$= g' \{1 + g'^{ij} F_{ij} + \tfrac{1}{2} g'^{ik} g'^{jl} (F_{ik}F_{jl} - F_{jk}F_{il}) + \dots\}.$$

The first order terms are $2\epsilon\phi g'$, as we have seen earlier, while the second order terms are

$$\epsilon^2 g' \{g'^{ij}\Phi_{ij} + 2g'^{ik} g'^{jl}(\phi_{ik}\phi_{jl} - \phi_{jk}\phi_{il})\},$$

i.e.
$$\epsilon^2 g' (\Phi + 2\phi^2 - 2\phi_{ij}\phi^{ij}),$$

where
$$\Phi \equiv g'^{ij}\Phi_{ij} = \Delta_1(v^\alpha, v_\alpha).$$

Hence $$\delta^2 g' = \epsilon^2 g'(\Phi + 2\phi^2 - 2\phi_{ij}\phi^{ij}). \tag{52}$$

Since $$\bar{g}' = g'\{1 + 2\epsilon\phi + \epsilon^2(\Phi + 2\phi^2 - 2\phi_{ij}\phi^{ij}) + \ldots\},$$

$$\sqrt{\bar{g}'} = \sqrt{g'}\{1 + \epsilon\phi + \tfrac{1}{2}\epsilon^2(\Phi + \phi^2 - 2\phi_{ij}\phi^{ij}) + \ldots\}. \tag{53}$$

If v is the volume of a closed region of $\overset{0}{V}_m$, and $\bar{v}$ the volume of the corresponding region of $\overset{0}{V}_m$,

$$\bar{v} = \int \ldots \int \sqrt{\bar{g}'}\, dx'^1 \ldots dx'^m.$$

Hence

$$\boxed{\delta v = \epsilon \int_v \phi\, dv;} \tag{54}$$

$$\boxed{\delta^2 v = \tfrac{1}{2}\epsilon^2 \int_v (\Phi + \phi^2 - 2\phi_{ij}\phi^{ij})\, dv.} \tag{55}$$

For the Riemannian case, which we are considering,

$$\phi_{ij} = \nabla_{(i} u_{j)} - \Omega_{ij}{}^{\alpha} v_{\alpha},$$

whence $$\phi = \operatorname{div} u_i - m N^{\alpha} v_{\alpha}.$$

Also, if f is the volume of the bounding $\overset{0}{V}_{m-1}$,

$$\int_v \operatorname{div} u_i dv = \int_f Y'^i u_i df,$$

where Y'^i is the unit normal to $\overset{0}{V}_{m-1}$ relative to $\overset{0}{V}_m$. Further, since Y'^i is contained in $\overset{0}{V}_m$ and u^i is the $\overset{0}{V}_m$-component of v^{α},

$$Y'^i u_i = Y^{\alpha} v_{\alpha},$$

where Y^{α} is the unit normal to $\overset{0}{V}_{m-1}$ relative to $\overset{0}{V}_n$ regarded as a vector in $\overset{0}{V}_n$. Hence

$$\boxed{\delta v = \epsilon \left[\int_f Y^{\alpha} v_{\alpha} df - m \int_v N^{\alpha} v_{\alpha} dv \right].} \tag{56}$$

Thus the first variation of the volume is zero if the deformation is perpen-

dicular to the mean curvature vector at each point and either vanishes or is normal to $\overset{0}{V}_m$ on the boundary*.

It follows that, if $m < n-1$, $\overset{0}{V}_m$ admits of an infinite number of (normal) volume-preserving deformations.

For $m = n-1$, and a normal deformation,

$$v^\alpha = \psi \chi^\alpha,$$

$$\phi_{ij} = \psi \omega_{ij}, \quad \phi = (n-1) J\psi,$$

$$\Phi = \Delta_1 \psi + \omega_{ij}\, \omega^{ij} \psi^2;$$

whence

$$\delta v = -(n-1)\, \epsilon \int_v J\psi \, dv; \tag{57}$$

$$\delta^2 v = \tfrac{1}{2} \epsilon^2 \int_v [\Delta_1 \psi + (n-1)(n-2)\, \bar{K} \psi^2]\, dv. \tag{58}$$

For a $\overset{0}{V}_2$ in E_3 these reduce to the well-known expressions†

$$\delta v = -2\epsilon \int_v J\psi \, dv;$$

$$\delta^2 v = \tfrac{1}{2} \epsilon^2 \int_v (\Delta_1 \psi + 2K' \psi^2)\, dv,$$

where K' is the Gaussian curvature of $\overset{0}{V}_2$.

If a minimal hypersurface is given an infinitesimal constant normal deformation (*i.e.* a normal parallel deformation),

$$\delta v = 0,$$

$$\delta^2 v = \tfrac{1}{2}(n-1)(n-2)\, \epsilon^2 \psi^2 \int_v \bar{K}\, dv.$$

But

$$\tfrac{1}{2}(n-1)(n-2)\, \bar{K} = \sum_{(hk)}{}' \underset{h}{\rho}\, \underset{k}{\rho},$$

where $\underset{h}{\rho}$ $(h = 1, 2, \ldots, n-1)$ are the principal normal curvatures of $\overset{0}{V}_{n-1}$

* For $m = 1$, *cf.* J. L. Synge, "The first and second variations of the length-integral in Riemannian space", *Proc. London Math. Soc.* (2), 25 (1926), 253.

† Gauss, *Ges. Werke*, 5, 65; Bianchi, *Lezioni di Geometria Differenziale*, 1 (2) (1927), 603; Blaschke, *loc. cit.*, 185.

relative to $\overset{0}{V}_n$ and $\Sigma'_{(hk)}$ denotes summation for all *pairs* of values h, k for which $h \neq k$. Since $\overset{0}{V}_{n-1}$ is minimal,

$$\Sigma_h \rho = 0,$$

and hence
$$\Sigma'_{(hk)} \rho_h \rho_k = -\tfrac{1}{2} \Sigma_h \rho^2,$$

and
$$\delta^2 v = -\tfrac{1}{2} \epsilon^2 \psi^2 \int_v \Sigma_h \rho^2 dv < 0;$$

that is, v is a maximum for normal parallel deformations.

Thus Steiner's theorem (given by Steiner* for a $\overset{0}{V}_2$ in E_3) is true for a $\overset{0}{V}_{n-1}$ in $\overset{0}{V}_n$.

* *Ges. Werke*, 2, 176; Bianchi, *loc. cit.*, 604.

ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS

(*Eighth Paper.*)

By Alfred Young.

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The first part of this paper, Sections I–VII, continues the general theory from the sixth paper. The last part, Sections VIII, IX, has to do with applications to the algebra of invariants in continuation of the seventh paper. The theory in the first part is developed mainly with a view to the applications made later; but I believe it to be of interest in itself.

The semi-normal group matrix introduced in Q.S.A., VI, presupposed a definite sequence of tableaux, taken to be that which was defined as the sequence from the last letter. In Section I it is shown that other sequences will do equally well, and lead to the same seminormal units. In fact, a whole class of sequences is defined as *principal sequences* for which this is true.

In Section II very simple expressions are found for those seminormal matrices which are unchanged on multiplication on the right or on the left by the permutations of the symmetric group of a set of consecutive letters; or else are changed in sign by the odd permutations. It is interesting to find that the matrix elements depend only on the tableau functions. And, further, that the numerical average of all the elements, in either case, is simply expressible in terms of the tableau functions of the first and last tableaux of the whole set.

It is important for applications to invariants, and I think of general interest, to find the expression for any seminormal unit of a set of consecutive letters in terms of the units of the whole. This problem is attacked in Section III, but it has only been solved in special cases. It leads at once to the notion of a distorted tableau, where the upper rows are bodily shifted to the right. In Section IV the special case when the tableaux contain

two rows only is discussed and solved. This is the case needed for binary covariants. In Section V there are introduced certain simple algebraic forms which represent the seminormal units, in that for a given unit we can put down a form which has the substitutional properties of the unit for left-hand multiplication and no others. Theorem XV, which connects the first term of such a form directly with the tableau, is of importance for its applications. Section VI obtains the matrix which transforms the regular group matrix of a symmetric group into its exact seminormal irreducible components.

Frobenius introduced a double group matrix $[x_{PQ^{-1}}+y_{Q^{-1}P}]$, and showed that it possessed the same number of irreducible components as the ordinary group matrix, but that the order of each irreducible matrix was f^2 instead of f. In Section VII the transformation effecting this reduction is discussed, and it is shown that the double group matrix has a remarkably simple form in a limited number of cases.

Section VIII deals with the expression of concomitants by means of seminormal units; it is shown that for concomitants of a single quantic the sets of numbers defining the concomitant and the leading gradient are identical. This establishes what was merely put forward as an empirical theorem in Q.S.A., VII. This may be regarded as the fundamental theorem of this presentation of the subject. One immediate result of it is that the exhibition of a form with a given leading gradient is proof of the irreducibility of the corresponding concomitant, whatever principal sequence is chosen. A second result is that a form whose leading gradient is the product of two leading gradients may be regarded as reducible. Section IX deals with applications to binary forms.

I. *Principal sequences.*

1. The seminormal matrix for $T_{a_1 a_2 \ldots a_k}$ was obtained by means of the quadratic invariant H of the symmetric group. The variables were written $x_1, x_2, \ldots, x_f$, the numbers referring to the tableaux $F_1, F_2, \ldots, F_f$; and the variables were connected to the natural units P_r, N_r, M_r defined by these tableaux. The first step was taken in § 6 of the sixth paper by expressing H as a sum of squares. This was done by first collecting all the terms which contained x_f and so obtaining

$$H = H' + A_f X_f^2,$$

where H' does not contain x_f.

Thus the actual form of the sum of squares, and hence also the seminormal matrix, depends on the sequence of tableaux. In general, when a different sequence is taken from that adopted, namely that which has been styled the sequence from the last letter, a different seminormal matrix will be obtained; and one, in fact, for which the very simple form for the transpositions of consecutive letters, obtained in the paper referred to, no longer holds good.

It may happen that an alteration in sequence does not alter the matrix except by interchanges of rows, and at the same time of columns. We write

$$H = \sum_{r=1}^{f} A_r X_r^2;$$

then, if X_{r+1} does not contain the variable x_r, the simple interchange of F_r and F_{r+1} in the sequence does not change the form of H, it only produces an interchange of a pair of rows and a pair of columns in the seminormal matrix, and the same change in the seminormal transforming matrix; and in this latter matrix the element $m_{r,r+1}$ is necessarily zero, like $m_{r+1,r}$ and all the elements below the leading diagonal; the seminormal units corresponding to the different tableaux are also unchanged. In order that a new sequence law may produce the same matrix except for the sequence of rows and columns, it is a necessary and sufficient condition that the element m_{rs} of the old seminormal matrix is zero when F_s precedes F_r according to the new law. It will be shown that certain other sequences may be adopted equally as well as that from the last letter to produce the same seminormal units. Thus, for instance, the sequence from the first letter is a case in point. As an example, for $T_{3,2}$,

$$H = \tfrac{1}{72} X_1^2 + \tfrac{1}{36} X_2^2 + \tfrac{1}{12} X_3^2 + \tfrac{1}{12} X_4^2 + \tfrac{1}{4} X_5^2,$$

where (see Q.S.A., VI, § 17)

$$X_1 = 12x_1, \quad X_2 = 9x_2 + 3x_1, \quad X_3 = 6x_3 + 3x_2 + 3x_1,$$

$$X_4 = 6x_4 + 3x_2 + 3x_1, \quad X_5 = 4x_5 + 2x_4 + 2x_3 + x_2 + 3x_1.$$

Here the sequence is taken from the last letter and the tableaux may be written

$$\begin{matrix} 123 & 124 & 134 & 125 & 135 \\ 45 & 35 & 25 & 34 & 24 \end{matrix}.$$

But, since X_4 does not contain x_3, the form of H is unaltered by the interchange of F_3 and F_4 in the tableau sequence; and this is the same as the sequence from the first letter.

2. It is possible to give many different laws of sequence of standard tableaux; among them there is a class of sequences which have important properties in common. We therefore define this class as follows:

Definition. A sequence of standard tableaux is called a *principal sequence* when it possesses the property that F_r precedes F_s whenever the permutation σ_{rs} which transforms F_s into F_r can be expressed as a product of transpositions $\tau_1, \tau_2, \ldots, \tau_k$, which is such that, when the permutation on F_s is made a transposition at a time, at each stage either the transposition raises the earlier letter in the tableau operated on and depresses the later letter, or else it merely interchanges two letters in the same row.

The sequence from the last letter and the sequence from the first letter are examples of a principal sequence. Other examples will be introduced later.

In the case of T_{32} the relative positions of all the tableaux are fixed for a principal sequence, except that of F_3 and F_4.

3. For the present purpose, and also for future use, it is necessary to consider more exactly the natural matrix C corresponding to an interchange of a pair of consecutive letters. In this case the seminormal matrix D is known, and the equation (Q.S.A., VI, § 6)

$$CM = MD$$

determines the seminormal transforming matrix M.

Let b, b' be the consecutive pair of letters. In a particular tableau F_r there are three ways in which these letters may appear:

(i) In the same row; if n_{rr} is the corresponding natural unit,

$$(bb')\, n_{rr} = n_{rr}.$$

(ii) In the same column:

$$(bb')\, n_{rr} = -n_{rr} + \ldots,$$

and the terms on the right need examination.

(iii) Obliquely, then there is a second standard tableau F_s obtained from F_r by the interchange of b and b', so that

$$(bb')\, n_{rr} = n_{sr}.$$

It is the second case that we have to examine.

The natural unit

$$n_{rs} = P_r \sigma_{rs} N_s M_s.$$

At the moment, the attention is confined to permutations applied to the left-hand side only of the units. The equations then are of the form

$$\tau n_{rs} = \Sigma \lambda_l n_{ls};$$

and the second suffix is unchanged, the equations being the same whatever its value provided that it is kept unaltered.

We require $(bb')n_{rs}$, where a pair of consecutive rows of F_r have the form

$$\ldots\ldots bc_1 c_2 \ldots c_k$$

$$a_1 \ldots a_l b' \ldots$$

the sequence of the letters exposed being

$$a_1 a_2 \ldots a_l bb' c_1 c_2 \ldots c_k.$$

After the transposition (bb') the standardizing equations must be used, and first of all

$$G[(bb')n_{rs}] = 0,$$

where $$G = \{a_1 a_2 \ldots a_l bb' c_1 \ldots c_k\}.$$

The result is $$\{bb'\} n_{rs} = -\Sigma q.$$

The terms q are those obtained from the different tableaux resulting from the operations of G on F_r. They are not necessarily all standard; but so far as the two rows at the moment considered are concerned, they are standard. Moreover, each q tableau is obtained from F_r by a permutation of the character described in the definition of the last paragraph. Hence those which are standard are earlier than F_r in any principal sequence.

Let us now consider a q tableau Φ which is not standard. Then a pair of consecutive rows exists (necessarily one of them is one of the rows already disturbed) for which at some point the letter in the upper row is later than that in the lower row. Starting from the left-hand side, we select the first pair of letters for which this is so, and apply the standardizing equation for this pair. As before,

$$q = -\Sigma q',$$

where the tableau for each q' is obtained from Φ by a permutation of the

character described in the definition of the last paragraph. Then, if this tableau is standard, it precedes F_r in any principal sequence. If not, the standardizing equations must be used again. That the process is finite we know; the argument is the same at each step; and hence we ultimately arrive at

THEOREM I. *When the letters a_p, a_{p+1} lie in the same column of the tableau F_r,*

$$\{a_p a_{p+1}\} n_{ru} = \Sigma \lambda_s n_{su},$$

where F_s precedes F_r in any principal sequence. Further, a_p, a_{p+1} lie in the same row of F_s, or else obliquely, in which case $\lambda_t = \lambda_s$ when

$$(a_p a_{p+1}) n_{su} = n_{tu};$$

but they never lie in the same column.

The equality of λ_s and λ_t in the last part of this statement is obvious when we operate on the equation with $\{a_p a_{p+1}\}$. For then the left-hand side is merely multiplied by two, and hence the same must be true for the right-hand side, since no linear relation exists between the units.

4. The sequences used so far have been that from the last letter and that from the first letter. It will be found that a principal sequence, which will be called the sequence from the last row, has advantages for certain purposes, in that the matrices thus arranged assume a regularity and symmetry which is lacking with other sequences. In the sequence from the last row the tableau F_r precedes the tableau F_s when the latest letter in the last row of F_r is later than the latest letter of the last row of F_s, or, when these are the same, the next latest letter in the last row of F_r precedes the next latest in the last row of F_s, and so on. When the last rows of F_r and F_s are the same, we consider the latest row which differs in the two tableaux and determine the sequence from that.

The corresponding sequence from the first row is also a principal sequence and appears to enjoy a like advantage. These sequences may be illustrated by giving the relative places of a set of six tableaux which are identical except for the positions of three consecutive letters a, b, c. The places occupied by these letters are supposed to be such that no two are in the same row or column. We write them in a row, the first in the row occupying that position which is highest in the tableaux and most to the right, and the last that which is lowest and most to the left. The

sequence from the last letter is

$$\begin{matrix} abc & bac & acb & bca & cab & cba \\ F_1' & F_2' & F_3' & F_4' & F_5' & F_6' \end{matrix}$$

That from the first letter is

$$F_1,\quad F_3,\quad F_2,\quad F_5,\quad F_4,\quad F_6.$$

That from the last row is

$$F_1,\quad F_2,\quad F_3,\quad F_5,\quad F_4,\quad F_6.$$

That from the first row is

$$F_1,\quad F_3,\quad F_2,\quad F_4,\quad F_5,\quad F_6.$$

Another form of sequence sometimes useful is obtained by grouping the n letters permuted into two consecutive sets. The first m are called the letters a, the last $n-m$ the letters b. Then, when the letters b are erased from a standard tableau F_r, we are left with a tableau F_r' of the a's only which is still standard. The tableau F_r' belongs to a representation T_r' of the symmetric group of $a_1 a_2 \dots a_m$. Then we may define F_r to precede F_s whenever T_r' precedes T_s'. When F_r' and F_s' belong to the same T', we may define the precedence of F_r and F_s by that of F_r' and F_s' according to any principal sequence. When $F_r' \equiv F_s'$, the sequence is defined by the letters b according to some principal sequence. A sequence thus defined is a principal sequence.

A moment's reflection will show the truth of

THEOREM II. *In every principal sequence the first tableau is the same, and also the last tableau.*

5. THEOREM III. *The element $m_{\xi\eta}$ of the seminormal transforming matrix M is zero, when F_η precedes F_ξ according to any one of the principal sequences.*

This is known to be the case when the sequence is from the last letter; we consider any other principal sequence S; and proceed by a double induction. We take the tableaux according to S, and notice that, the first tableau F_1 being the same in all the principal sequences, the theorem is true when $\eta = 1$. Then the theorem is assumed for every value of η prior to the given one in S, and the tableaux F_ξ are considered beginning with the last F_f which is the same in all principal sequences. The theorem then is known to be true when $\xi = f$; it is then assumed for all values of ξ

later in S than the given one and the particular η in question, and also for all earlier values of η in S. Consider a particular F_η; except in the trivial cases T_n and T_{1^n}, a consecutive pair of letters can be found in F_η which lie obliquely.

Indeed, it is possible to pass from F_1 to F_f by successive transpositions of such pairs of consecutive letters.

Let a, a' be such a consecutive pair of letters. Let C be the natural matrix for (aa'), and D the seminormal matrix; let M be the seminormal transforming matrix. Then the results of Theorem I must be applied to the equation

$$\text{(I)} \qquad CM = MD.$$

Let F_p have a, a' in one row, let F_r have them in one column, and let $F_s = (aa')F_t$ have them obliquely, where a' lies lower in F_s than in F_t. In this case F_s precedes F_t in every principal sequence.

Then $$\{aa'\} n_{rv} = \Sigma \lambda_{ru} n_{uv},$$

where F_u precedes F_r in every principal sequence. In the columns F_p, F_r of D every element is zero except that in the primary diagonal, which is ± 1. In the pair of columns F_s, F_t of D the elements are zero except for the primary diagonal matrix

$$\begin{bmatrix} -\rho & 1-\rho^2 \\ 1 & \rho \end{bmatrix}.$$

Consider the element (ξ, η) in (I).

On the left we have four cases $\xi = p, r, s, t$:

$$m_{p\eta} + \Sigma \lambda_{\varpi p} m_{\varpi \eta}, \quad -m_{r\eta} + \Sigma \lambda_{\varpi r} m_{\varpi \eta},$$
$$m_{t\eta} + \Sigma \lambda_{\varpi s} m_{\varpi \eta}, \quad m_{s\eta} + \Sigma \lambda_{\varpi t} m_{\varpi \eta}.$$

Here F_ϖ has a, a' in the same column, and $\lambda_{\varpi\xi}$ is zero unless F_ξ precedes F_ϖ in every principal sequence. Hence, on the assumption of the induction, $\lambda_{\varpi\xi} m_{\varpi\eta}$ is zero in every case; for we are considering the case in which F_η precedes F_ξ according to S. Thus on the left-hand side we have

$$m_{p\eta}, \quad -m_{r\eta}, \quad m_{t\eta}, \quad m_{s\eta},$$

in the four cases.

In F_η the letters a, a' lie obliquely, we write $(aa')\, F_\eta = F_\zeta$ and suppose that F_ζ precedes F_η. Then on the right we have

$$(1-\rho^2) m_{\xi\zeta} + \rho m_{\xi\eta}.$$

Then, for $\xi = p$,

$$m_{p\eta} = (1-\rho^2)\, m_{p\zeta} + \rho m_{p\zeta} = (1+\rho)\, m_{p\zeta} = 0,$$

for $\xi = r$, $$m_{r\eta} = -(1-\rho)\, m_{r\zeta} = 0,$$

for $\xi = s$, $$m_{l\eta} - \rho m_{s\eta} = (1-\rho^2)\, m_{s\zeta} = 0,$$

for $\xi = t$, $$m_{s\eta} - \rho m_{t\eta} = (1-\rho^2)\, m_{l\zeta} = 0,$$

and hence $$m_{s\eta} = m_{l\eta} = 0.$$

It has been assumed that F_η is the later of the pair of F_η and F_ζ. The consecutive pair a, a' can always be chosen so that F_η is later than its fellow F_ζ unless $\eta = 1$. For consider any tableau F_η, and each letter in order in connection with its consecutive. If a_2 is in the second row, let a_{r+1} be the first letter in the second column, then a_r, a_{r+1} will be a suitable pair to take. If a_2 is in the first row, let a_r be the first letter not in the first row, then let a_s be the letter which follows a_{r-1} in the first row, then a_{s-1}, a_s will be a suitable pair. The only exception to the success of this is when the first row letters are all consecutive; we then proceed to the second row, and obtain a suitable pair unless these are consecutive, and so on.

Thus the induction is proved, and the theorem is true. As an immediate result we obtain

THEOREM IV. *The seminormal matrix is the same except for row and column interchanges whichever of the principal sequences originally is adopted; and the seminormal units are the same in each case.*

II. *Positive and negative symmetric groups of a set of consecutive letters.*

6. Consider an expression S given in terms of the *exact* seminormal units $\varpi_{u,v}$; we are to be concerned with permutations applied on the left-hand side, and so are not concerned with the second suffix of ϖ which is supposed always the same, *i.e.* we write

$$S = \Sigma \lambda_u \varpi_u,$$

in effect an exact seminormal matrix consisting of a single column and zeros elsewhere. In the exact seminormal matrix, for a transposition of a pair of consecutive letters, the diagonal quadratic matrices take the form

$$\begin{bmatrix} -\rho & 1-\rho \\ 1+\rho & \rho \end{bmatrix}.$$

Multiply S on the left by the transposition (aa'). Then, using the notation of the last paragraph,

$$\lambda_p \varpi_p \quad \text{becomes} \quad \lambda_p \varpi_p,$$

$$\lambda_r \varpi_r \quad \text{becomes} \quad -\lambda_r \varpi_r,$$

$$\lambda_s \varpi_s + \lambda_t \varpi_t \quad \text{becomes} \quad \{-\rho\lambda_s + (1-\rho)\lambda_t\}\varpi_s + \{(1+\rho)\lambda_s + \rho\lambda_t\}\varpi_t.$$

When S is unchanged, $\lambda_r = 0$,

$$(1+\rho)\lambda_s = (1-\rho)\lambda_t.$$

Hence (Q.S.A., VI, § 16)

$$\lambda_s [F_s]_1 [F_s]_2 = \lambda_t [F_t]_1 [F_t]_2;$$

$[F]_1$, $[F]_2$ being the first and second tableau functions.

Thus when S is unchanged by the permutations of the symmetric group of m consecutive letters $b_1, b_2, \ldots, b_m$, then S is a sum of sets of terms $\Sigma \lambda_u \varpi_u$, a set being such that the letters b collectively occupy the same position in the tableaux F_u, no two b's lie in the same column, and the coefficient

$$\lambda_u = \frac{A}{[F_u]_1 [F_u]_2},$$

where A is a constant for the set.

We can now prove that the sum of the coefficients λ_u is $AN/[F_1]_1 [F_t]_2$, where N is their number, F_1 is the first and F_t the last tableau of the set.

The proof is by induction; the theorem is assumed true for the case of m consecutive letters $b_1, \ldots, b_m$, and then it is established for the symmetric group of $b_1, \ldots, b_{m+1}$. For definiteness the case is first taken for which each of the letters b is in a different row in the tableaux F_u. The first and the last tableaux will be called F_1 and F_t. For the group of the first m letters there are just $m+1$ sets, and the first and the last tableaux of the r-th set will be denoted by $F_{r,1}$, $F_{r,t}$. In F_1 the axial projections of $b_1 b_2$, $b_2 b_3$, ..., $b_m b_{m+1}$ will be denoted by $a_1, a_2, \ldots, a_m$. Then, of course, that of $b_1 b_3$ is $a_1 + a_2$, and so on. Then, by Q.S.A., VI, § 15,

$$[F_1]_1 = [F_{1,1}]_1 = \{1+1/a_m\}[F_{2,1}]_1 = \{1+1/a_{m-1}\}\{1+1/(a_{m-1}+a_m)\}[F_{3,1}]_1$$

$$= \ldots = \{1+1/a_1\}\{1+1/(a_1+a_2)\}\ldots\{1+1/(a_1+a_2+\ldots+a_m)\}[F_{m+1,1}]_1;$$

$$[F_t]_2 = [F_{m+1,t}]_2 = \{1-1/a_1\}[F_{m,t}]_2 = \{1-1/a_2\}\{1-1/(a_2+a_1)\}[F_{m-1,t}]_2$$

$$= \ldots = \{1-1/a_m\}\{1-1/(a_m+a_{m-1})\}\ldots\{1-1/(a_m+\ldots+a_2+a_1)\}[F_{1,t}]_2.$$

Then, by hypothesis,

$$\sum_{r=1}^{l} \frac{A}{[F_r]_1 [F_r]_2} = \sum_{r=1}^{m+1} \frac{A.m!}{[F_{r,1}]_1 [F_{r,l}]_2} = \frac{Am!}{[F_1]_1 [F_l]_2} X,$$

where

$$X = \sum_{r=1}^{m+1} \{1+1/a_{m+2-r}\}\{1+1/(a_{m+2-r}+a_{m+3-r})\}\dots \times\{1-1/a_{m+1-r}\}\{1-1/(a_{m+1-r}+a_{m-r})\}\dots.$$

To calculate X we first make a_r very small; there are two terms in which the very large factor $1/a_r$ appears. The coefficient of $1/a_r$ is seen to be (on neglecting a_r)

$$-\{1-1/a_{r-1}\}\{1-1/(a_{r-1}+a_{r-2})\}\dots\{1-1/(a_{r-1}+\dots+a_1)\} \times\{1+1/a_{r+1}\}\{1+1/(a_{r+1}+a_{r+2})\}\dots\{1+1/(a_{r+1}+\dots+a_{m+1})\}$$

$$+\{1+1/a_{r+1}\}\dots\{1+1/(a_{r+1}+\dots+a_{m+1})\} \times\{1-1/a_{r-1}\}\dots\{1-1/(a_{r-1}+\dots+a_1)\},$$

that is, zero. Hence, when the expression $X-m-1$ is brought to a common denominator D, so that

$$X-m-1 = Y/D,$$

then a_r is a factor of Y.

Next let us make a_r+a_{r-1} very small; then the coefficient of the large factor $1/(a_r+a_{r-1})$ becomes

$$-\{1-1/a_r\}\{1-1/a_{r-2}\}\dots\{1-1/(a_{r-2}+\dots+a_1)\} \times\{1+1/a_{r+1}\}\dots\{1+1/(a_{r+1}+\dots+a_{m+1})\}$$

$$+\{1+1/a_{r-1}\}\{1+1/a_{r+1}\}\dots\{1+1/(a_{r+1}+\dots+a_{m+1})\} \times\{1-1/a_{r-2}\}\dots\{1-1/(a_{r-2}+\dots+a_1)\}$$

$$= 0,$$

since $$a_{r-1} = -a_r.$$

Hence a_r+a_{r-1} is a factor of Y.

In the same way we find that every factor of D is a factor of Y, and hence, since Y is of lower order than D,

$$Y = 0.$$

Therefore $$X = m+1,$$

and the induction is proved; for it is quite simple to prove the result for $m = 1$ or 2.

To prove the more general case, we suppose that the letters $b_1, \ldots, b_{m+1}$ occupy r_1 places in one row, r_2 in a lower, and so on, ending with r_k in the lowest row touched. Then

$$[F_1]_1 = [F_{1,1}]_1$$
$$= \{1+1/(\alpha_{k-1}+r_k-1)\}\{1+1/(\alpha_{k-1}+r_k-2)\}\ldots\{1+1/\alpha_{k-1}\}[F_{2,1}]_1;$$

here α_{k-1} is the axial projection of a pair of letters occupying the extreme right-hand positions in the last two rows. Then

$$[F_1]_1 = \{1+r_k/\alpha_{k-1}\}[F_{2,1}]_1 = \{1+r_{k-1}/\alpha_{k-2}\}\{1+r_k/(\alpha_{k-2}+\alpha_{k-1})\}[F_{3,1}]_1 = \ldots$$

and

$$[F_l]_2 = [F_{k,l}]_2 = \{1-r_1/\alpha_1\}[F_{k-1,l}]_2 = \{1-r_2/\alpha_2\}\{1-r_1/(\alpha_2+\alpha_1)\}[F_{k-2,l}]_2 = \ldots.$$

The total number of tableaux is

$$N = (m+1)!/\Pi r!,$$

and the number of tableaux when b_{m+1} is in the s-th row is

$$N \cdot r_s/(m+1) = N_{k+1-s}.$$

We have to sum $$\sum_{s=1}^{k} N_s/[F_{s,1}]_1 [F_{s,l}]_2.$$

The result follows exactly as before.

THEOREM V. *A substitutional expression S, which is unaltered by left-hand multiplication with the permutations of the positive symmetric group of m consecutive letters, may be expressed in exact seminormal units as a sum of sets of terms*

$$\Sigma A\varpi_u/([F_u]_1 [F_u]_2),$$

where A is constant for each set, and the tableaux of a set are those tableaux obtained from each other by operations of the group, no two of the letters of the group lying on the same column. Further, when F_1, F_l are the first and last tableaux of the set, the average of the coefficients is

$$A/([F_1]_1 [F_l]_2).$$

Multiplication by transpositions of consecutive letters on the right shows that for the symmetric group of m letters the columns corresponding to the tableaux F_u of a set are all identical. Consider then the matrix representation of $G/m!$, where G is the positive symmetric group of m consecutive letters.

The form is determined by Theorem V, and the fact that the u columns are all identical. The constant A is given by the fact that $G/m!$ multiplied by itself simply reproduces itself.

But this multiplication by itself simply reproduces each element multiplied by the sum of the elements. This sum is

$$NA/\{[F_1]_1\,[F_l]_2\} = 1,$$

which gives A.

7. In exactly the same way it is found that, for multiplication of S on the left by a transposition, S is changed in sign when

$$\lambda_p = 0, \quad \lambda_s = -\lambda_t.$$

For multiplication on the right we have to consider a row in the matrix or

$$S = \sum_v \lambda_v \varpi_{uv};$$

and then S is changed in sign provided that

$$\lambda_p = 0, \quad (1-\rho)\lambda_s = -(1+\rho)\lambda_t,$$

i.e.

$$\lambda_s/\{[F_s]_1\,[F_s]_2\} = -\lambda_t/\{[F_t]_1\,[F_t]_2\}.$$

Thus we obtain

THEOREM VI. *A substitutional expression S, which is changed in sign by right-hand multiplication with any odd permutation of the symmetric group of m consecutive letters, may be expressed in exact seminormal units as a sum of sets of terms*

$$\sum_v \pm A\,[F_v]_1\,[F_v]_2\,\varpi_{uv},$$

where A is constant for each set, the tableaux F_v of a set are those obtained by permuting the m letters in one of them, no two of the letters being in the same row. Further, when F_1, F_l are the first and last tableaux of the set, the average numerical value of the coefficients is

$$A\,[F_l]_1\,[F_1]_2.$$

The proof of the last fact follows exactly the same lines as that of the corresponding fact of Theorem V. The expression for $\Gamma/m!$, where Γ is the negative symmetric group of m consecutive letters, is obtained as before; it is necessary to remember in this case that the signs are alternately positive and negative both in the rows and columns of the matrix.

III. *The expression for a unit belonging to a consecutive set of letters.*

8. It is necessary for certain applications of this analysis to express the units $\Pi_{r,s}$ of a representation $T_{\beta_1\beta_2\ldots}$ of the symmetric group of $b_1 b_2 \ldots b_n$ in terms of the units $\varpi_{\rho\sigma}$ of a representation $T_{\gamma_1\gamma_2\ldots}$ of the symmetric group of $a_1 a_2 \ldots a_m b_1 \ldots b_n$. Exact seminormal units will be used, so that a transposition of consecutive letters will take the form of quadratic matrices

$$\begin{bmatrix} -\rho & 1-\rho \\ 1+\rho & \rho \end{bmatrix},$$

or linear matrices $[1]$ or $[-1]$ on the leading diagonal and zero elsewhere. The letter sequence will be taken as that written above, so that all the letters b are consecutive.

Consider any tableau F_ρ of T_γ. Let all the letters b be erased and there will remain a tableau of the a's alone; this will be standard and we shall call it F_{ρ_1}. Now erase the letters a in F_ρ, there remains a tableau of the b's alone, which, however, differs in general from the ordinary tableau in that some of the lower rows begin on an earlier column than the higher rows. It is a tableau distorted by moving some of the upper rows to the right. It must be standard as it is, but will not, in general, be standard when the rows are moved into the regular position. We call the distorted b tableau F'_{ρ_2}.

Then we use, instead of F_ρ, the double suffix notation F_{ρ_1,ρ_2}.

The quadratic matrices which appear in the representation of a transposition of consecutive letters have elements whose row and column tableaux differ only by this transposition. Hence any substitutional expression S_b in the b's can be expressed in the form

$$(\text{II}) \qquad S_b = \sum_{\rho_1} \Big[\sum_{\rho_2,\sigma_2} \lambda_{\rho_2\sigma_2}\, \varpi_{\rho_1\rho_2:\rho_1\sigma_2} \Big].$$

That is, each set of terms with the tableau F_{ρ_1} in common is really a separate representation so far as the letters b are concerned.

Similarly we find

$$S_a = \sum_{\rho_3} \Big[\sum_{\rho_1, \sigma_1} \lambda_{\rho_1 \sigma_1} \varpi_{\rho_1 \rho_2 : \sigma_1 \rho_2} \Big].$$

When S_a belongs to the representation $T_{\alpha_1 \alpha_2 \ldots}$, each of the tableaux F_{ρ_1} belongs to this representation, and, in fact,

$$S_a = \sum \lambda_{\rho_1 \sigma_1} \varpi_{\rho_1 \sigma_1}.$$

When all the rows of F_{ρ_1} are equal, and in number equal to that of the rows in the whole F_{ρ_1, ρ_2} tableau, F'_{ρ_2} is no longer distorted. The expression for S_b is obtained then directly from the b matrices. Hence we have

THEOREM VII. *Let the representation τ_{a^h} (which has tableaux with equal rows) have units ϖ_{rs} and let*

$$S_1 = \sum \lambda_{rs} \varpi_{rs};$$

let $T_{\beta_1 \beta_2 \ldots \beta_k}$, a representation of a set of independent letters, have units ϖ_{uv} and let

$$S_2 = \sum \mu_{uv} \varpi_{uv}.$$

Then

$$S_1 S_2 T_{\alpha+\beta_1, \alpha+\beta_2, \ldots} = \sum \lambda_{rs} \mu_{uv} \varpi_{ru : sv},$$

h being equal to or greater than k, and the sequence of letters being such that those of S_1 precede those of S_2.

9. Consider now equation (II) when the tableau F'_{ρ_2} is distorted.

Those terms for which the first tableau F_{ρ_1} are the same form a set. An operation, on the left or right, with any permutation of the b's on a unit of this set gives a linear function of units of the same set. Thus the matrix formed by the coefficients of the set is a representation of the symmetric group of $b_1 b_2 \ldots b_n$. This representation corresponds to the distorted tableaux F'_{ρ_2}; let g be its order. Then, in general, this is not an irreducible representation and so a matrix Λ exists such that $\Lambda^{-1}[\lambda_{\rho_2 \sigma_2}]\Lambda$ is a matrix with partial square matrices on the leading diagonal (which are irreducible group matrices of the b's), and zeros elsewhere. Now let $S_b = [\mu_{rs}]$ be the group matrix of the b's for the representation $T_{\beta_1 \beta_2 \ldots \beta_k}$. Then the corresponding matrix $[\lambda_{\rho_2 \sigma_2}]$ for the particular set defined by F_{ρ_1} is transformable into a matrix which has $[\mu_{rs}]$ for a diagonal matrix, possibly repeated, and zeros elsewhere.

Thus

$$[\lambda_{\rho_2\sigma_2}]\Lambda = \Lambda \begin{bmatrix} [\mu_{rs}] & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots \end{bmatrix}. \tag{III}$$

Let f be the order of $[\mu_{rs}]$; then, if we select the f columns of Λ which correspond to the position of $[\mu_{rs}]$ in the last matrix (the first f columns as it is written), and delete the rest, we shall obtain a matrix Λ_1 of g rows and f columns such that

$$[\lambda_{\rho_2\sigma_2}]\Lambda_1 = \Lambda_1[\mu_{rs}].$$

The value of a transposition of consecutive letters is known in terms of both λ and μ matrices, and the equations so obtained give the means of calculating Λ_1. When the solution is unique, except for a proportional factor, the matrix appears but once in the matrix on the right of (III). When it is repeated there, the solution for Λ_1 is not unique.

In the same way we may find a matrix Λ_1' of f rows and g columns, which is part of the matrix Λ^{-1}, such that

$$\Lambda_1'[\lambda_{\rho_2\sigma_2}] = [\mu_{rs}]\Lambda_1'.$$

And from the relation of Λ_1, Λ_1' to Λ, Λ^{-1} we obtain

$$\Lambda_1'\Lambda_1 = E_f.$$

10. Consider, then, the equation

$$\Lambda D = \Delta\Lambda.$$

A transposition τ of consecutive letters is represented in D and Δ on a given row (or column) by $+1$ or -1 on the diagonal, or else on a pair of rows (or columns), by a quadratic matrix of the form

$$\begin{bmatrix} -p & 1-p \\ 1+p & p \end{bmatrix} \equiv \{p\}_{r,s},$$

where the suffixes denote the rows and columns concerned. Then the above equation indicates the equality of corresponding elements; it is convenient to group the elements according to the elementary diagonal matrices. There are thus nine cases.

(i) As regards D, τ may have the value $+1$ or -1 on the r-th place on the diagonal or the matrix $\{p\}_{r,s}$ where the r-th and s-th rows and columns intersect.

(ii) Similarly for Δ, there are three cases

$$\{1\}_\rho, \quad \{-1\}_\rho, \quad \{q\}_{\rho,\sigma}.$$

The simultaneous consideration of $\{1\}_r$ and $\{1\}_\rho$, or of $\{-1\}_r$ and $\{-1\}_\rho$, gives no information about $\lambda_{\rho r}$.

There are six cases of interest:

(I) $$\Lambda_\rho\{1\}_r = \{-1\}_\rho\Lambda_r, \quad \text{or} \quad \Lambda_\rho\{-1\}_r = \{1\}_\rho\Lambda_r,$$

gives $$\lambda_{\rho r} = 0.$$

(II) $$\Lambda_{\rho,\sigma}\{1\}_r = \{q\}_{\rho,\sigma}\Lambda_r$$

gives $$(1+q)\lambda_{\rho r} = (1-q)\lambda_{\sigma r}.$$

(III) $$\Lambda_{\rho,\sigma}\{-1\}_r = \{q\}_{\rho,\sigma}\Lambda_r$$

gives $$\lambda_{\rho r} = -\lambda_{\sigma r}.$$

(IV) $$\Lambda_\rho\{p\}_{r,s} = \{1\}_\rho\Lambda_{r,s}$$

gives $$\lambda_{\rho r} = \lambda_{\rho s}.$$

(V) $$\Lambda_\rho\{p\}_{r,s} = \{-1\}_\rho\Lambda_{r,s}$$

gives $$(1-p)\lambda_{\rho r} = -(1+p)\lambda_{\rho s}.$$

(VI) $$\Lambda_{\rho\sigma}\{p\}_{r,s} = \{q\}_{\rho,\sigma}\Lambda_{r,s}$$

gives four equations, each one of which omits one of the four quantities concerned, giving what is, in general, a unique relation between the other three; for only two of the equations are independent. The relation

$$\lambda_{\rho r}+\lambda_{\sigma r} = \lambda_{\rho s}+\lambda_{\sigma s}$$

may be deduced from them.

For our purposes, the equations of most importance are those obtained when it is known otherwise that one of the four elements is zero. These are:

(A) $\lambda_{\rho r} = 0$: $\quad \lambda_{\rho s}/(1-q) = \lambda_{\sigma r}/(1+p) = \lambda_{\sigma s}/(p+q).$

(B) $\lambda_{\rho s} = 0$: $\quad \lambda_{\rho r}/(1-q) = \lambda_{\sigma s}/(1-p) = \lambda_{\sigma r}/(q-p).$

(C) $\lambda_{\sigma r} = 0$: $\quad \lambda_{\rho r}/(1+p) = \lambda_{\sigma s}/(1+q) = \lambda_{\rho s}/(p-q).$

(D) $\lambda_{\sigma s} = 0$: $\quad \lambda_{\rho s}/(1-p) = \lambda_{\sigma r}/(1+q) = -\lambda_{\rho s}/(p+q).$

(VI a) The special case $p = q$ yields

$$\lambda_{\rho s}/(1-p) = \lambda_{\sigma r}/(1+p) = (\lambda_{\sigma s} - \lambda_{\rho r})/2p.$$

11. Let D be a matrix of order f of $T_{\beta_1 \beta_2 \ldots \beta_k}$ and Δ the corresponding matrix of order g which results by distorting the tableau, the number of letters in each row remaining the same. Then Λ is a matrix with g rows and f columns. Consider the first row; this corresponds to the distorted tableau F_1' in which the first β_1 letters lie in the first row, and so on. For the transposition of any consecutive pair of letters from the first β_1 the diagonal matrix for the first row is $\{1\}_1$. Let F_r be a tableau with b_2 below the first row, then a diagonal matrix of D for $(b_1 b_2)$ is $\{-1\}_r$.

Hence, by (I), $$\lambda_{1r} = 0.$$

Then, from $(b_2 b_3)$ and (IV),

$$\lambda_{1s} = 0,$$

where $$(b_2 b_3) F_r = F_s.$$

In the same way we see that, when F_s has any one of the letters $b_2, b_3, \ldots, b_{\beta_1}$ in the first place of the second row,

$$\lambda_{1s} = 0.$$

Now b_{β_1+1} is the latest letter which can occupy this position, and when it does so the first row is necessarily the same as in F_1. We next observe that, when the first row in F_r is as in F_1 but b_{β_1+2} is in the third row, then

$$\lambda_{1r} = 0.$$

Proceeding as before, we find that

$$\lambda_{1r} = 0$$

unless the first two rows of F_r are the same as in F_1. The next row can be then considered, and we finally see that

$$\lambda_{1r} = 0$$

unless $r = 1$.

The tableaux F_r of D are arranged according to some principal sequence, and the same sequence is used for the tableaux F_ρ'. Two tableaux F_r, F_ρ' are said to correspond when the rows of the two tableaux are the same, and this will be written

$$F_\rho' = F_r.$$

In general, F_{r+1} and $F'_{\rho+1}$ do not correspond, since between $\bar{F}_r$ and $\bar{F}_{r+1}$ there may be distorted tableaux which become non-standard when the rows are shifted back into the usual position. It will now be proved that $\lambda_{\rho r}$ is zero when $\bar{F}_r$ is a tableau later than F_ρ' in any principal sequence. This has been proved for $\rho = 1$; it will be assumed for all values of ρ less than the given one. Since $\rho > 1$, there is some letter b_u which is in a lower row in F_ρ' than b_{u+1}.

Then $$(b_u b_{u+1}) F_\rho' = F_\pi',$$

where $\pi < \rho$.

Let $\bar{F}_r$ be a tableau later than F_ρ'; then, by hypothesis,

$$\lambda_{\pi r} = 0;$$

and, unless b_u, b_{u+1} lie obliquely in F_r, it follows at once that

$$\lambda_{\rho r} = 0.$$

When b_u, b_{u+1} lie obliquely in F_r, let

$$(b_u b_{u+1}) F_r = F_s.$$

When $s > r$, $$\lambda_{\pi s} = 0;$$

and hence again $$\lambda_{\rho r} = 0.$$

When $s < r$, there is always some principal sequence according to which F_π' precedes $\bar{F}_s$. For it is possible to arrange a principal sequence that regards b_u and b_{u+1} as identical letters in all cases except that of the determination of the relative position of a pair of tableaux which are identical except for the interchange of these letters. Thus, when F_ρ' precedes $\bar{F}_r$ according to any principal sequence, then a principal sequence can be found for which F_π' precedes $\bar{F}_s$.

Hence, by hypothesis,

$$\lambda_{\pi r} = 0, \quad \lambda_{\pi s} = 0.$$

When two of the four elements in (VI) are zero, the other two are zero, and hence

$$\lambda_{\rho r} = 0, \quad \lambda_{\rho s} = 0,$$

which proves the statement.

Let $$F_\rho' = \bar{F}_r, \quad F_\sigma' = \bar{F}_s$$

be tableaux such that $r > s$ and

$$\tau F_r = F_s;$$

where τ is a transposition of consecutive letters. Then $\lambda_{\rho s} = 0$, and hence, by (VI), (B),

$$\lambda_{\rho r}/(1-q) = \lambda_{\sigma s}/(1-p) = \lambda_{\sigma r}/(p-q);$$

whence $$\frac{\lambda_{\rho r}}{\lambda_{\sigma s}} = \frac{[F_r]_2}{[F_\rho']_2} \Big/ \frac{[F_s]_2}{[F_\sigma']_2}.$$

Thus the "skin" elements, those dividing the zero part of the matrix from the rest on the rows whose tableaux correspond to standard undistorted tableaux, have the form

$$\lambda_{\rho r} = \lambda\,[F_r]_2/[F_\rho']_2.$$

In exactly the same way it may be shown for the matrix Λ', for which

$$D\Lambda' = \Lambda'\Delta,$$

that $\lambda'_{r\rho} = 0$, when F_r is later than F_ρ' and that, for the skin elements $\lambda'_{r\rho}$,

$$\lambda'_{r\rho} = \lambda'[F_\rho']_1/[F_r]_1.$$

THEOREM VIII. *The group matrix D of order f for the representation $T_{\beta_1\beta_2\ldots\beta_k}$ is represented as a matrix Δ of order g obtained by lateral distortion of the tableaux, the sizes of the rows being unchanged. Then a matrix Λ of g rows and f columns exists, which is unique but for a proportional factor, such that*

$$\Lambda D = \Delta\Lambda.$$

The element $\lambda_{\rho r}$ of Λ is zero when the distorted tableau F_ρ' precedes $\bar{F}_r$ in any principal sequence, and the skin elements are

$$\lambda_{\rho r} = \lambda[F_r]_2/[F_\rho']_2.$$

Also there is a unique matrix Λ' of f rows and g columns, for which

$$D\Lambda' = \Lambda'\Delta.$$

The element $\lambda'_{r\rho}$ of Λ' is zero when F_ρ' precedes $\bar{F}_r$ in any principal sequence,

and the skin elements are

$$\lambda'_{r\rho} = \lambda'[F_\rho']_1/[F_r]_1.$$

Also

$$\Lambda'\Lambda = E_f.$$

The only statement in the enunciation not yet proved is that the matrices Λ, Λ' are unique.

If Λ is not unique, let M be a second matrix fulfilling the conditions, then

$$(\Lambda - M)D = \Delta(\Lambda - M).$$

The proportional factors may be chosen so that in

$$\Lambda - M = [\mu_{\rho r}]$$

any definite element is zero.

This will be taken to be μ_{11}. Then $[\mu_{\rho r}]$ satisfies all the equations obtained for $[\lambda_{\rho r}]$, and hence the whole of the first row is zero. But the equations of § 10 obtain all the elements of a row defined by F_ρ' in terms of those of any row defined by F_σ' for which

$$F_\sigma' = \tau F_\rho',$$

where τ is a transposition of consecutive letters. The elements of any row, by repeated application of this, can then be expressed in terms of those of the first row; and thus all the rows are zero.

The matrix Λ is then unique, and also Λ'. This principle may be used in all cases to which the equations of § 10 apply; and the general result is that the number of undetermined constants in every row is the same. That is, the number of times a given group matrix appears in a distorted tableau representation is the same as the number of independent elements in any row of Λ as given by the equations of § 10. The group matrix is repeated only when the sizes of the rows are changed.

12. The last three paragraphs may be illustrated by the representation T_{32}. Let us make a distortion by shifting the second row one place to the left. There are nine distorted tableaux with the sequence from the last letter; their second rows are:

45,	35,	25,	15,	34,	24,	14,	23,	13
$\rho = 1$	2	3	4	5	6	7	8	9.

Then $$F_r' = \bar{F}_r \quad (r = 1, 2, 3).$$

$$F_5' = \bar{F}_4, \quad \bar{F}_6' = \bar{F}_5.$$

Also

$$36\Lambda = \begin{bmatrix} 18 & 0 & 0 & 0 & 0 \\ -2 & 16 & 0 & 0 & 0 \\ -4 & -4 & 12 & 0 & 0 \\ -12 & -12 & -12 & 0 & 0 \\ -4 & -4 & 0 & 12 & 0 \\ -8 & 1 & -3 & -3 & 9 \\ -24 & 3 & 3 & -9 & -9 \\ 0 & -9 & -9 & -9 & -9 \\ 0 & -27 & 9 & -27 & 9 \end{bmatrix}$$

and

$$36\Lambda' = \begin{bmatrix} 60 & -4 & -4 & -4 & -4 & -4 & -4 & 0 & 0 \\ 0 & 64 & -8 & -8 & -8 & 1 & 1 & -3 & -3 \\ 0 & 0 & 72 & -24 & 0 & -9 & 3 & -9 & 3 \\ 0 & 0 & 0 & 0 & 72 & -9 & -9 & -9 & -9 \\ 0 & 0 & 0 & 0 & 0 & 81 & -27 & -27 & 9 \end{bmatrix}.$$

13. Let ϖ_{uv} be a unit of $T_{a_1 a_2 \ldots a_h}$ of the letters $a_1, a_2, \ldots, a_m$, and Π_{rs} a unit of $T_{\beta_1 \beta_2 \ldots \beta_k}$ of $b_1, b_2, \ldots, b_n$. The product $\varpi_{uv}\Pi_{rs}$, expressed in terms of units of $a_1, a_2, \ldots, a_m, b_1, \ldots, b_n$, has terms which are units from many different representations $T_{\gamma_1 \gamma_2 \ldots}$. For our purposes the terms which belong to $T_{a_1+\beta_1, a_2+\beta_2, \ldots}$ have a special importance; they are

$$\varpi_{uv}\Pi_{rs}T_{a_1+\beta_1, a_2+\beta_2, \ldots}.$$

In § 8 it was seen that this is

$$\Sigma A_{\rho,\sigma}\varpi_{u,\rho;v,\sigma},$$

where the tableau $F_{u,\rho}$ becomes the "a" tableau F_u on removal of the b's, and the distorted "b" tableau F_ρ' on the removal of the a's.

In § 9 it was seen that a non-singular matrix L exists such that

$$L^{-1}[A_{\rho,\sigma}]L = K,$$

a matrix which has matrices D_1, D_2, ... on the leading diagonal and zeros elsewhere, where D_1, D_2, ... are the particular values for the case in point of the irreducible group matrices of the b's. Here every D is zero except those which arise from the representation $T_{\beta_1\beta_2\ldots\beta_k}$. These D's are, of course, all equal, and, indeed, in the case before us have the element $d_{rs}=1$ and all the other elements zero. They may be placed in any desired position on the diagonal.

Theorem VIII tells us that there is just one D, which we suppose to occupy the intersections of the first rows and columns in K, the rest of the matrix being zero. Moreover, it gives us the first f columns of L, and the first f rows of L^{-1}, thus

$$[A_{\rho\sigma}]=LKL^{-1}.$$

Hence
$$A_{\rho\sigma}=\lambda_{\rho r}\lambda'_{s\sigma}.$$

This is zero when F_ρ' precedes $\bar{F}_r$ or F_σ' precedes $\bar{F}_s$ in any principal sequence. It is not zero when F_ρ' corresponds to F_r, and F_σ' to F_s. This result is important when considering products of functions.

THEOREM IX. *The seminormal unit ϖ_{uv} belongs to $T_{\alpha_1\ldots\alpha_h}$ of the letters a; the unit Π_{rs} belongs to $T_{\beta_1\ldots\beta_k}$ of the letters b. Then*

$$\varpi_{uv}\,\Pi_{rs}\,T_{\alpha_1+\beta_1,\,\alpha_2+\beta_2,\ldots}=\Sigma A_{\rho\sigma}\,\varpi_{u,\rho;\,v,\sigma},$$

where $A_{\rho\sigma}$ is zero when F_ρ' precedes $\bar{F}_r$ or F_σ' precedes $\bar{F}_s$ in any principal sequence, and is not zero when F_ρ' corresponds to F_r, and F_σ' to F_s.

IV. *The case when the tableaux have two rows only.*

14. When the tableaux contain only two rows the matrices Λ, Λ' may be calculated as follows. We consider the most general case of this, that is, the product of a matrix belonging to $T_{n-k,\,k}$ of $b_1, b_2, \ldots, b_n$ by a matrix belonging to $T_{\alpha+\gamma,\,\alpha}$ of $a_1, a_2, \ldots, a_m$ $(m=2\alpha+\gamma)$ expressed in terms of a matrix belonging to $T_{n-k+\alpha+\gamma-l,\,k+\alpha+l}$. Obviously there will be no such matrix when

$$l>n-2k \quad \text{or} \quad l>\gamma.$$

As before, we have to consider a matrix D belonging to $T_{n-k,\,k}$ and a matrix Δ corresponding to a distortion of γ places to the left in

$$T_{n-k-l,\,k+l}.$$

The equations for λ, λ' are simplified by writing

$$\lambda_{\rho r}[F_\rho']_1\,[F_\rho']_2 = \mu_{\rho r},$$

$$\lambda'_{r\rho}[F_r]_1\,[F_r]_2 = \mu'_{r\rho}.$$

Then, from the equations of § 10, it follows that, when a_t, a_{t+1} lie in the same row of F_ρ', but their interchange changes F_r into F_s,

$$\mu_{\rho r} = \mu_{\rho s}, \quad \mu'_{r\rho} = \mu'_{s\rho}.$$

When $(a_t a_{t+1})$ changes F_ρ' into F_σ' but leaves F_r unchanged,

$$\mu_{\rho r} = \mu_{\sigma r}, \quad \mu'_{r\rho} = \mu'_{r\sigma}.$$

When a_t, a_{t+1} lie in the same row of F_ρ' and the same column of F_r,

$$\mu_{\rho r} = 0 = \mu'_{r\rho}.$$

The tableaux will be defined by their second rows:

$$F_\rho' : b_{x_1} b_{x_2} \dots b_{x_{k+l}}; \quad x_1 < x_2 \dots < x_{k+l}.$$

$$F_r : b_{y_1} b_{y_2} \dots b_{y_k}; \qquad y_1 < y_2 \dots < y_k.$$

Let F_r be a tableau in which $y_{u-1} < y_u - 1$, so that $b_{y_u - 1}$ lies in the upper row. Let its place be the ξ-th; then

$$\xi + u = y_u,$$

and the axial projection of $b_{y_u - 1} b_{y_u}$ is

$$\xi - u + 1 = y_u - 2u + 1.$$

In the case of the distorted tableau F_ρ' this becomes

$$x_u - 2u + 1 + \gamma.$$

Thus

$$[F_r]_1 = \prod_{u=1}^{k} (y_u - 2u + 2);$$

$$[F_\rho']_2 = \prod_{u=1}^{k+l} (x_u - 2u + 1 + \gamma).$$

When $y_1 = 2$ and $x_1 > 2$,

$$\lambda_{\rho r} = 0 = \lambda'_{r\rho}.$$

When $x_1 > 3$, and $(b_2 b_3)$ changes F_r to F_s,

$$\lambda_{\rho s} = 0 = \lambda'_{s\rho}.$$

Thus, when $y_1 < x_1$, $\lambda_{\rho r}$ and $\lambda'_{r\rho}$ are zero.

Let us assume that, provided that $u < v$, $\lambda_{\rho r}$ and $\lambda'_{r\rho}$ are zero when $y_u < x_u$. Let

$$y_v = 2v < x_v, \quad y_{v-1} = 2(v-1) \geqslant x_{v-1}.$$

Then $\lambda_{\rho r}$, $\lambda'_{r\rho}$ are zero, for b_{2v}, b_{2v-1} lie in the same row of F_ρ' and the same column of F_r; and hence $\lambda_{\rho s}$ and $\lambda'_{s\rho}$ are also zero when

$$y_{v-1} = 2(v-1) \geqslant x_{v-1}, \quad y_v < x_v.$$

Taking $x_{v-1} < y_{v-1}$, we may use the transpositions $(b_{2v-2} b_{2v-1})$, etc., to show that the pair of elements is zero when $y_v < x_v$ and

$$x_{v-1} < 2(v-1) < y_{v-1}.$$

When $x_{v-1} = y_{v-1}$, we use $(b_{2v-2} b_{2v-1}) = \tau_1$, but now it is case (VI) of § 10 involving four elements

$$\lambda_{\rho s}, \quad \lambda_{\rho t}, \quad \lambda_{\sigma s}, \quad \lambda_{\sigma t},$$

where $$\tau F_\rho' = F_\sigma', \quad \tau F_s = F_t.$$

In $\lambda_{\sigma s}$, $$y_{v-1} = 2(v-1) < x_{v-1},$$

and, by hypothesis, $\lambda_{\sigma s}$ is zero; also $\lambda_{\rho s}$ is zero, hence $\lambda_{\rho t}$ and λ_{st} are zero. Proceeding thus, we see that $\lambda_{\rho r}$ is zero when $y_v < x_v$, provided that neither x_{v-2} nor y_{v-2} is greater than $2v-3$. This last proviso can be removed step by step as before. Hence, in all cases, $\lambda_{\rho r}$ and $\lambda'_{r\rho}$ vanish when $y_u < x_u$. Let

$$t+1 = y_u = x_{u+v}, \quad y_u > y_{u-1}+1, \quad x_{u+v} > x_{u+v-1}+1, \quad \tau = (b_t b_{t+1}),$$

$$\tau F_r = F_s, \quad \tau F_\rho' = F_\sigma'.$$

Then τ gives the equations

$$\Lambda_{\rho,\sigma}\{(t-2u+2)^{-1}\}_{r,s} = \{(t-2u-2v+2+\gamma)^{-1}\}_{\rho,\sigma} \Lambda_{r,s},$$

$$\{(t-2u+2)^{-1}\}_{r,s} \Lambda'_{\rho,\sigma} = \Lambda'_{r,s} \{(t-2u-2v+2+\gamma)^{-1}\}_{\rho,\sigma}.$$

As was seen in § 10, each equation gives two independent equations for the

λ elements. These equations are satisfied when

$$\mu_{\rho\tau} = (t-2u-v+2)(t-2u-v+3+\gamma) = \mu'_{\tau\rho},$$

$$\mu_{\sigma s} = (t-2u-v+1)(t-2u-v+2+\gamma) = \mu'_{s\sigma},$$

$$\mu_{\rho s} = -v(\gamma-v+1) = \mu'_{s\rho},$$

$$\mu_{\sigma\tau} = -(v+1)(\gamma-v) = \mu'_{\tau\sigma}.$$

When $v=0$, $\mu_{\rho s}=0$, as is already known, for here $y_u < x_u$; and in this case the ratios of the other three elements are uniquely given by the equations. We take then

$$\mu_{\rho\tau} = \mu'_{\tau\rho} = \prod_{u=1}^{k} B_u^{\rho\tau},$$

where $$B_u = (y_u-2u-v+1)(y_u-2u-v+2+\gamma),$$

when $$y_u = x_{u+v};$$

$$B_u = -(v+1)(\gamma-v),$$

when $$x_{u+v+1} > y_u > x_{u+v};$$

$$B_u = 0,$$

when $$y_u < x_u;$$

and we observe that the result has been proved for $v=0$. Let

$$x_{u+v+1} = t+2, \quad y_{u+1} > t+2,$$

$$(b_{t+1} b_{t+2})\, F_r = F_w.$$

Then $$\mu_{\rho\tau} = \mu_{\rho w}.$$

But in $\mu_{\rho w}$, $$y_u = t+2 = x_{u+v+1};$$

the factor B_u, as given above, is unchanged in value, hence if the value is true for v it is true for $v+1$, as regards the case $y_u = x_{u+v}$. The other case follows from the fact that, as we have seen, the set of values for v verifies the equations of (VI), § 10.

The equations are thus uniquely solved. Thus, in general, the group representation by the matrix for the tableaux of $T_{n-k-l,\,k+l}$ distorted by a lateral shift of the lower row γ places to the left is made up of each

irreducible representation $T_{n-g,\,g}$, $g = 0, 1, \ldots, k+l$, or

$$g = k+l-\gamma,\ k+l-\gamma+1,\ \ldots,\ k+l$$

when $\gamma < k+l$, once and once only. This may be further verified by noticing that the order of the reducible representation is equal to the sum of the orders of the irreducible representations. In the case when $l = 0$, the matrices Λ, Λ' thus obtained satisfy the equation

$$\Lambda'\Lambda = E.$$

The equations have determined the matrices except for a proportional factor. It is then sufficient to consider a single element of the product to determine them completely. We take

$$S = \sum_{\rho} \lambda'_{1\rho}\lambda_{\rho 1} = \{[F_1]_1[F_1]_2\}^{-1} \sum_{\rho} \{[F_\rho{}']_1[F_\rho{}']_2\}^{-1} \prod_{u} [B_u^{\rho,1}]^2.$$

In the first place the value of γ will be supposed greater than k; the modifications for smaller values of γ will be indicated later.

The tableau F_1 has

$$y_1 = n-k+1,\quad y_2 = n-k+2,\quad \ldots,\quad y_k = n.$$

Let $F_\rho{}'$ have

$$x_{k-r} < n-k+1,\quad x_{k-r+1} \geqslant n-k+1.$$

Then

$$\prod_{u}\ {}_{u} B^{\rho,1} = (-)^{k-r}(k-r)!\ \gamma_{k-r}(n-2k+r)_r(\gamma+n-2k+r+1)_r.$$

This is independent of the particular values of $x_1, x_2, \ldots, x_{k-r}$, provided that all are less than $n-k+1$, and of those of $x_{k-r+1}, \ldots, x_k$, provided that each is greater than $n-k$. The factors of $[F_\rho{}']_1$ due to the first $k-r$ suffixes $x_1, \ldots$ will be denoted by $[F'_{\rho_1}]_1$, and those due to the last r suffixes by $[F'_{\rho_2}]_1$. Thus

$$[F_\rho{}']_1 = [F'_{\rho_1}]_1[F'_{\rho_2}]_1, \qquad [F_\rho{}']_2 = [F'_{\rho_1}]_2[F'_{\rho_2}]_2.$$

The variation of each set of factors is independent of the other set. Hence

$$\sum_{\rho} \{[F_\rho{}']_1[F_\rho{}']_2\}^{-1} = \left[\sum_{\rho_1} \{[F'_{\rho_1}]_1[F'_{\rho_1}]_2\}^{-1}\right]\left[\sum_{\rho_2} \{[F'_{\rho_2}]_1[F'_{\rho_2}]_2\}^{-1}\right].$$

Each of these sums is given by Theorem V, and thus their product is

$$\frac{\binom{n-k}{k-r}}{(\gamma+n-2k+r+1)_{k-r}\,\gamma_{k-r}} \cdot \frac{\binom{k}{r}}{(\gamma+n-2k+r+1)_r\,(\gamma+n-3k+2r)_r}.$$

Thus

$$(n-k+1)_k\,(n-k)_k\,S$$

$$= \sum_{r=0}^{k} \frac{(n-k)!\;k!\;\gamma_{k-r}\,(n-2k+r)_r\,(\gamma+n-3k+2r+1)}{(n-2k)!\;r!\;(\gamma+n-2k+1)_{k-r+1}}$$

$$= (n-k)_k \sum_{r=0}^{k} \frac{k!\;\gamma!\;(\gamma+p-k+r)!}{(\gamma-k+r)!\;(\gamma+p+1)!}\binom{p+r}{r}(\gamma+p-k+2r+1),$$

where $$p = n-2k.$$

It will now be shown that

$$\sum_{r=0}^{k}(\gamma+p-k+2r+1)\binom{p+r}{p}\binom{\gamma+p-k+r}{p}$$

$$= (p+1)\sum_{r=0}^{k}\left[\binom{p+r}{p}\binom{\gamma+p-k+r}{p+1}+\binom{p+r+1}{p+1}\binom{\gamma+p-k+r}{p}\right]$$

$$= (p+1)\,\Sigma = (p+1)\binom{\gamma+p+1}{\gamma}\binom{p+k+1}{k}.$$

This is equivalent to $S = 1$.

The result for Σ is true for $k = 0$ and for $k = 1$, irrespective of the values of γ and p.

It will be assumed to be true for a particular value of k. Let Σ become Σ' when k is replaced by $k+1$, and at the same time γ by $\gamma+1$; then

$$\Sigma'-\Sigma = \binom{p+k+1}{p}\binom{\gamma+p+1}{p+1}+\binom{p+k+2}{p+1}\binom{\gamma+p+1}{p}$$

$$= \binom{\gamma+p+2}{\gamma+1}\binom{p+k+2}{k+1}-\binom{\gamma+p+1}{\gamma}\binom{p+k+1}{k}.$$

Thus the result is true in general. It is to be noticed that the number of terms in Σ is finite, and the whole can be expressed as an algebraic identity of degree $p+1$ in γ for positive integral values of p and k. When $\gamma \leqslant k$ the argument as given breaks down, for when $\gamma < k-r$ the number of terms ρ_1 is diminished, and indeed Theorem V is no longer applicable to these terms. But in this case the general result has a vanishing factor affecting these terms, and thus the general result is still true.

Let us now consider the general case when the tableau F_ρ' has $k+l$ letters in the second row. When

$$x_{k+i-r+1} \geqslant n-k+1 > x_{k+l-r},$$

$$\prod_{u=1}^{k} B_{iu}^{\rho_1} = (-)^{k-r}(k+l-r)_{k-r}\,\gamma_{k-r}\,(n-2k-l+r)_r\,(\gamma+n-2k-l+r+1)_r.$$

Also, for all those tableaux F_ρ' for which these inequalities are true,

$$\Sigma\{[F_\rho']_1[F_\rho']_2\}^{-1} = \frac{\binom{n-k}{\gamma+l-r}}{(\gamma+n-2k-l+r+1)_{k+l-r}\,\gamma_{k+l-r}} \times \frac{\binom{k}{r}}{(\gamma+n-2k-2l+r+1)_r(\gamma+n-3k-2l+2r)_r}.$$

Now

$$(k+l-r)_{k-r}\,(n-2k-l+r)_r\binom{n-k}{k+l-r} = (n-k)_k\binom{n-2k}{l}.$$

Also

$$(\gamma+n-2k-l+1)_{k+l-2r}\,(\gamma+n-2k-2l+r+1)_r\,(\gamma+n-3k-2l+2r)_r$$
$$= (\gamma+n-2k-l+1)_l\,(\gamma+n-2k-2l+r+1)_{k+1}(\gamma+n-3k-2l+2r+1)^{-1}.$$

Let $$S = \sum_\rho \lambda'_{1\rho}\lambda_{\rho 1} = \Sigma\,[B^{\rho,1}_u]^2\{[F_\rho']_1[F_\rho']_2[F_1]_1[F_1]_2\}^{-1}.$$

Then

$$S(n-k+1)_k\,(\gamma+n-2k-l+1)_l\,\gamma_l\left\{\binom{n-2k}{l}\right\}^{-1}$$
$$\cdot \quad (k+l-r)_{k-r}\,(\gamma-l)_{k-r}\,(n-2k-l+r)_r$$
$$= \sum_{r=0}^{k}\binom{k}{r}\frac{\times(\gamma+n-2k-l+r+1)_r(\gamma+n-3k-2l+2r+1)}{(\gamma+n-2k-2l+r+1)_{k+1}}$$
$$= \frac{k!\,l!\,(p-l)!\,(\gamma-l)!}{(\gamma+p-l+1)!}\,Q,$$

where $p = n-2k$, and

$$Q = \phi(\gamma, p, l, k)$$
$$= \sum_{r=0}^{k}\binom{k+l-r}{l}\binom{\gamma+p-l+r+1}{l}\left[\binom{p-l+r+1}{p-l+1}\binom{\gamma+p-k-2l+r}{p-l} + \binom{p-l+r}{p-l}\binom{\gamma+p-k-2l+r}{p-l+1}\right].$$

It will now be shown that

$$S = \binom{n-2k}{l}\Big/\{(\gamma+n-2k-l+1)_l\,\gamma_l\};$$

this gives unity when $l = 0$, a result already established. It requires

$$Q = \frac{(\gamma+p-l+1)!}{(p-l)!\,(\gamma-l)!\,l!}\binom{p+k+1}{k}.$$

Now

$$\binom{k+l-r+1}{l}\binom{\gamma+p-l+r+2}{l}$$

$$=\binom{k+l-r}{l}\binom{\gamma+p-l+r+1}{l}+\frac{\gamma+p+k-l+3}{l}\binom{k+l-r}{l-1}\binom{\gamma+p-l+r+1}{l-1}.$$

Hence

$$\phi(\gamma+1, p, l, k+1)-\phi(\gamma, p, l, k)$$
$$=l^{-1}(\gamma+p+k-l+3)\,\phi(\gamma, p-1, l-1, k+1).$$

It is a simple matter to verify that the proposed result satisfies this equation. If then the result is true up to a given value of k, also for $k+1$ and $l-1$, irrespective of γ and p, then it is true for $k+1$ and l, and hence for $k+1$ and $l+1$, etc. Now it is true for $l=0$ and any k. Hence it is true for $k+1$ if it is true for k. It is easy to see that it is true for $k=0$, 1, or 2, and any l. Thus the result is true universally.

THEOREM X. *The exact seminormal unit* Π_{uv} *belongs to* $T_{\alpha+\gamma,\alpha}$ *of* $a_1, a_2, \ldots, a_m$; *the unit* π_{rs} *belongs to* $T_{n-k,k}$ *of* $b_1, b_2, \ldots, b_n$. *Then*

$$\Pi_{u,v}\pi_{r,s}T_{n-k+\alpha+\gamma-1,\alpha+k+l}$$
$$=\gamma_l(\gamma+n-2k-l+1)_l\left\{\binom{n-2k}{l}\right\}^{-1}\sum_{\rho,\sigma}\lambda_{\rho r}\lambda'_{s\sigma}\varpi_{u\rho;v\sigma},$$

where ϖ *is the unit whose row tableau is formed by the combination of* F_u *and* F_ρ' *and column tableau by the combination* F_v *and* F_σ'; *where the second row letters defining the tableaux are*

$$F_\rho': \quad b_{x_1}b_{x_2}\ldots b_{x_{k+l}}; \quad x_1<x_2\ldots<x_{k+l};$$
$$F_r: \quad b_{y_1}b_{y_2}\ldots b_{y_k}; \quad y_1<y_2\ldots<y_k;$$
$$\lambda_{\rho r}[F_\rho']_1[F_\rho']_2=\lambda'_{r\rho}[F_r]_1[F_r]_2=\prod_{t=0}^{k}B_t^{\rho,r};$$

and where

$$B_t^{\rho,r}=0, \quad y_t<x_t,$$
$$B_t^{\rho,r}=(y_t-2t-w+1)(\gamma+y_t-2t-w+2), \quad y_t=x_{t+w},$$
$$B_t^{\rho,r}=-(w+1)(\gamma-w), \quad x_{t+w+1}>y_t>x_{t+w}.$$

15. Two results are given here without proof. They are, I think, of sufficient interest and value to be put on record; for they show the

manner in which the numerical values of the matrix elements depend on the tableaux, and particularly on the tableau functions; and they are of use in calculation. On the other hand, once the form of these results is known, the verification is merely a matter of repeated application of processes already given.

In Q.S.A., VI, § 17, the seminormal transforming matrix M for the representation on $T_{n-k,\,k}$ was obtained. When D is the seminormal and C the natural matrix,

$$D = M^{-1} C M.$$

The following theorem gives the reciprocal matrix M^{-1}. It has been demonstrated by induction in a manner which is practically a repetition of the argument of § 17 just referred to. Probably the methods of this paper would verify its truth more quickly.

THEOREM XI. *The reciprocal M^{-1} of the seminormal transforming matrix for the representation $T_{n-k,\,k}$ is*

$$M^{-1} = [m_{\lambda\mu}] = [p_{\lambda\mu}/p_{\lambda\lambda}],$$

where the rows and columns are defined by the tableaux F_λ, F_μ, with second row suffixes

$$\lambda_1, \lambda_2, \ldots, \lambda_k; \quad \mu_1, \mu_2, \ldots, \mu_k;$$

and where

$$p_{\lambda\lambda} = [F_\lambda]_2 = \prod_{r=1}^{k} (\lambda_r - 2r + 1), \quad p_{\lambda\mu} = \prod_{r=1}^{k} B_r^{\lambda\mu},$$

$$B_r^{\lambda\mu} = 0, \quad \mu_r > \lambda_r.$$

$$B_r^{\lambda\mu} = \lambda_{r-s} - 2r + s + 1, \quad \mu_r = \lambda_{r-s},$$

$$B_r^{\lambda\mu} = -s, \quad \lambda_{r-s+1} > \mu_r > \lambda_{r-s}.$$

The second result may be verified by the same method as the proof of Theorem X given here. It gives the expression for the permutation

$$(a_1 a_n)(a_2 a_{n-1}) \ldots (a_r a_{n+1-r}) \ldots,$$

which inverts the sequence of letters and may be called the inverting permutation; the result is as follows.

THEOREM XII. *The exact seminormal matrix*

$$K = [\kappa_{xy}]$$

is the expression for the inverting permutation in the representation $T_{n-k,k}$, *where*

$$[F_x]_1 [F_x]_2 \kappa_{xy} = \varpi_{xy},$$

$$\varpi_{xy} = \prod_{r=1}^{k} B_r^{xy},$$

$$B_r^{xy} = 0, \quad y_r < n+1-x_{k-r+1},$$

$$B_r^{xy} = (x_{k-r+1-s} - 2k + 2r - 1 + s)(n - 2r + 2 - s - x_{k-r+1-s}),$$

$$y_r = n+1-x_{k+1-r-s},$$

$$B_r^{xy} = -s(n-2k+s+1), \quad x_{k-r+1-s} < n+1-y_r < x_{k-r+2-s}.$$

V. *Algebraic equivalents of seminormal units.*

16. The relations between the seminormal units and the natural units are, of course, inherent in the equation

$$MDM^{-1} = C$$

connecting the seminormal matrix D to the natural matrix C. In a few cases the relations between the units may be very simply expressed.

THEOREM XIII. *The seminormal units* ϖ_{11}, ϖ_{ff}, ϖ_{1f}, ϖ_{f1} *are given by*

$$\varpi_{11} = f/(n!\, g) P_1 N_1 P_1,$$

$$\varpi_{ff} = f/(n!\, \gamma) N_f P_f N_f,$$

$$\varpi_{1f} = p_{1f} = f/n!\, P_1 \sigma_{1f} N_f,$$

$$\varpi_{f1} = 1/g\gamma N_f \sigma_{f1} P_1,$$

where g *is the product of the orders of the groups of* P, γ *is that of the orders of the groups of* N *and* p_{1f} *is the natural unit.*

In every case the matrices M, M^{-1} have the property that their elements m_{rs}, m'_{rs} are zero for $r > s$, and unity for $r = s$.

To find the seminormal unit ϖ_{11} in terms of the natural units, we put for D that matrix which has $d_{11} = 1$, and all other elements zero. Then

$$\varpi_{11} = \Sigma m'_{1r} p_{1r}.$$

Hence

$$\varpi_{11} P_1 = p_{11} P_1 = f/n!\, P_1 N_1 P_1.$$

Now every permutation of P_1, expressed as a seminormal matrix, has zero for every element of the first row and column, except for that on the leading diagonal which is unity. Hence

$$\varpi_{11} P_1 = g\varpi_{11};$$

this establishes the value of ϖ_{11}. The value of ϖ_{ff} is obtained in the same way; that of ϖ_{1f} comes even more directly from the matrix equation.

Finally,

$$N_f \varpi_{f1} P_1 = g\gamma\varpi_{f1}.$$

Hence

$$\varpi_{f1} = \lambda N_f \sigma_{f1} P_1.$$

The value of λ is obtained at once from the product

$$\varpi_{f1}\varpi_{1f} = \varpi_{ff}.$$

The *exact* units are the same for ϖ_{11} and ϖ_{ff}, but for ϖ_{1f} we must multiply by $[F_f]_1/[F_1]_1$ and for ϖ_{f1} by $[F_1]_1/[F_f]_1$; see Q.S.A., VI, §18.

COROLLARY. *It follows from the above reasoning that*

$$TP_1 = g\varpi_{11}, \quad TN_f = \gamma\varpi_{ff}.$$

17. A representation $T_{a_1 a_2 \dots a_k}$ is defined by the lengths of the rows of its tableaux, it might equally well be defined by the lengths of the columns $\beta_1 = h, \beta_2, \dots, \beta_k$. Let us suppose the letters $a_1, a_2, \dots, a_n$ which are the subject of permutations to be really h-ary variables. Then with any column $b_1 b_2 \dots b_\beta$ in the tableau may be associated a determinant

$$\{b_1 b_2 \dots b_\beta\}' b_{11} b_{22} \dots b_{\beta\beta} = (b_1 b_2 \dots b_\beta),$$

and with the whole tableau the product of these determinant factors.

THEOREM XIV. *The product*

$$X = (a_1 a_2 \dots a_{\beta_1})(a_{\beta_1+1} a_{\beta_1+2} \dots a_{\beta_1+\beta_2}) \dots$$

has the substitutional properties of ϖ_{ff} and no others.

It is seen at once that

$$N_f X = \gamma X.$$

Hence $T'X$ is zero when the first of the differences $\beta_1 - \beta_1', \beta_2 - \beta_2', \dots$ which does not vanish is positive. Also when $\beta_1' > \beta_1$,

$$N'X = 0,$$

for the variables contain no more than β_1 possible second suffixes. Similarly, this equation is true when the first of the differences

$$\beta_1' - \beta_1,\ \beta_2' - \beta_2,\ \dots$$

which does not vanish is positive. Thus it is true unless $T' = T$. Hence

$$T'X = 0,$$

unless

$$T' = T.$$

Hence

$$X = TX = 1/\gamma\, TN_f X = \varpi_{ff} X,$$

which proves the statement.

COROLLARY. *The form, in single non-homogeneous variables,*

$$Y = \Delta_{1,2\ \dots,\beta_1} \Delta_{\beta_1+1,\beta_1+2,\dots,\beta_1+\beta_2} \cdots \Delta_{\beta_1+\beta_2+\dots+\beta_{k-1}+1,\dots,\beta_1+\beta_2+\dots+\beta_k},$$

where

$$\Delta_{1,2,\dots,\beta_1} = \{a_1 a_2 \dots a_{\beta_1}\}'\, a_1^{\beta_1-1} a_2^{\beta_1-2} \dots a_{\beta_1-1},$$

has the same property.

18. By operating with different permutations on the function X of the last paragraph, all kinds of products like X may be obtained; their substitutional form is given by the operations of these permutations on ϖ_{ff}. Thus there may be obtained a linear function X_s of such products corresponding to every unit ϖ_{sf}; this will be called the symbolical product equivalent of the tableau F_s. In exactly the same way we may obtain from Y a determinantal equivalent of the tableau F_s.

The symbolical products may be multiplied out, in which case X_s becomes a sum of products $\prod_{r=1}^{n} a_{r,u}$ with numerical coefficients. These terms may be arranged in a sequence just as the sequence of tableaux, the second suffix u_r defining the row of the letter a_r.

THEOREM XV. *The first term of the symbolical product equivalent of any tableau F_s according to any principal sequence is that defined by the tableau itself.*

Here a principal sequence is one defined as such in § 2. The term defined by the tableau itself is that obtained by writing for the second suffix u_r the number of the row in which the letter a_r lies in the tableau.

The theorem is obviously true for the tableau F_f. We assume its truth for all tableaux F_τ when $\tau \geqslant \sigma$. Then, when $\sigma < f$, there exists a transposition $(a_r a_{r+1})$ which changes F_σ into F_τ where $\tau > \sigma$.

Then $$(a_r a_{r+1}) X_\tau = (1-\rho) X_\sigma + \rho X_\tau,$$

where ρ is the inverse of the axial projection of $a_r a_{r+1}$ in either tableau. The earliest term of X_τ is S_τ, that given by the tableau F_τ. Hence the term S_σ given by the tableau F_σ, which is obtained by the transposition $(a_r a_{r+1})$ from S_τ, is certainly a term of $(a_r a_{r+1}) X_\tau$, while it is not a term of X_τ, hence it is a term of X_σ. Moreover, if possible, let S be an earlier term of X_σ than that defined by F_σ. Then $(a_r a_{r+1}) S = S'$ is a term of F_τ.

By hypothesis, S' is a later term than S according to any principal sequence. Taking the sequence from the first letter, if S' differs from S_τ in the first $r-1$ letters, then S' (and also S) is later than S_τ and also S_σ in this sequence. In the same way, taking the sequence from the last letter, if S' differs from S by a permutation affecting one of the last $n-r-1$ letters, it is later than S_τ in this sequence, and hence also S is later than S_σ in this sequence. Thus S cannot exist. And, in all cases, the term defined by the tableau F_s is the earliest term of the function X_s.

COROLLARY. *The function Y_s defined by the tableau F_s in the last paragraph has for its first term the term defined by the tableau.*

The above reasoning applies to this case.

VI. *The regular group matrix.*

19. The regular group matrix R of the symmetric group of n letters contains $n!$ rows and columns. It is usually defined as the matrix $[x_{PQ^{-1}}]$, where P is the same for each row, Q for each column, and P and Q are respectively each operation of the group in turn; x_P then represents a set of $n!$ arbitrary variables. Then there is a non-singular matrix Λ for which

$$\Lambda^{-1} R \Lambda = D,$$

where D is a matrix which consists of Σf smaller matrices about the principal diagonal and zeros elsewhere. These smaller matrices may be made to consist of a succession of sets each set containing f identical group matrices of order f. Further, we may suppose Λ chosen so that these group matrices are the exact seminormal group matrices. The matrix Λ

consists of a series of bands of columns. Corresponding to a particular group matrix of D of order f, there is a band of f columns which occupy the same position in Λ as the columns on which this matrix lies in D. And further, corresponding to the f identical group matrices in D, there will be f bands, of f columns each, which, however, are not identical since Λ is non-singular. In the same way there are bands of rows in Λ^{-1}. Schur has shown that the only matrices permutable with D are of the form K, where in K the sub-matrix defined by the f^2 rows and columns, occupied by a set of f identical group matrices of order f in D, is

$$[B_{rs}\, E_f],$$

where B_{rs} are f^2 arbitrary constants, and all the other elements are zero.

Then Λ may be replaced by ΛK, as the most general form of Λ. This, in effect, replaces the f bands corresponding to a particular representation by any f linear functions of them which are independent.

Let us now consider a transposition $\varpi = (a_a a_{a+1})$ of a pair of consecutive letters in R; the coefficient of x_ϖ is represented by $\frac{1}{2}n!$ quadratic matrices about the leading diagonal of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In D we must consider each group matrix separately, and the coefficient of x_ϖ is either ± 1 on the leading diagonal or else a quadratic matrix of the form

$$\begin{bmatrix} -\rho & 1-\rho \\ 1+\rho & \rho \end{bmatrix}$$

about this diagonal, and zeros elsewhere. Thus the equation

$$R\Lambda = \Lambda D$$

gives us a series of equations, of one of the two forms

(i) $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{r\sigma} \\ \lambda_{s\sigma} \end{bmatrix} = \pm \begin{bmatrix} \lambda_{r\sigma} \\ \lambda_{s\sigma} \end{bmatrix},$$

whence $$\lambda_{r\sigma} = \pm \lambda_{s\sigma}.$$

(ii) $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{r\sigma} & \lambda_{r\tau} \\ \lambda_{s\sigma} & \lambda_{s\tau} \end{bmatrix} = \begin{bmatrix} \lambda_{r\sigma} & \lambda_{r\tau} \\ \lambda_{s\sigma} & \lambda_{s\tau} \end{bmatrix} \begin{bmatrix} -\rho & 1-\rho \\ 1+\rho & \rho \end{bmatrix},$$

whence

$$\left.\begin{aligned}(1-\rho)\lambda_{s\sigma} &= -\rho\lambda_{s\tau}+\lambda_{r\tau},\\ (1-\rho)\lambda_{r\sigma} &= -\rho\lambda_{r\tau}+\lambda_{s\tau}.\end{aligned}\right\}, \qquad \left.\begin{aligned}\lambda_{r\sigma} &= -\rho\lambda_{s\sigma}+(1+\rho)\lambda_{s\tau},\\ \lambda_{s\sigma} &= -\rho\lambda_{r\sigma}+(1+\rho)\lambda_{r\tau}.\end{aligned}\right\}.$$

Similarly, if $$\Lambda^{-1} = [\mu_{rs}],$$

$$(1+\rho)\mu_{\sigma r} = -\rho\mu_{\tau r}+\mu_{\tau s},$$

$$(1+\rho)\mu_{\sigma s} = -\rho\mu_{\tau s}+\mu_{\tau r}.$$

Now the rows and columns of R may be supposed to be defined by Φ, which is nothing more than the n letters $a_1, a_2, \ldots, a_n$ arranged in some particular sequence. Then, in the above, if r is the numerical equivalent of Φ, s is that of $(a_\alpha a_{\alpha+1})\Phi$. Consider the last column of a band in Λ, the remaining $f-1$ columns of this band are obtained from it by the equations (ii) above.

The elements of this last column will be written λ_{rf}, where r has the values $1, 2, \ldots, n!$, corresponding to the $n!$ sequences Φ. Corresponding to $r=1$, the letters in Φ are arranged in order of the suffixes. An arbitrary function of n variables has, in general, $n!$ different values corresponding to the possible $n!$ permutations of the variables. We may then suppose λ_{rf} to be such a function ϕ_r of

$$A_1, A_2, \ldots, A_n,$$

derived from a function $\lambda_{1f} = \phi_1$ by permutations. But we define the connection between ϕ_r and ϕ_s thus. Let $(a_\alpha a_{\alpha+1}) = \varpi$ change Φ_r to Φ_s, let a_α occupy the p-th place in Φ_r and the q-th place in Φ_s, then

(IV) $$\phi_s = (A_p A_q)\phi_r.$$

Corresponding to any permutation ϵ_r for which

$$\Phi_r = \epsilon_r \Phi_1,$$

there exists a permutation $\bar{\epsilon}_r$ for which

$$\phi_r = \bar{\epsilon}_r \phi_1.$$

Then, if ϵ_r changes a_α into a_{u_α}, $\bar{\epsilon}_r$ will change A_{u_α} into A_α, as can be easily proved by induction, so that

$$\bar{\epsilon}_r = \epsilon_r^{-1}.$$

Let $\beta_1. \beta_2. \ldots$ be the column lengths in the tableaux of the representations considered; then the equations (i) above show that ϕ_1 is a function which changes sign for every odd permutation of the groups

$$\{A_1 A_2 \ldots A_{\beta_1}\}, \quad \{A_{\beta_1+1} \ldots A_{\beta_1+\beta_2}\}, \quad \ldots.$$

Also when S is a substitutional expression which satisfies the equation

$$TS = 0,$$

the sum $$[S\Phi, f] = 0,$$

where $$[\Phi_r, f] \equiv \lambda_{rf},$$

and S is supposed written out with the appropriate numerical coefficients. This follows from the equation

$$R\Lambda = \Lambda D$$

by giving the variables x_Q in R the values corresponding to the coefficients of S, and then considering the f column in the product; this column vanishes in D with TS. Let $\bar{S}$ be the result of replacing the permutations of a in S by the inverse permutations of A, then

$$\bar{S}\lambda_{rf} = 0.$$

Hence $$T'\lambda_{rf} = 0,$$

where T' is any representation other than T, and hence $\lambda_{1f} = \phi_1$ is of the nature ϖ_{ff}. The appropriate form of ϕ_1 is then that defined in § 18 as Y_f.

The remaining columns of this band are given by the equations (ii). It is to be observed that certain substitutional equations are involved.

That these are actually satisfied may be seen as follows. Let us assume that

$$\lambda_{r\tau} = \varpi_{\tau\tau}\lambda_{r\tau},$$

and that $$\lambda_{s\tau} = (a_\alpha a_{\alpha+1})\,\varpi_{\tau\tau}\lambda_{r\tau}.$$

Then
$$\varpi_{\sigma\sigma}(a_\alpha a_{\alpha+1}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\rho & 1-\rho \\ 1+\rho & \rho \end{bmatrix}$$

$$= -\rho\varpi_{\sigma\sigma} + (1-\rho)\,\varpi_{\sigma\tau},$$

and $$\varpi_{\tau\tau}(a_\alpha a_{\alpha+1}) = (1+\rho)\,\varpi_{\tau\sigma} + \rho\varpi_{\tau\tau}.$$

Now, by equations (ii),

$$(1-\rho)\lambda_{r\sigma} = -\rho\lambda_{r\tau}+\lambda_{s\tau},$$

whence $$(1-\rho)\varpi_{\sigma\sigma}\lambda_{r\sigma} = \varpi_{\sigma\sigma}\lambda_{s\tau} = (1-\rho)\varpi_{\sigma\tau}\lambda_{r\tau};$$

and similarly $$\varpi_{\tau\tau}\lambda_{r\sigma} = 0.$$

In the same way it may be shown that every

$$\varpi_{\xi\xi}\lambda_{r\sigma} = 0, \quad \xi \neq \sigma;$$

hence $$\varpi_{\sigma\sigma}\lambda_{r\sigma} = \lambda_{r\sigma}.$$

It is to be remarked that it does not signify that $\lambda_{r\tau}$ is not a function of a but of A; all that matters is that a permutation of the a's on $\lambda_{r\tau}$ has a definite meaning. There are f independent bands connected with T, and all others are expressible as a linear function of such f bands. We conclude that it is possible to choose f sets of quantities

$$A_{r,1}, A_{r,2}, \ldots, A_{r,n} \quad (r = 1, 2, \ldots, f);$$

such that, for any other set,

$$Y_f = B_1 Y_f^{(1)} + B_2 Y_f^{(2)} + \ldots + B_f Y_f^{(f)}.$$

Next consider $\Lambda^{-1} = [\mu_{rs}]$. There are f bands of rows connected with T. The last row of a band must also have the form Y_f; but equations (ii) show that the $[\mu]$ rows differ from the $[\lambda]$ columns in that they are changed by the transposition $(a_\alpha a_{\alpha+1})$ in the ratio

$$\lambda_{-\sigma} : \mu_{\sigma-} = 1+\rho : 1-\rho = [F_\sigma]_1 [F_\sigma]_2 : [F_\tau]_1 [F_\tau]_2.$$

THEOREM XVI. *The matrix Λ of degree $n!$ which transforms the regular group matrix of the symmetric group of degree n into its irreducible seminormal components is made up of a set of f bands of f columns for each representation T. The elements of a column are obtained by permutation from a function Y_σ of n variables, corresponding to the tableau F_σ which defines the column. A similar result is true for the rows of the inverse matrix Λ^{-1}.*

20. LEMMA. *In the orthogonal group matrix, if every permutation is replaced by its inverse the unit ϖ_{rs} becomes ϖ_{sr}.*

It was shown (Q.S.A., VI, Theorem V) that in the orthogonal matrix for a transposition of consecutive letters, the quadratic matrices

which appear have the form

$$\begin{bmatrix} -\rho & \sqrt{(1-\rho^2)} \\ \sqrt{(1-\rho^2)} & \rho \end{bmatrix};$$

hence any transposition of consecutive letters is of the form

$$\Sigma \lambda_r \varpi_{rr} + \Sigma \lambda_{rs} (\varpi_{rs} + \varpi_{sr}).$$

If any permutation is

$$\Sigma \mu_r \varpi_{rr} + \Sigma \mu_{rs} \varpi_{rs} + \Sigma \mu_{sr} \varpi_{sr},$$

it may be obtained as a product of consecutive transpositions; its inverse is obtained by taking this product in the reverse order and is therefore

$$\Sigma \mu_r \varpi_{rr} + \Sigma \mu_{rs} \varpi_{sr} + \Sigma \mu_{sr} \varpi_{rs}.$$

Hence, also, the result of replacing each permutation by its inverse is to change ϖ_{rs} into ϖ_{sr} and to leave ϖ_{rr} unchanged.

The change from the orthogonal units to the seminormal units is effected by replacing each unit ϖ_{rs} in an expression given in orthogonal units by

$$\varpi_{rs} \left(\frac{[F_s]_1 [F_s]_2}{[F_r]_1 [F_r]_2} \right)^{\frac{1}{2}}.$$

Thus when each permutation is changed into its inverse the seminormal unit ϖ_{rr} is unaltered, but

$$[F_s]_1 [F_s]_2 \varpi_{rs} \quad \text{is changed into} \quad [F_r]_1 [F_r]_2 \varpi_{sr}.$$

21. Consider a product of two functions

$$\phi(x_1, x_2, \ldots, x_n) \psi(y_1, y_2, \ldots, y_n).$$

Let G be the simultaneous symmetric group of x and y. That is when x_r and x_s are interchanged, then y_r and y_s are also interchanged, and G is the sum of these $n!$ permutations. Let k be a permutation of the x's and κ the same permutation of the y's, so that $k\kappa$ is contained in G. Also let p_{rr} be a seminormal unit of the x's, and π_{rr} the same unit of the y's. Then

$$G\phi\, \kappa^{-1}\psi = Gk\kappa\phi\, \kappa^{-1}\psi = Gk\phi\, \psi;$$

and hence, by the last paragraph,

$$G\phi\, \varpi_{rr}\psi = Gp_{rr}\phi\, \psi.$$

Hence $$Gp_{rr}\phi\,\varpi_{ss}\psi = 0,$$

unless the units not only belong to the same representation, but are identical.

Hence the sum of products of corresponding elements of two different columns of Λ is zero unless the two columns are defined by the same tableau of the same representation. Again, the second pair of equation (ii) § 19, give, on squaring and adding,

$$(\lambda_{r\sigma}^2+\lambda_{s\sigma}^2)(1-\rho^2)-(1+\rho)^2(\lambda_{r\tau}^2+\lambda_{s\tau}^2) = -2(1+\rho)\{\lambda_{s\sigma}\lambda_{s\tau}+\lambda_{r\sigma}\lambda_{r\tau}\}.$$

Now sum for the whole length of the columns; the sum on the right is zero and we have

$$(1-\rho)\sum_r \lambda_{r\sigma}^2 = (1+\rho)\sum_r \lambda_{r\tau}^2,$$

whence $\sum_r \lambda_{r\sigma}^2$ is proportional to $[F_\sigma]_1\,[F_\sigma]_2$.

THEOREM XVII. *The functions Y_σ of $A_1, A_2, \ldots, A_n$ and Z_σ of $B_1, B_2, \ldots, B_n$ are formed from the tableau F_σ. Then*

$$\Sigma\, Y_\sigma Z_\tau = 0 \quad (\sigma \neq \tau);$$

$$\Sigma\, Y_\sigma Z_\sigma = \Phi\,[F_\sigma]_1\,[F_\sigma]_2;$$

where the sum extends to all simultaneous permutations of A and B, and Φ is a function which depends only on the particular representation T.

This corresponds to the fact that the elements of Λ and Λ^{-1} can each be put into a form Y_σ while the product of the two matrices is E.

VII. *The double group matrix.*

22. The regular group matrix $[x_{PQ^{-1}}]$ is reducible to Σf irreducible matrices which are identical in k sets of f matrices each. Frobenius* showed that the matrix $[x_{PQ^{-1}}+y_{Q^{-1}P}]$ is reducible to k irreducible matrices of order f^2, each of which reduces to f identical matrices in x or y when the other set of variables is suppressed. It follows from the results of Schur† that, if D_1 is the double regular group matrix transformed so that its k components occupy positions as principal minors, the only matrix permutable with D_1 is that which has the place of these minors

* *Berliner Sitzungsberichte*, 1896, 1343–1382 (1372).

† *Berliner Sitzungsberichte*, 1905, 406–432, § 3.

filled by matrices BE_{f^2} and zeros elsewhere. Thus it has k arbitrary constants, one for each irreducible representation of the symmetric group. Then for the double regular group matrix there is a matrix Λ_1 such that

$$R_1\Lambda_1 = \Lambda_1 D_1.$$

This matrix Λ_1 is obtained by giving particular values to the constants in the transforming matrix Λ already discussed. It consists of k bands of f^2 columns in a band, and each band as a whole is unique except for one arbitrary constant which multiplies every element of the band.

Now $[y_{Q^{-1}P}]$ may be looked upon as a group matrix, indeed, if $y_P = z_{P^{-1}}$,

$$[y_{Q^{-1}P}] = [z_{P^{-1}Q}].$$

This is the same as the matrix $[x_{PQ^{-1}}]$ except for a rearrangement of the rows and columns. The regular matrix R is orthogonal in the sense that the coefficient of every variable x_P is an orthogonal matrix, and hence it gives an orthogonal representation of the group. When the matrix Λ is orthogonal the matrix D is also orthogonal. In this case Λ^{-1} is obtained from Λ by changing columns into rows and *vice versa*. Thus, when in D_1 the irreducible minors of order f^2 are orthogonal so far as the x's alone are concerned, they are also so as regards the y's alone. Does it follow that when this minor appears as f separate minors in x about the leading diagonal, the variables y being suppressed, the same property will be preserved for the y's when the variables x are suppressed? The most obvious arangement is as follows. Consider the principal minor of D_1 of f^2 rows and columns corresponding to a particular representation T. The rows and columns are made to correspond to a double set of tableaux FF' one of which refers to the x units and the other to the y units. The elements follow the rule that that element whose row and column tableaux are $F_r F'_{r'}$, $F_c F'_{c'}$ respectively contains x, only when $F'_{r'}$ and $F'_{c'}$ are the same, and then as the function given by the unit ω_{rc}; and similarly it contains y, only when F_r and F_c are the same, and then as the function given by $\omega'_{r'c'}$, it being remembered that the y variables are associated with the permutations in the inverse manner to the x variables. Thus the elements in the leading diagonal are functions of both x and y, and are unique in this. It will be shown that for all representations of unit rank the double group matrix can be transformed into this form, and also in some other cases, but that it is not possible in general.

Consider then the elements of Λ_1 in that column which corresponds to the double tableau $F_f F_f'$, supposing that this arrangement is possible. Having regard to the transformation of the x's alone, each element must have the form Y_f, just as in the seminormal case; for the distinction

between the seminormal and the orthogonal cases arises only when passing from one row to another. As regards the variables y, we suppose that the elements λ_{rf} can be alternatively expressed as functions of a set of quantities A_1', A_2', ..., A_n'. Then, whereas in the former case the rule (IV), § 19, was obtained, here we must have

$$(A_\alpha' A_{\alpha+1}') \lambda_{rf}' = \lambda_{sf}',$$

where λ_{rf}' is λ_{rf} differently expressed.

The first element of the column λ_{1f} is that function Y_f derived directly from F_f by writing the variables A_α in the tableau; and λ_{1f}' is the result of replacing A_α by A_α'. But this relation between λ_{rf} and its equivalent λ_{rf}' does not hold in general. Consider a representation $T_{n-k+1, 1^{k-1}}$,

$$\lambda_{1f} = Y_f = \Delta_{1, 2, \ldots, k} = \Delta_{1, 2, \ldots, k}'.$$

The suffixes denote the variables present. The function $\Delta_{1, 2, \ldots, k-1, k+1}'$ is the value of λ_{rf}' when r corresponds to even permutations ϵ of

$$\{a_1 a_2 \ldots a_{k-1} a_{k+1}\},$$

where

$$\Phi_r = \epsilon \Phi_{r_1} = \epsilon (a_k a_{k+1}) \Phi_1;$$

and of $-\lambda_{rf}'$ for odd permutations ϵ. On the other hand, the value of λ_{rf}, while governed by the same rule of signs, has each of the variables $A_1, A_2, \ldots, A_{k-1}$ omitted in turn in this range. Hence, in order that the two forms λ, λ' of the element may co-exist, we obtain the equations (after division by a common factor)

$$\frac{\Delta_{2, 3, \ldots, k}}{A_1 - A_{k+1}} = \frac{-\Delta_{1, 3, \ldots, k}}{A_2 - A_{k+1}} = \ldots.$$

Whence

$$\sum_{\alpha=1}^{k} A_\alpha = k A_{k+1}, \quad \sum_{\alpha=1}^{k} A_\alpha^2 = k A_{k+1}^2, \quad \ldots, \quad \sum_{\alpha=1}^{k-1} A_\alpha^{k-1} = k A_{k+1}^{k-1}.$$

And therefore $A_1, A_2, \ldots, A_k$ are the roots of

$$(x - A_{k+1})^k = (-A_{k+1})^k + B.$$

Let ϖ be a primitive k-th root of unity, we may write

$$A_\alpha = A_{k+1} + \varpi^\alpha \xi.$$

In the same way A_{k+2} may be treated with the same result, so that

$$A_{k+1} = A_{k+2} = \ldots = A_n.$$

Since the functions are functions of the differences only, they are unaffected by the subtraction of A_{h+1} from each variable; thus we have

$$A_\alpha = \varpi^\alpha \xi \quad (\alpha = 1, 2, \ldots, h),$$

$$A_\alpha = 0 \quad (\alpha > h).$$

For orthogonal transformations the relations between different columns are given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{r\sigma} & \lambda_{r\tau} \\ \lambda_{s\sigma} & \lambda_{s\tau} \end{bmatrix} = \begin{bmatrix} \lambda_{r\sigma} & \lambda_{r\tau} \\ \lambda_{s\sigma} & \lambda_{s\tau} \end{bmatrix} \begin{bmatrix} -\rho & \sqrt{(1-\rho^2)} \\ \sqrt{(1-\rho^2)} & \rho \end{bmatrix},$$

whence $$\sqrt{(1-\rho^2)}\,\lambda_{r\sigma} = -\rho\lambda_{r\tau} + \lambda_{s\tau}.$$

The remaining f^2-1 columns may be calculated by means of this equation from that already determined.

23. As an example, we consider the double regular group matrix of degree 3. The sequence of arrangements Φ is taken to be

abc,	*bac*,	*acb*,	*bca*,	*cab*,	*cba*
1	2	3	4	5	6

The suffix r of each variable represents the permutation which changes *abc* into the corresponding arrangement Φ_r. Then

$$R_1 = \begin{bmatrix} x_1+y_1 & x_2+y_2 & x_3+y_3 & x_5+y_5 & x_4+y_4 & x_6+y_6 \\ x_2+y_2 & x_1+y_1 & x_4+y_5 & x_6+y_3 & x_3+y_6 & x_5+y_4 \\ x_3+y_3 & x_5+y_4 & x_1+y_1 & x_2+y_6 & x_6+y_2 & x_4+y_5 \\ x_4+y_4 & x_6+y_3 & x_2+y_6 & x_1+y_1 & x_5+y_5 & x_3+y_2 \\ x_5+y_5 & x_3+y_6 & x_6+y_2 & x_4+y_4 & x_1+y_1 & x_2+y_3 \\ x_6+y_6 & x_4+y_5 & x_5+y_4 & x_3+y_2 & x_2+y_3 & x_1+y_1 \end{bmatrix},$$

$$\Lambda_1 = \frac{1}{\sqrt{(12)}} \begin{bmatrix} \sqrt{2} & 2 & 0 & 0 & 2 & \sqrt{2} \\ \sqrt{2} & 2 & 0 & 0 & -2 & -\sqrt{2} \\ \sqrt{2} & -1 & \sqrt{3} & \sqrt{3} & 1 & -\sqrt{2} \\ \sqrt{2} & -1 & -\sqrt{3} & \sqrt{3} & -1 & \sqrt{2} \\ \sqrt{2} & -1 & \sqrt{3} & -\sqrt{3} & -1 & \sqrt{2} \\ \sqrt{2} & -1 & -\sqrt{3} & -\sqrt{3} & 1 & -\sqrt{2} \end{bmatrix}.$$

The orthogonal double group matrix T_{21} is then

$$\begin{bmatrix} X_{11}+Y_{11} & X_{12} & Y_{12} & 0 \\ X_{21} & X_{22}+Y_{11} & 0 & Y_{12} \\ Y_{21} & 0 & X_{11}+Y_{22} & X_{12} \\ 0 & Y_{21} & X_{21} & X_{22}+Y_{22} \end{bmatrix},$$

where

$$X_{11} = x_1+x_2-\tfrac{1}{2}(x_3+x_4+x_5+x_6),$$

$$X_{22} = x_1-x_2-\tfrac{1}{2}(x_4+x_5-x_3-x_6);$$

and Y_{11}, Y_{22} are the same functions of y:

$$X_{12} = \tfrac{1}{2}\sqrt{3}(x_4-x_5+x_3-x_6), \quad X_{21} = \tfrac{1}{2}\sqrt{3}(x_5-x_4+x_3-x_6),$$

$$Y_{12} = \tfrac{1}{2}\sqrt{3}(y_5-y_4+y_3-y_6), \quad Y_{21} = \tfrac{1}{2}\sqrt{3}(y_4-y_5+y_3-y_6).$$

It is necessarily irreducible.

24. Consider next the representation T_{22}. Here

$$\lambda_{1f} = (A_1-A_2)(A_3-A_4) = (A_1'-A_2')(A_3'-A_4').$$

There are four values of r for which λ'_{rf}, and four for which $-\lambda'_{rf}$, is

$$(A_1'-A_3')(A_2'-A_4').$$

In the first four cases, λ_{rf} is $(A_1-A_3)(A_2-A_4)$, in the other four λ_{rf} is $(A_1-A_4)(A_2-A_3)$; hence

$$(A_1-A_3)(A_2-A_4)+(A_1-A_4)(A_2-A_3) = 0.$$

That is, if A_1, A_2, A_3, A_4 represent points on a straight line, the pairs A_1A_2 and A_3A_4 must be harmonically conjugate. This will, of course, give virtually the same double group matrix as in the former case; there are, indeed, four times as many variables, but they correspond in sets of four to each of the former variables.

Similarly, for the representation T_{33} (which again is practically the same as T_{32}), the pairs A_1A_2, A_3A_4, A_5A_6, on which λ_{rf} depends, must represent pairs of points mutually harmonic two and two. Such pairs are known to exist and hence the double group matrix can be transformed into the required form. But for a representation T_{kk}, where $k > 3$, since it is not possible to find more than three pairs of mutually harmonic points, it is not possible to express the group matrix in the form suggested.

It is not difficult to show that, apart from the representations of unit rank, the only representations for which this is possible are $T_{n-2,2}$, T_{33},

and the corresponding representations when rows and columns are interchanged in the tableaux, *i.e.* $T_{2^2 1^k}$ and T_{2^3}.

VIII. *Concomitants given by seminormal units.*

25. In what follows the results which have been obtained here are applied to the theory of algebraic invariants. The general method of the applications of this analysis to invariants has been explained (Q.S.A., VII). In the first place, with the use of natural units n_{rs}, concomitant types take the form

$$Gn_{rs}X,$$

where G is a product of positive symmetric groups, expressing the facts that the first n_1 letters are equivalent as all representing the first quantic, the next n_2 letters are equivalent and so on. Here we are dealing with the seminormal units ϖ_{rs} and concomitant types of the form

$$G\varpi_{rs}X.$$

In both cases the second suffix s of the unit may be always taken as the same; in such applications we are only concerned with one column of the matrix. Now the operator G really means taking a sum of different terms, or else a number of elements in the column. These are obtained, from the one element expressed, by permuting the letters belonging to each quantic among themselves in all possible ways; the actual numerical coefficients attached to the different elements and units are given by Theorem V; and we see that in F_r no two elements belonging to any quantic may lie in the same column. It is not contemplated to write out the expression in full, but the power to do so gives complete definiteness to the abbreviated expression.

26. A concomitant of a single quantic requires another substitutional operator Γ in either set of units. This represents the fact that all the quantics concerned are the same, or that each set of n letters is interchangeable with any other set. Thus the full expression is $\Gamma Gn_{rs}X$, or $\Gamma G\varpi_{rs}X$. Like G, the operator Γ represents a sum. Any one of the units occurring in this sum might be taken for n_{rs} (or ϖ_{rs}) to give the expression, but for the sake of the developments of the theory the earliest unit according to the sequence of tableaux selected is chosen. This unit or its tableau defines also a gradient, a product of the coefficients of the quantic; and, conversely, the gradient defines the concomitant. Every gradient defines a concomitant in this way, some of these concomitants are zero, and some can be equally well expressed by earlier gradients. A

concomitant as thus defined by the earliest possible gradient is called irreductible; naturally the term implies also a definite sequence. There is a distinction to be remembered between the natural and seminormal units; in the natural units the letters of permutation are all on the same footing and their sequence is of no particular consequence until the sequence of tableaux has to be determined; in the seminormal units the sequence of the letters is of the fundamental texture of the theory. The terms of the sum given by Γ in this latter case are to be supposed calculated as from the values of the transpositions of consecutive letters, they are not arrived at by mere permutation. Further, in the natural units any pre-arranged sequence of tableaux might be used, and thus any sequence of gradients might be conceived as the foundation for the representation of concomitants. In seminormal units a practical limitation in this matter is laid upon us by the sequence used in the transformation from natural to seminormal units. There is still a considerable choice, as is shown by Theorem IV, but in seminormal units the sequence of tableaux must be a principal sequence, such as is defined in § 2.

27. Consider the transformation from natural to seminormal units. The natural matrix C becomes the seminormal matrix D, where

$$D = M^{-1}CM,$$

and M, M^{-1} are matrices with nothing but zero elements below leading diagonals of positive units. Thus the natural unit n_{rs} becomes

$$\sum_{uv} m'_{ur} m_{sv} \varpi_{uv},$$

where m, m' are elements of M, M^{-1}, respectively, and $\varpi_{u,v}$ are the seminormal units. The coefficient of ϖ_{uv} is zero unless $u \leqslant r$, $v \geqslant s$; also the coefficient of ϖ_{rs} is unity. Hence an algebraic function given in natural units in the form

$$n_{rs} X$$

becomes

$$\sum_{u=1}^{r} \lambda_u \varpi_{uv} X',$$

where $\lambda_r = 1$, when we change to seminormal units. As has been observed, we are only really concerned with one column of the matrix; the transformation leads to several columns of the seminormal matrix, but they are all proportional to each other, and they may be absorbed into one by the change from X to X'; thus the

coefficient λ_u only really relates to the first suffix of the unit. We deduce at once from the above:

THEOREM XVIII. *The difference between the algebraic forms or concomitants given by the same given tableau by natural and by seminormal units is a linear function of forms given by earlier tableaux in any principal sequence in either set of units.*

The result as to concomitants follows at once when n_{rs} is taken as the earliest unit in the concomitant $\Gamma G n_{rs} X$.

28. It is necessary to be clear as to what constitutes a principal sequence. This last theorem shows that, in the case of concomitants of a single form, the sequence, and consequently the determining tableau F_r, may be selected for the natural units first, and then the sequence of letters of permutation determined accordingly as a necessary preliminary to the transformation to seminormal units. Thus, for binary forms under certain conditions, a gradient

$$A_0^{\lambda_0} A_2^{\lambda_2} \dots A_\delta^{\lambda_\delta}$$

has been found to represent an irreductible seminvariant. This is first polarized $\delta - 1$ times in order to obtain δ different sets of letters; and a particular term in the result is selected. In the sequence of letters of permutation the letters corresponding to a coefficient suffix zero are taken first, then those of another, and so on, until these are exhausted, then those of suffix 2, and so on. With such a method both the sequences A and B are principal sequences. Quite another sequence is used for the invariants of the quaternary cubic (Q.S.A., VII, section XI), but it is not difficult to see that the sequence of the letters of permutation may be selected, so that this also is a principal sequence. The sequences C and D would require the separation of the letters of the different sets; the sequence of letters of permutation being begun with one letter from each set, then a second, and so on. They do not appear to be suitable sequences for use with seminormal units.

29. An empirical theorem has been stated (Q.S.A., VII, section XII, §51). No proof had been found, but very strong experimental evidence of its truth had been obtained. This theorem will now be demonstrated. With the demonstration its scope is greatly extended, and accurately defined. It may be regarded as the fundamental theorem of this approach to the

algebra of invariants. It may be enunciated thus:

Theorem XIX. *With any principal sequence, the leading gradient of an irreductible concomitant given in seminormal units is that which defines the concomitant. And the same gradients define the irreductible concomitants in natural units.*

The last part of the enunciation is an immediate consequence of Theorem XVIII. The first part follows from Theorem XV, § 18, when it is remembered that the tableau taken as representative of the whole form is that of the earliest unit in the sum. The theorem quoted applies to every term of the sum, and hence the leading gradient is the leading gradient of the first term. The second empirical theorem of the last paper follows at once from this, as was pointed out there. It thus becomes a demonstrated theorem and is no longer empirical. It is of much less importance.

30. The product of two concomitants has for its leading gradient the product of their leading gradients. And thus it follows that a concomitant given in seminormal units by the product of two leading gradients differs from the product of the corresponding concomitants by concomitants which are given by *later* gradients. This fact is given by Theorem IX, independently of the theorem just proved.

In looking for the complete irreducible system of concomitants of a quantic, all leading gradients have to be considered; but those which are products of other leading gradients may be rejected at once as reducible.

In binary forms it is often easy to show that a certain gradient is a leading gradient, by a method explained in the last paper. When the number of leading gradients found in this way is equal to the full number of linearly independent forms, as given by generating functions, then, as these are also the gradients giving irreductible covariants, the covariant given by any other gradient is reductible.

31. It was pointed out in § 51 (Q.S.A., VII) that the covariant type given by a binary gradient

$$A_0^{(1)} A_{\lambda_2}^{(2)} A_{\lambda_3}^{(3)} \dots A_{\lambda_s}^{(s)}$$

in seminormal units is the continued transvectant

$$(\dots ((f_1, f_2)^{\lambda_2} f_3)^{\lambda_3} \dots f_s)^{\lambda_s};$$

and hence also the covariant of a single quintic given by the irreductible form

$$A_0 A_{\lambda_2} \dots A_{\lambda_5}$$

is

$$(\dots((f, f)^{\lambda_2} f)^{\lambda_3} \dots f)^{\lambda_5}.$$

Hence we deduce:

THEOREM XX. *The continued transvectant*

$$(\dots((f, f)^{\lambda_2} f)^{\lambda_3} \dots f)^{\lambda_5}$$

has the leading gradient

$$A_0 A_{\lambda_2} A_{\lambda_3} \dots A_{\lambda_5}$$

according to a principal sequence, when and only when it is irreductible according to that sequence.

In the case of ternary forms the coefficients of the ground form require two suffixes for their expression. They are connected with the tableau by regarding the first suffix as giving the number of letters of the corresponding set which lie in the second row, while the second suffix gives the number in the third row. The addition of the last quantic in a concomitant type may be described as a two-indexed transvectant (in a manner already known). And the result of this transvectant can be obtained by differential operators. In the same way for quaternary forms, a three-indexed transvectant is to be used. Using the word transvectant in this extended sense, we see that an irreductible concomitant given by a gradient is equal to the corresponding continued transvectant, no matter how many variables there are. Further, Theorem XX is true for all cases when the multiple indexes of the transvectants and the corresponding multiple suffixes of the coefficients are inserted, and not merely for binary forms alone.

IX. *Binary forms.*

32. For binary forms the product of two seminvariants is given by Theorem X. It is the case of the product of the form

$$T_{\alpha_1 \alpha_2} T_{\beta_1 \beta_2} T_{\alpha_1+\beta_1,\ \alpha_2+\beta_2}.$$

It does not follow that all the terms in the product represent separate seminvariants; some of them will unite under the operation of ΓG. But when a leading gradient may be expressed as a product of two leading gradients the difference of the products will give us at once a reduction

for a later gradient. The case of the product of the form

$$T_{\alpha_1\alpha_2}\, T_{\beta_1\beta_2}\, T_{\alpha_1+\beta_1-\kappa,\ \alpha_2+\beta_2+\kappa},$$

dealt with by the same theorem, gives us the transvectant

$$(C_\alpha,\ C_\beta)^\kappa.$$

Incidentally, we learn at once from the result what is the leading gradient of this transvectant.

33. That exactly the same analysis may be applied to seminvariants as functions of differences of the roots has been already pointed out. (Q.S.A., VII, § 53.) The results obtained here, so far as binary forms are concerned, may be interpreted this way. A direct connection between the seminvariants expressed in terms of the coefficients and the seminvariants expressed in terms of the roots, which was suggested in the place quoted, will now be established. A leading gradient, which therefore represents an irreductible seminvariant, is considered, $A_0 A_{\lambda_2} \dots A_{\lambda_\delta}$; where, as always, $\lambda_\delta \geqslant \lambda_{\delta-1} \geqslant \lambda_{\delta-2} \dots$. A tableau of $\delta - 1$ rows all beginning with the same column is written down, the first row being $\alpha_1 \alpha_2 \dots \alpha_{\lambda_\delta}$, the second $\alpha_1 \alpha_2 \dots \alpha_{\lambda_{\delta-1}}$, and so on. Then the gradient may be looked upon as the symbolical product

$$a_1^{\,0} a_2^{\lambda_2} \dots a_\delta^{\lambda_\delta}.$$

The tableau as constructed yields a *root* product

$$\alpha_1^{\delta-1} \dots,$$

where each index is the length of the corresponding column. The two products will be called reciprocal; one is always given by the rows of a tableau, the other by the columns of the same tableau. The symbolical products are arranged in a definite sequence, this determines a definite sequence for the reciprocal products, which will be called the reciprocal sequence.

34. In a seminvariant expressed by substitutional forms, the letters of permutation are regarded as the symbolical letters and they appear in sets of n (the order of the ground form), each set representing one symbol; the equivalence of members of a set gives rise to the operator G, the equivalence of sets as a whole to the operator Γ. If the seminvariant is regarded as a function of the roots, the ground form is written as a product of n linear factors, each factor giving rise to one letter of permutation for each degree of the seminvariant. The letters again appear in sets of n

and the operator G is still required because for each coefficient of the ground form in the seminvariant the roots appear symmetrically. The seminvariant expressed in the roots is, in fact, of the same form as its expression in the coefficients, except that a new operator H must be introduced. This is the product of the symmetric groups (divided by the order $\delta!$ in each case) of the letters which represent each individual root separately, there being one group for each letter. Thus the seminvariant

$$\Gamma G \varpi_{u,v} X$$

becomes the function of the differences of the roots

$$H \Gamma G \varpi_{u,v} X.$$

Conversely, we may start with a function of the differences of the roots $\Gamma' G' \varpi_{u,v} X$, and obtain the seminvariant in the coefficients

$$H' \Gamma' G' \varpi_{u,v} X.$$

Here G' represents the fact that the letters really represent a function of degree δ in the coefficients of each of the linear factors; Γ' represents the fact that the function is symmetrical in the different roots; and, finally, H' puts the letters in sets of n proper to the symbolical representation.

The two forms of expression of a seminvariant are thus entirely complementary. To a leading gradient on one hand,

$$A_0 A_{\lambda_2} \dots A_{\lambda_\delta},$$

corresponds a first term on the other,

$$\alpha_2^{\lambda_2} \dots \alpha_\delta^{\lambda_\delta},$$

where $\alpha_1, \alpha_2, \dots, \alpha_\delta$ are the roots of a binary δ-ic. Anything which can be proved about the one can be interpreted in the other case. With this in view, the term leading gradient may be used with either meaning.

35. THEOREM XXI. *The reciprocal of a leading gradient for any principal sequence is a leading gradient in the reciprocal sequence.*

Consider a leading gradient

$$g = A_0 A_{\lambda_2} \dots A_{\lambda_\delta},$$

the irreductible covariant

$$(\dots((f, f)^{\lambda_2} f)^{\lambda_3} \dots f)^{\lambda_\delta}$$

corresponds to it. For n, the order of f, we have

$$n \geqslant \lambda_\delta \geqslant \lambda_{\delta-1} \ldots \geqslant \lambda_2.$$

The covariant exists for $n = \lambda_\delta$, and hence in this case there is a function ϕ_{λ_δ} of the differences of the λ_δ roots, which is such that $\Sigma\phi_{\lambda_\delta}$ is equal to the seminvariant in question. For every irreductible covariant of f of this degree and weight there is such a function of the differences of the λ_δ roots. And the several functions $\Sigma\phi_{\lambda_\delta}$ are all non-zero and linearly independent. When the order of f is increased, the seminvaraints corresponding to the leading gradients g remain irreductible. The functions $\Sigma\phi_{\lambda_\delta}$, the same in form but with extension in summation, remain also non-zero and linearly independent. Hence a definite non-zero covariant may be obtained from g in the following way. The factors of f are arranged in fixed sequence $\alpha_{1_x}\alpha_{2_x}\ldots\alpha_{n_x}$. The product of the first μ factors is written f_μ, that of the next $\nu-\mu$ factors is written $f_{\nu,-\mu}$; then the sum for all permutations of the roots

$$\Sigma\,(f_{\lambda_2},\ f_{\lambda_2})^{\lambda_2} f^2_{\lambda_3,-\lambda_2},\quad f_{\lambda_3})^{\lambda_3} f^3_{\lambda_4,-\lambda_3},\quad f_{\lambda_4})^{\lambda_4}\ldots,\quad f_{\lambda_\delta})^{\lambda_\delta}\quad f^\delta_{n,-\lambda_\delta}$$

is a definite non-zero covariant.

The first term of its seminvariant considered as a function of the roots, according to the reciprocal sequence, is the reciprocal of g. And hence the theorem is proved.

By an obvious application of the argument used in § 22 (Q.S.A., VII) we see that this seminvariant expressed in terms of the roots whose first term is g', the reciprocal of g, differs from the seminvariant defined by g as a continued transvectant by seminvariants defined by earlier gradients than g in the principal sequence used, and hence in any principal sequence. Thus

THEOREM XXII. *The seminvariant defined by a leading gradient differs from that defined as above by a product of differences of the roots with the first term reciprocal to this gradient by seminvariants defined by earlier leading gradients in any principal sequence.*

36. We now see that a covariant may be defined either by a leading gradient in a principal sequence, or by one in a reciprocal sequence in terms of the coefficients or else in terms of the roots. Thus every leading gradient stands for two covariants, one of degree δ of an n-ic, the other of degree n of a δ-ic. When a complete system of leading gradients in a principal or a reciprocal sequence can be obtained, that in the other can be written down. The difficulty is to obtain the law by which it may be

known at once whether a given gradient is a leading one or not. It has been shown (Q.S.A., VII, § 32) that the condition in a reciprocal sequence is that the coefficient of highest index appears with an index equal to 2 at least: the proof, which was there given for a particular reciprocal sequence, applies equally to all. But this statement loses sight of the fact that such a gradient is only a leading gradient when the order of the ground form exceeds some lower limit, and that no statement has yet been given as to what this limit is. Reciprocally, if g is any gradient whatever which does not contain A_1, there is some number γ for which $A_0^{\gamma} g$ is a leading gradient for a given principal sequence and $A_0^{\gamma-1} g$ is not one; γ may be called the *index* of g for this sequence.

37. The leading gradients for sequence A of degree 5 have been obtained (Q.S.A., VII, invariants, Section VII; covariants, Section X). By taking the reciprocals we have the leading gradients for the quintic, for the sequence D reciprocal to A.

The invariants

$$I_4 = A_3^2 A_2^2, \quad I_8 = A_4^2 A_3^2 A_2^2 A_1^2, \quad I_{12} = A_4^2 A_3^2 A_2^8,$$

$$I_{18} = A_4^2 A_3^5 A_2^{11}.$$

Moreover, the relation

$$I_{18}^2 = I_{12}^2 I_4^3$$

indicates the syzygy.

The reciprocal of a form obtained by adding the arguments of two forms sequence A is evidently the product of the reciprocal forms.

Covariants of order 1.

$$C_5^1 = A_3^2 A_2^3, \quad C_7^1 = A_3^3 A_2^4, \quad C_{11}^1 = A_4^2 A_3^2 A_2^6 A_1,$$

$$C_{13}^1 = A_4^2 A_3^2 A_2^9, \quad C_{17}^1 = A_4^2 A_3^5 A_2^9 A_1.$$

Here

$$I_{12} C_5^1 - I_4 C_{13}^1 = 0,$$

so far as their representative gradients are concerned, but the left-hand side cannot be accurately zero; it is, therefore, a covariant of order one given by a later gradient sequence D. It can only be C_{17}^1 which is therefore reducible. We thus have the four well-known linear covariants.

Covariants of order 2.

$$C_2^2 = A_2^2, \quad C_6^2 = A_3^3 A_2^2 A_1, \quad C_8^2 = A_3^3 A_2^5, \quad C_{12}^2 = A_4^2 A_3^2 A_2^7 A_1,$$

$$C_{12t-2}^2 = A_3^{8t-2} A_2^{2t} A_1^{2t}, \quad C_{12t+16}^2 = A_3^{8t+9} A_2^{2t+5} A_1^{2t+2}.$$

A reduction for C_{12}^2 arises from

$$I_4 C_8^2 - C_5^1 C_7^1,$$

which cannot be accurately zero, and C_{12}^2 is the only form which it can be. The reductions of C_{12t-2}^2, C_{12t+16}^2 must follow in the same way.

Covariants of order 3.

$$C_{6\sigma+2\rho-3}^3 = A_3^{4\sigma+\rho-3} A_2^{\sigma+\rho} A_1^{\sigma},$$

$$C_{12\sigma+2\rho+3}^3 = A_4^{2\sigma} A_3^{2\sigma+\rho} A_2^{8\sigma+\rho+3} \quad (\rho \neq 1),$$

$$C_{12\sigma+2\rho+1}^3 = A_4^{2\sigma} A_3^{2\sigma+\rho} A_2^{8\sigma+\rho} A_1 \quad (\rho \neq 1).$$

Covariants of order greater than 3.

$$A_4^{2\lambda} A_3^{2\lambda+\xi} A_2^{2\lambda+\eta} A_1^{\zeta},$$

where $\xi \neq 1, \quad \eta \neq 1, \quad 3\zeta - 6\lambda - \xi + \eta > 3.$

THE INTERSECTION OF TWO PRIMALS IN THREE SURFACES

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In space of four dimensions two primals f and f' which contain a general surface F have for residual intersection a surface F' meeting F in a curve C; if f and f' have no other singularities than the nodes which they must possess if they are to contain F, the characters of F' and C may be calculated by means of the formulae due to Severi*. It has been shown by Severi that F and F' have no isolated intersections, and that C has quadruple points at the improper nodes of F, which are also improper nodes of F'; and, further, that the nodes of f and f' are situated at the improper nodes of F and at certain other points of C, which are simple on F and F'; also, that f and f' touch along C. In this note, Severi's formulae are extended to the case in which F is replaced by a pair of surfaces intersecting in a general curve (*i.e.* without singularities) and, possibly, in a number of isolated points.

1. Let F be a general surface of order μ_0, rank μ_1, class μ_2, with ν_2 apparent pinch-points and d improper nodes, where

$$2d = \mu_0(\mu_0-1)-\mu_1-\nu_2.$$

Let F' be a general surface, with characters denoted by corresponding accented letters, which meets F in a curve C of order ϵ_0 and rank ϵ_1. We suppose that C is without singularities, and that it does not pass through the improper nodes of F or F'. If C has classes of immersion z, z' in F, F', and X is the class of the developable of spaces containing the pairs of tangent planes to F, F' at points of C, then†

$$z+z' = X+\epsilon_1. \tag{1}$$

* Severi, *Memorie di Torino* (2), 52 (1903), § 20. More general formulae, including the case considered in the present note, have been obtained independently by Semple, *Proc. Royal Irish Acad.*, 41 (1933), 70.

† See Severi, *op. cit.*

In general, F and F' meet in a number I of points P external to C, given by

$$I = \mu_0\mu_0' - \epsilon_0 - X. \tag{2}$$

Now suppose that f and f' are two primals of orders ρ and ρ' respectively, which contain F and F'. The residual intersection is a surface F'', say, which meets F in a curve Γ and F' in a curve Γ'. From the above remarks it follows that F'' has no isolated intersections with F or F', and that it has improper nodes at the improper nodes of F and F' and also at the I intersections of F and F'; moreover, Γ has a quadruple point at each node of F and a node at each point P, and similarly for Γ'. The primals f and f' have nodes at the nodes of F and F' and at the points P, and at certain points situated on C, Γ, or Γ'. Thus, if δ, δ' are the numbers of nodes which f must have at simple points of F, F', these will include the points P; hence, if δ_{12} is the number of nodes which f must have at points of C, the number of nodes of f at simple points of Γ is $\delta - \delta_{12} - I$, and similarly for Γ'. Further, f and f' touch along C, Γ, and Γ'.

Let Γ be of order η_0 and rank η_1, and Γ' of order η_0' and rank η_1'. Let the developables of spaces containing tangent planes at points of Γ and Γ' be of classes Y and Y' respectively. Then, as in (1), we have

$$y + y_1'' = Y + \eta_1, \tag{1}'$$

$$y' + y_2'' = Y' + \eta_1', \tag{1}''$$

where y, y_1'' are the classes of immersion of Γ in F, F'', and y', y_2'' are the classes of immersion of Γ' in F', F''.

2. If the characters of F'' are μ_0'', μ_1'', μ_2'', ν_2'', we have

$$\mu_0'' = \rho\rho' - \mu_0 - \mu_0'.$$

By taking a space section, we get

$$\mu_1 = \mu_0(\rho + \rho' - 2) - \epsilon_0 - \eta_0, \tag{3}$$

$$\mu_1' = \mu_0'(\rho + \rho' - 2) - \epsilon_0 - \eta_0', \tag{3}'$$

$$\mu_1'' = \mu_0''(\rho + \rho' - 2) - \eta_0 - \eta_0'. \tag{3}''$$

Also

$$\nu_2 + 2d = \mu_0(\mu_0 - 1) - \mu_1, \tag{4}$$

$$\nu_2' + 2d' = \mu_0'(\mu_0' - 1) - \mu_1', \tag{4}'$$

$$\nu_2'' + 2(d + d' + I) = \mu_0''(\mu_0'' - 1) - \mu_1''. \tag{4}''$$

These equations determine μ_1'', ν_2'', η_0, and η_0'. To find Y and Y' we consider the intersections of F and F' with the first polars of an arbitrary

point O with respect to f and f'. The surface common to the first polars meets F in the points P (which are nodes on f and f'), the d nodes counted twice, and the ν_2 apparent pinch-points; and, since f and f' touch along C and Γ, in the points of contact with C and Γ of the tangent spaces to f (and f') passing through O. Thus

$$I+2d+\nu_2+X+Y=\mu_0(\rho-1)(\rho'-1). \tag{5}$$

Similarly,

$$I+2d'+\nu_2'+X+Y'=\mu_0'(\rho-1)(\rho'-1). \tag{5}'$$

The corresponding equation for F'' merely leads to that obtained by adding (5) and (5)'.

3. Now consider the intersections of C with the first polar of O with respect to f. If δ_{12} is the number of nodes of f which lie on C, we have

$$X+\delta_{12}=\epsilon_0(\rho-1). \tag{6}$$

The intersections of Γ with the first polar consist of the nodes of F counted four times (since they are double on f and quadruple on Γ), the points P counted twice (since they are double on f and Γ), and the nodes of f at simple points of Γ; and, finally, the points of contact of the Y tangent spaces to f. Hence

$$4d+2I+(\delta-\delta_{12}-I)+Y=\eta_0(\rho-1). \tag{6}'$$

Similarly,

$$4d'+2I+(\delta'-\delta_{12}-I)+Y'=\eta_0'(\rho-1). \tag{6}''$$

Eliminating X and Y by means of (5), (6), and (6)', and using (3), we obtain, for the number of nodes of f at simple points of F,

$$\delta=\mu_0(\rho-1)^2-\mu_1(\rho-1)+\nu_2-2d,$$

a result given by Severi.

4. For the next group of formulae it is necessary to introduce the characters i, i', giving the number of lines through O which meet C and F or F' elsewhere. These are at once obtained by projection from O; thus

$$z+i=\epsilon_0(\mu_0-1), \quad z'+i'=\epsilon_0(\mu_0'-1). \tag{7}$$

The corresponding characters j, j_1'' for Γ are given by

$$y+j+4d=\eta_0(\mu_0-1), \quad y_1''+j_1''+4d+2I=\eta_0(\mu_0''-1). \tag{7}'$$

In obtaining this last equation we notice that the points P, which are nodes of Γ, project into points of the double curve on the projection of F''.

Similarly, the characters j', j_2'' of Γ' are given by

$$y'+j'+4d'=\eta_0'(\mu_0'-1), \quad y_2''+j_2''+4d'+2I=\eta_0'(\mu_0''-1). \quad (7)''$$

From (1) and (7), we have

$$i+i'+X+\epsilon_1=\epsilon_0(\mu_0+\mu_0'-2). \quad (8)$$

From (1)′ and (7)′,

$$j+j_1''+Y+\eta_1+8d+2I=\eta_0(\mu_0+\mu_0''-2). \quad (8)'$$

Similarly,

$$j'+j_2''+Y'+\eta_1'+8d'+2I=\eta_0'(\mu_0'+\mu_0''-2). \quad (8)''$$

The cone of secants to C which pass through O meets f in a residual curve of order $\epsilon_0(\rho-1)$. Consider the intersections of this curve with f'; they consist of the nodes of f lying on C, the points X counted twice, the points where the secants to C through O meet F and F' again, and the points where they meet F''. We observe that F'' meets C in a number of points ξ, say, which lie on Γ and Γ' since they are common to all three surfaces. Thus

$$\delta_{12}+2X+i+i'+(\epsilon_0\mu_0''-\xi)=\epsilon_0(\rho-1)\rho'. \quad (9)$$

Since δ_{12} is known from (6), this equation determines ξ.

In constructing the corresponding equation for Γ, we note that the improper nodes of F count eight times in the intersections and that the points P count four times; also that the number of apparent intersections of Γ and F' is $\eta_0\mu_0'-\xi-2I$. Hence

$$(\delta-\delta_{12}-I)+8d+4I+2Y+j+j_1''+(\eta_0\mu_0'-\xi-2I)=\eta_0(\rho-1)\rho'. \quad (9)'$$

Similarly,

$$(\delta'-\delta_{12}-I)+8d'+4I+2Y'+j'+j_2''+(\eta_0'\mu_0-\xi-2I)=\eta_0'(\rho-1)\rho'. \quad (9)''$$

Eliminating $j+j_1''$ between (8)′ and (9)′ we determine η_1; and from (8)″ and (9)″ we determine η_1'.

5. The class μ_2'' of F'' and the classes of immersion y are now determined as follows. The tangents to F meeting a given line l touch F in a curve of order μ_1. Consider the intersection of this curve with the Jacobian of f, f' and three spaces containing a line l'; the intersections, in number $\mu_1(\rho+\rho'-2)$, consist of the points of contact of tangent planes to F which meet l and l', and of tangent planes to F at points of C and Γ

which meet l. Since the number of tangent planes meeting two given lines is $\mu_2+\nu_2$, we have

$$z+y=\mu_1(\rho+\rho'-2)-(\mu_2+\nu_2). \tag{10}$$

Similarly,
$$z'+y'=\mu_1'(\rho+\rho'-2)-(\mu_2'+\nu_2'), \tag{10}'$$

$$y_1''+y_2''=\mu_1''(\rho+\rho'-2)-(\mu_2''+\nu_2''). \tag{10}''$$

Equations (10) and (10)′ determine y and y' and (1)′ and (1)″ then determine y_1'', y_2'', so that μ_2'' is known. Alternatively, we may add the three equations given above and, using equations (1), obtain

$$X+Y+Y'+\epsilon_1+\eta_1+\eta_1'=(\rho+\rho'-2)(\mu_1+\mu_1'+\mu_1'')-(\mu_2+\mu_2'+\mu_2'')$$
$$-(\nu_2+\nu_2'+\nu_2''), \tag{11}$$

from which μ_2'' may be found. Thus the characters of F'', Γ, and Γ' are determined in terms of those of F, F', and C.

6. We proceed to give some interesting applications* of these results; we consider first the projection of Veronese's surface on [4]. It is known that this surface is not residual to any other of the same or lower order; we shall show that it is, however, residual to a pair of surfaces of lower order. In the above formulae take F to be a cubic scroll and F' a quadric, and let the curve C of intersection be a pair of generators. We thus have

$$\mu_0=3,\quad \mu_1=4,\quad \mu_2=3,\quad \nu_2=2,\quad d=0,$$
$$\mu_0'=\mu_1'=\mu_2'=2,\quad \nu_2'=0,\quad d'=0.$$

Also $\epsilon_0=2$, $\epsilon_1=0$, $z=z'=2$, so that $X=4$, $I=0$.

Suppose that f and f' are cubic primals; we then have for the characters of F'', $\mu_0''=4$, $\mu_1''=6$, $\nu_2''=6$, from (3)″ and (4)″.

From equations (3) and (3)′, $\eta_0=6$, $\eta_0'=4$.

From (5) and (5)′, $Y=6$, $Y'=4$.

Hence $\delta=6$, $\delta'=4$, $\delta_{12}=0$, from (6), (6)′, (6)″.

Then, from (9), $\xi=6$, *i.e.* each generator common to F and F' is a trisecant of F''. From (8)′ and (9)′ we obtain $\eta_1=12$ and, similarly, $\eta_1'=6$. Thus Γ is an elliptic sextic and Γ' a rational quartic.

Finally, from (11) we find that $\mu_2''=3$, so that F'' is the projection of Veronese's surface.

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* The case in which F and F' have only point intersections is readily dealt with by putting $\epsilon_0=\epsilon_1=z=z'=0$ in the above results.

As a second example consider the elliptic sextic scroll in [4]. If we take F to be a rational quartic scroll and F' a quadric meeting F in a conic, and draw through F and F' a cubic and a quartic primal, the residual surface F'' is the scroll in question. In this case we have $\rho = 3$, $\rho' = 4$,

$$\mu_0 = 4, \quad \mu_1 = 6, \quad \mu_2 = 4, \quad \nu_2 = 4, \quad d = 1,$$

$$\mu_0' = \mu_1' = \mu_2' = 2, \quad \nu_2' = 0, \quad d = 0.$$

$$\epsilon_0 = \epsilon_1 = 2, \quad z = 4, \quad z = 2,$$

so that $$X = 4, \quad I = 2.$$

We then find that

$$\mu_0'' = 6, \quad \mu_1'' = 12, \quad \nu_2'' = 12.$$

Also $$\eta_0 = 12, \quad \eta_0' = 6, \quad Y = 12, \quad Y' = 6.$$

Hence $$\delta = 6, \quad \delta' = 4, \quad \delta_{12} = 0,$$

so that $$\xi = 6, \quad \eta_1 = 34, \quad \eta_1' = 14,$$

and, finally, $$\mu_2'' = 6.$$

Thus Γ is of order 12 and genus 6, with one quadruple and two double points. Γ' is of order 6 and genus 2, with two double points; it is thus a particular (3, 3) curve on F'.

The Del Pezzo sextic in [4] is residual to this sextic scroll; but it may also be obtained as the residual intersection of a cubic and a quartic primal, the surfaces F and F' being the same as before, and the common curve C a pair of generators.

ON FLEXURE IN PRISMS WITH CROSS-SECTIONS OF UNIAXIAL SYMMETRY

By B. R. Seth.

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The well-known Saint-Venant solution of the problem of flexure has been limited to a relatively few cases. All the cross-sections discussed by Saint-Venant and many other writers possess biaxial symmetry, there being an axis of symmetry in the plane of loading and also an axis perpendicular to this plane. In the case of uniaxial symmetry we can have the axis of symmetry (1) perpendicular to the plane of loading, or (2) in the plane of loading. Young, Elderton, and Pearson have dealt with the first case in a Draper's Company Research memoir*. I may also mention an attempt by S. Timoschenko†, in which the cross-section is supposed to be an isosceles triangle. The only definite result which he obtains, though he does not mention it in the paper, is for the special case of an equilateral triangle when η, the Poisson's ratio, is equal to $\frac{1}{2}$. In such a case μ and E both vanish, and the displacement tends to infinity. For all practical purposes, therefore, the solution is not of much use. For the second case, when the axis of symmetry is in the plane of loading, Seeger and Pearson‡ have obtained a solution for the flexure of cantilevers of cross-section in the form of complete and curtate sectors. Thus, as far as I know, there exists only one Saint-Venant flexure solution when the direction of loading has no relation to any plane of symmetry at all, the cross-section of the prism possessing uniaxial symmetry only.

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* Draper's Company Research Memoirs, Technical Series, No. VII (1918).

† *Proc. London Math. Soc.* (2), 20 (1922), 398.

‡ *Proc. Roy. Soc.* (A), 96 (1920), 211.

The object of this short paper is to give a complete solution of a similar problem relating to the flexure of a prism whose cross-section is an isosceles right-angled triangle. The torsion solution of this cross-section has already been obtained by C. Kolossoff*. Since the second case, that is, when the axis of symmetry is in the plane of loading, gives quite a simple result in the present problem, we take it first.

(*a*) *Axis of symmetry in the plane of loading.*

We take the origin at the vertex O of the triangle. The weight W is supposed to act along Ox, the axis of symmetry of the cross-section.

Let $OD = c$, D being the mid-point of AB. If G is the centre of gravity of the triangle, $OG = \frac{2}{3}c$.

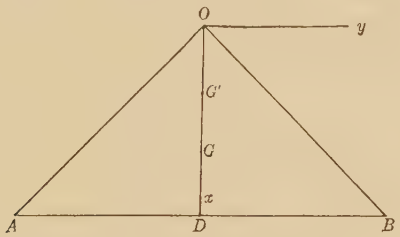

Fig. 1.

An investigation, following the usual method adopted in problems on flexure, gives the displacements u, v, w as

$$u = \frac{W}{EI}\{(l-z)[\tfrac{1}{2}\eta(x-\tfrac{2}{3}c)^2-\tfrac{1}{2}\eta y^2]+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\}+\beta z+\alpha, \qquad (1.1)$$

$$v = \frac{W}{EI}[\eta y(x-\tfrac{2}{3}c)(l-z)], \qquad (1.2)$$

$$w = \chi(x, y)-\frac{W}{EI}[(x-\tfrac{2}{3}c)(lz-\tfrac{1}{2}z^2)+(x-\tfrac{2}{3}c)y^2]-\beta x+\gamma, \qquad (1.3)$$

where χ is a harmonic function, α, β, and γ are constants to be determined by the fixing, l is the length of the beam, and I is the moment of inertia of the cross-section about an axis through G parallel to Oy.

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* *Comptes rendus*, 178 (1924), 2057.

All the stresses except $\widehat{yz}$, $\widehat{zx}$, and $\widehat{zz}$ vanish. A short calculation gives

$$\widehat{yz} = \mu\left[\frac{\partial\chi}{\partial y} - \frac{W}{EI}\{(2+\eta)(x-\tfrac{2}{3}c)y\}\right], \tag{2.1}$$

$$\widehat{zx} = \mu\left[\frac{\partial\chi}{\partial x} - \frac{W}{EI}\{\tfrac{1}{2}\eta(x-\tfrac{2}{3}c)^2 + (1-\tfrac{1}{2}\eta)y^2\}\right], \tag{2.2}$$

$$\widehat{zz} = -\frac{W}{I}[(l-z)(x-\tfrac{2}{3}c)]. \tag{2.3}$$

The boundary condition is given by

$$\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu) = 0. \tag{3}$$

Two of the equations of equilibrium give $\partial\widehat{xz}/\partial z = 0$, and $\partial\widehat{yz}/\partial z = 0$. These are obviously satisfied, since χ is a function of x and y only. The third of the equations of equilibrium becomes

$$\frac{\partial\widehat{xz}}{\partial x} + \frac{\partial\widehat{yz}}{\partial y} + \frac{W}{I}(x-\tfrac{2}{3}c) = 0, \tag{4}$$

and this is satisfied in virtue of (2.1) and (2.2).

The tractions on the elements of area of a cross-section are statically equivalent to a certain force at the centroid of the section and a certain couple. The force, of course, should be of magnitude W, and directed parallel to the axis of x, and the couple of moment $W(l-z)$ with its axis parallel to the axis of y. These statements are equivalent to the equations

$$\iint \widehat{xz}\,dx\,dy = W, \quad \iint \widehat{yz}\,dx\,dy = 0, \quad \iint \widehat{zz}\,dx\,dy = 0, \tag{5}$$

$$\iint y\widehat{zz}\,dx\,dy = 0, \quad -\iint x\widehat{zz}\,dx\,dy = W(l-z), \quad \iint (x\widehat{yz} - y\widehat{xz})\,dx\,dy = 0. \tag{6}$$

Using (3) and (4), we have

$$\iint \widehat{xz}\,dx\,dy = \iint \left\{\widehat{xz} + (x-\tfrac{2}{3}c)\left[\frac{\partial\widehat{xz}}{\partial x} + \frac{\partial\widehat{yz}}{\partial y}\right] + \frac{W}{I}(x-\tfrac{2}{3}c)^2\right\} dx\,dy,$$

i.e.

$$\iint \widehat{xz}\,dx\,dy = W + \int \{\widehat{xz}\cos(x\nu) + \widehat{yz}\cos(y\nu)\}(x-\tfrac{2}{3}c)\,ds$$

$$= W.$$

In like manner we may establish the rest of the equations in (5) and (6).

It remains to determine the flexure function χ. We assume that

$$\chi = A_1 x + A_2(x^2 - y^2) + A_3(x^3 - 3xy^2) + A_4(x^4 - 6x^2y^2 + y^4). \tag{7}$$

For the base AB of the triangle the boundary condition (3) becomes

$$\widehat{xz} = 0 \quad \text{over} \quad x = c.$$

Hence, by (2.2) and (7), we have

$$A_1 + 2A_2 c + 3A_3(c^2 - y^2) + 4A_4(c^3 - 3cy^2) - \frac{W}{EI}\left[\tfrac{1}{18}\eta c^2 + (1 - \tfrac{1}{2}\eta)y^2\right] = 0.$$

Equating the constant term and the coefficient of y^2 separately to zero, we get

$$A_1 + 2A_2 c + 3A_3 c^2 + 4A_4 c^3 - \frac{c^2 \eta W}{18EI} = 0, \tag{8.1}$$

$$3A_3 + 12c A_4 + \frac{W}{EI}(1 - \tfrac{1}{2}\eta) = 0. \tag{8.2}$$

For the side OB of the triangle the boundary condition (3) becomes

$$\widehat{xz} - \widehat{yz} = 0 \quad \text{over} \quad x - y = 0.$$

(2.1), (2.2), and (7) now give

$$A_1 + 2A_2 y - 8A_4 y^3 - \frac{W}{EI}\left[\tfrac{1}{2}\eta(y - \tfrac{2}{3}c)^2 + (1 - \tfrac{1}{2}\eta)y^2\right]$$

$$- \left[-2A_2 y - 6A_3 y^2 - 8A_4 y^3 - \frac{W}{EI}(2 + \eta)y(y - \tfrac{2}{3}c)\right] = 0.$$

The coefficient of y^3 vanishes. Equating the constant term and the coefficient of y and y^2 separately to zero, we get

$$A_1 - \tfrac{2}{9}c^2 \eta \frac{W}{EI} = 0, \tag{8.3}$$

$$4A_2 + \tfrac{2}{3}c\eta \frac{W}{EI} - \tfrac{2}{3}c(2 + \eta)\frac{W}{EI} = 0, \tag{8.4}$$

$$6A_3 + (1 + \eta)\frac{W}{EI} = 0. \tag{8.5}$$

It can easily be seen that the boundary condition for the side OA also gives (8.3), (8.4), and (8.5). Thus we have the five equations (8) to determine

the four constants A_1, A_2, A_3, and A_4. But it so happens that only four out of the five equations (8) are independent. Hence the constants A_1, A_2, A_3, and A_4 can be uniquely determined. Their values are easily seen to be

$$A_1 = \frac{2c^2\eta W}{9EI}, \quad A_2 = \frac{cW}{3EI}, \quad A_3 = -\frac{(1+\eta)W}{6EI}, \quad A_4 = -\frac{\frac{1}{2}-\eta}{12cEI}.$$

Substituting these values in (7), we have the flexure function χ given by

$$\chi = \frac{W}{3EI}\left[\tfrac{2}{3}c^2\eta x + c(x^2-y^2) - \tfrac{1}{2}(1+\eta)(x^3-3xy^2) - \tfrac{1}{4}(\tfrac{1}{2}-\eta)(x^4-6x^2y^2+y^4)/c\right]. \tag{9}$$

Substituting this value of χ in (1) and (2), we get the displacements and the stresses. Thus we get the solution for the case when the axis of symmetry is in the plane of loading. We now proceed to the second case.

(*b*) *Axis of symmetry perpendicular to the plane of loading.*

As in (*a*) we take the origin at the vertex O of the triangle. The weight W is supposed to act parallel to Ox, which is perpendicular to the axis of symmetry Oy. $OD = c$, G is the centre of gravity of the triangle, G' being the mid-point of OG, and D that of AB.

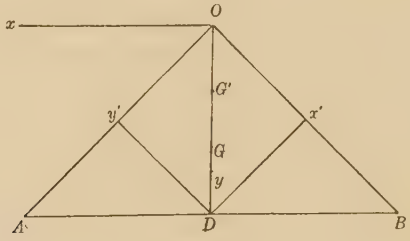

Fig. 2.

Since this case involves what is called the "associated flexural torsion", we shall include the terms resulting from pure torsion in the values of u, v, w, and $\widehat{yz}$ and $\widehat{xz}$. Thus we have

$$u = -\tau yz + \frac{W}{EI}\left[(l-z)\{\tfrac{1}{2}\eta x^2 - \tfrac{1}{2}\eta(y-\tfrac{2}{3}c)^2\} + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\right] + \beta z + \alpha, \tag{10.1}$$

$$v = \tau zx + \eta x(l-z)(y-\tfrac{2}{3}c)\frac{W}{EI}, \tag{10.2}$$

$$w = \tau\phi + \chi(x, y) - \frac{W}{EI}\left[x(lz-\tfrac{1}{2}z^2) + x(y-\tfrac{2}{3}c)^2\right] - \beta x + \gamma, \tag{10.3}$$

and

$$\widehat{yz} = \mu\left[\tau\left(\frac{\partial\phi}{\partial y}+x\right)+\frac{\partial\chi}{\partial y}-\frac{W}{EI}x(2+\eta)(y-\tfrac{2}{3}c)\right], \tag{11.1}$$

$$\widehat{xz} = \mu\left[\tau\left(\frac{\partial\phi}{\partial x}-y\right)+\frac{\partial\chi}{\partial x}-\frac{W}{EI}\{\tfrac{1}{2}\eta x^2+(1-\tfrac{1}{2}\eta)(y-\tfrac{2}{3}c)^2\}\right]. \tag{11.2}$$

The boundary condition is, of course, given by (3). For the flexure function χ in this case, we assume that

$$\chi = A_1x+2A_2xy+A_3(x^3-3xy^2)+\chi_1, \tag{12}$$

where χ_1 is also a harmonic function.

From the boundary condition (3), we have

$$\left[\tau\left(\frac{\partial\phi}{\partial y}+x\right)+\frac{\partial\chi}{\partial y}-\frac{W}{EI}x(2+\eta)(y-\tfrac{2}{3}c)\right]\cos(y\nu)$$

$$+\left[\tau\left(\frac{\partial\phi}{\partial x}-y\right)+\frac{\partial\chi}{\partial x}-\frac{W}{EI}\{\tfrac{1}{2}\eta x^2+(1-\tfrac{1}{2}\eta)(y-\tfrac{2}{3}c)^2\}\right]\cos(x\nu)=0. \tag{13}$$

The torsion function ϕ satisfies the equation

$$\frac{\partial\phi}{\partial x}\cos(x\nu)+\frac{\partial\phi}{\partial y}\cos(y\nu) = y\cos(x\nu)-x\cos(y\nu)$$

over the boundary of the cross-section. We choose χ_1 so that

$$\left[\frac{\partial\chi_1}{\partial y}+\tfrac{2}{3}cx(2+\eta)\frac{W}{EI}\right]\cos(y\nu)+\left[\frac{\partial\chi_1}{\partial x}+\tfrac{4}{3}cy(1-\tfrac{1}{2}\eta)\frac{W}{EI}\right]\cos(x\nu)=0 \tag{14}$$

over the boundary. (13) now reduces to

$$\left[2A_2x-6A_3xy-xy(2+\eta)\frac{W}{EI}\right]\cos(y\nu)$$

$$+\cos(x\nu)\left[A_1+2A_2y+3A_3(x^2-y^2)-\frac{W}{EI}\{\tfrac{1}{2}\eta x^2+(1-\tfrac{1}{2}\eta)(y^2+\tfrac{4}{9}c^2)\}\right]=0. \tag{15}$$

Over the base AB of the triangle, (15) gives

$$2A_2-6A_3c-c(2+\eta)\frac{W}{EI}=0. \tag{16.1}$$

Over the side OA, (15) reduces to

$$A_1+6A_3y^2+\frac{W}{EI}[y^2(2+\eta)-\{y^2+\tfrac{4}{9}c^2(1-\tfrac{1}{2}\eta)\}]=0.$$

Equating the constant term and the coefficient of y^2 separately to zero, we get

$$A_1-\tfrac{4}{9}c^2(1-\tfrac{1}{2}\eta)\frac{W}{EI}=0, \tag{16.2}$$

$$6A_3+(1+\eta)\frac{W}{EI}=0. \tag{16.3}$$

(16.1) now gives

$$A_2=\tfrac{1}{2}c\frac{W}{EI}. \tag{16.4}$$

(16.2), (16.3), and (16.4) determine the unknown constants A_1, A_2, and A_3. We have still to find out the value of χ_1. Putting

$$\chi_1=\chi_2-\tfrac{4}{3}cxy\frac{W}{EI},$$

we have, from (14),

$$\frac{\partial\chi_2}{\partial x}\cos(x\nu)+\frac{\partial\chi_2}{\partial y}\cos(y\nu)=\tfrac{2}{3}c\eta\frac{W}{EI}[y\cos(x\nu)-x\cos(y\nu)]$$

to be satisfied over the boundary of the cross-section. Comparing it with the equation satisfied by the torsion function ϕ, we see that we can put

$$\chi_2=\frac{2c\eta\phi W}{3EI},$$

so that

$$\chi_1=\frac{2c\eta\phi W}{3EI}-\frac{4cxyW}{3EI}.$$

χ is, therefore, given by

$$\chi=\frac{W}{3EI}[\tfrac{4}{3}c^2x(1-\tfrac{1}{2}\eta)-cxy-\tfrac{1}{2}(1+\eta)(x^3-3xy^2)+2c\eta\phi]. \tag{17}$$

We now consider the value of ϕ. With Dx' and Dy' as the axes of coordinates in Fig. 2, Kolossoff has obtained

$$\phi=-\tfrac{1}{2}(x^2-y^2)+a(x-y)$$

$$-2\left(\frac{2}{\pi}\right)^3a^2\sum_0^\infty\left[\frac{(-1)^n}{(2n+1)^3\sinh(n+\frac{1}{2})\pi}\right.$$

$$\left.\times\left\{\sin(n+\tfrac{1}{2})\frac{\pi x}{a}\cosh(n+\tfrac{1}{2})\frac{\pi y}{a}-\sin(n+\tfrac{1}{2})\frac{\pi y}{a}\cosh(n+\tfrac{1}{2})\frac{\pi x}{a}\right\}\right],$$

where $c=a\sqrt{2}$.

Referred to the coordinate axes which we have taken, the above value of ϕ becomes

$$\phi = -x(y-c)$$

$$+\left(\frac{2}{\pi}\right)^3 c^2 \sum_0^\infty \left[\frac{(-1)^n}{(2n+1)^3 \sinh(n+\frac{1}{2})\pi}\right.$$

$$\times \left\{\sin(2n+1)\frac{(x+y-c)\pi}{2c}\cosh(2n+1)\frac{(x-y+c)\pi}{2c}\right.$$

$$\left.\left.+\sin(2n+1)\frac{(x-y+c)\pi}{2c}\cosh(2n+1)\frac{(x+y-c)\pi}{2c}\right\}\right]. \quad (18)$$

As is well known, we may take any convenient set of coordinate axes in a torsion problem. If the z-axis passes through the centroids of the cross-sections, we can change it to any parallel position without affecting the stress-system. When we have taken one set of axes at the beginning of the problem and obtained the corresponding ϕ, the ϕ referred to the new set of axes which we want to take is obtained by merely changing the values of the coordinates to their new values in the expression for the torsion function*. But such a method cannot be adopted in a Saint-Venant flexure problem. If we want to change the axes, we must go back to the values of the displacements u, v, and w, change their values, and then calculate the new χ afresh. Hence it is that we have to change Kolossoff's formula giving the torsion function ϕ to the axes which we have taken for our flexure problem. Apparently, of course, it would have been simpler to change the form of the remaining terms in the flexure function rather than that of ϕ. But, as I have already pointed out, this would be incorrect.

Substituting the value of ϕ given by (18) in (17), we get the final form for χ. This completes the determination of the flexure function. α, β, γ, the unknown constants in the displacements u and w, are still to be found. We now proceed to determine them.

Determination of the constants α, β, γ.

As already stated, these constants are to be determined by the fixing. We shall at first treat the two cases separately.

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* It is assumed that the origin remains the same. If it is changed to the point (x', y') the new φ' is given by $\varphi' = \varphi - xy' + yx'$.

(a) If, after Saint-Venant, we fix a small area enclosing G, we have

$$\left.\begin{aligned} u = v = w &= 0 \\ \frac{\partial w}{\partial x} &= 0 \end{aligned}\right\} \quad \text{at} \quad x = \tfrac{2}{3}c,\ y = z = 0.$$

(1) now gives

$$\alpha = 0, \quad \gamma - \tfrac{2}{3}\beta c + \chi_0(x, y) = 0, \quad \beta - \frac{\partial \chi_0}{\partial x} = 0,$$

where $\chi_0(x, y)$ and $\partial\chi_0/\partial x$ are the values of $\chi(x, y)$ and $\partial\chi/\partial x$ at the point $x = \frac{2}{3}c$, $y = 0$. Using (9), we get

$$\chi_0(x, y) = \frac{2c^3 W}{3^5 EI}(14\eta + 11),$$

$$\frac{\partial \chi_0}{\partial x} = \frac{2c^2 W}{3^4 EI}(4\eta + 7).$$

Hence we have

$$\left.\begin{aligned} \alpha &= 0, \\ \beta &= \frac{2c^2 W}{3^4 EI}(4\eta + 7), \\ \gamma &= \frac{2c^3 W}{3^4 EI}(1 - 2\eta). \end{aligned}\right\} \tag{19}$$

But, since this fixing is not possible in practice, we shall try to fix two points on the x-axis. If we fix O, it can easily be seen that we cannot fix any other point on the x-axis. If we fix D, it becomes possible to fix G' as well, G' being the mid-point of OG. In such a case we have

$$u = v = w = 0 \quad \text{both at} \quad x = c, \quad y = z = 0, \quad \text{and at} \quad x = \tfrac{1}{3}c, \quad y = z = 0.$$

(1) now gives

$$\alpha + \frac{c^2 l \eta W}{18 EI} = 0,$$

$$\gamma - \beta c + \frac{c^3 W}{4EI}(\tfrac{1}{2} + \tfrac{5}{9}\eta) = 0,$$

$$\gamma - \tfrac{1}{3}\beta c + \frac{c^3 W}{2^3 . 3^5 EI}(59 + 134\eta) = 0,$$

and hence

$$\left.\begin{aligned} \alpha &= -\frac{c^2 l \eta W}{18 EI}, \\ \beta &= \frac{c^2 W}{2 . 3^4 EI}(23 + 17\eta), \\ \gamma &= \frac{11 c^3 W}{2^3 . 3^4 EI}(1 - 2\eta). \end{aligned}\right\} \tag{20}$$

(*b*) We now consider the other case. Here, too, it is found that we can fix the points D and G' on the axis of symmetry, Oy, in Fig. 2. From (18) we can easily see that the torsion function ϕ vanishes for all points on Oy. Hence (17) gives $\chi = 0$ for all points on Oy. The conditions for fixing D and G' are

$$u = v = w = 0 \quad \text{both at} \quad y = c, \quad x = z = 0, \quad \text{and at} \quad y = \tfrac{1}{3}c, \quad x = z = 0.$$

Using (10), we get

$$\alpha = \frac{c^2 l \eta W}{18 E I},$$

$$\gamma = 0,$$

and β disappears from the equations.

It appears, therefore, that the two points D and G' on the axis of symmetry can be fixed whatever be the direction in which the load W acts.

The flexure solution given in the foregoing pages is quite a simple one. So far as rectilinear cross-sections go, it is the only solution after that for a rectangular cross-section which has as yet been obtained. The forms of cross-sections which we quite often encounter in the solution of engineering problems possess uniaxial and not biaxial symmetry. Hence it is that problems of the type treated in these pages should be given more attention than they have hitherto received.

ON THE DEFORMATION OF TENSOR MANIFOLDS

By P. DIENES.

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Notation. All the suffixes vary from 1 to n, with the usual summation convention. The system of n contravariant and n covariant vectors $\underset{\alpha}{e}$ and $\overset{\alpha}{e}$ respectively, whose components are all 0 but the α-th, which is 1, is the mathematical frame or base; vectors and tensors are forms in $\underset{\alpha}{e}$ and $\overset{\alpha}{e}$. In a system of vectors $(\underset{1}{v}, \ldots, \underset{n}{v})$, $\underset{i}{v^\alpha}$ denotes the α-th component of the i-th vector $\underset{i}{v}$; $\delta_{ij} = \delta_j^{\,i}$ is equal to 0 if $i \neq j$ and to 1 if $i = j$.

1. *Point and base deformation.* Following Einstein (**2**) and Levi-Civita (**4**), to every point $P(x^1, \ldots, x^n) \equiv P(x) \equiv (x)$ we assign a geometrical base, *i.e.* a system of fundamental contravariant and covariant vectors $\underset{i}{e^\alpha}(x)$, $\underset{i}{e_\alpha}(x)$, where the latter set is determined by the equations

$$\underset{i}{e_\alpha}\, e^\alpha = \delta_{ij}, \tag{1}$$

which also involves

$$\underset{i}{e_\alpha}\, \underset{i}{e^\beta} = \delta_\alpha^\beta. \tag{2}$$

In addition to the hypothesis that they are differentiable, the functions $\underset{i}{e^\alpha}$ are subjected to the sole condition that their determinant $||\underset{i}{e^\alpha}||$ should not vanish. The functions $\underset{i}{e^\alpha}$ and $\underset{i}{e_\alpha}$ can also be interpreted as an n-fold congruence of curves and their so-called moments. (See Levi-Civita, **3**, 265.)

We propose to complete the usual point deformation

$$\bar{x} = x^\alpha + \epsilon \xi^\alpha, \tag{3}$$

which moves $P(x)$ to $\bar{P}(\bar{x})$, by a deformation of the base (congruence) as

given by

$$\underset{i}{\bar{e}^{\alpha}} = \underset{i}{e^{\alpha}} + \epsilon \underset{i}{k^{\alpha}}, \quad \underset{i}{\bar{e}_{\alpha}} = \underset{i}{e_{\alpha}} - \epsilon \underset{i}{k_{\alpha}}, \tag{4}$$

where

$$\underset{i}{k_{\alpha}} = \underset{i}{e_{\beta}}\, \underset{j}{e_{\alpha}}\, \underset{j}{k^{\beta}}, \tag{5}$$

in order that we should have

$$\underset{i}{\bar{e}^{\alpha}}\, \underset{j}{\bar{e}_{\alpha}} = \delta_{ij}, \quad \text{and thus also} \quad \underset{i}{\bar{e}^{\alpha}}\, \underset{i}{e_{\beta}} = \delta^{\alpha}_{\beta}. \tag{6}$$

If v^{α} and v^{i} are the components of a vector in the mathematical and geometrical axes respectively, we have

$$v^{\alpha} = \underset{i}{v^{i}}\, \underset{i}{e^{\alpha}}, \tag{7}$$

so that the base deformation changes v^{α} into

$$\bar{v}^{\alpha} = \underset{i}{v^{i}}\, \underset{i}{\bar{e}^{\alpha}} = v^{\beta}(\delta^{\alpha}_{\beta} + \epsilon \eta^{\alpha}_{\beta}), \tag{8}$$

where

$$\eta^{\alpha}_{\beta} = \underset{i}{e_{\beta}}\, \underset{i}{k^{\alpha}}. \tag{9}$$

In like manner,

$$\bar{v}_{\beta} = v_{\alpha}(\delta^{\alpha}_{\beta} - \epsilon \eta^{\alpha}_{\beta}). \tag{10}$$

The effect of the point deformation (3) on vectors and tensors is usually deduced from its effect on the standard contravariant vector dx^{α}, *i.e.* from the relation

$$d\bar{x}^{\alpha} = dx^{\alpha} + \epsilon \partial_{\beta} \xi^{\alpha}\, dx^{\beta} = (\delta^{\alpha}_{\beta} + \epsilon \partial_{\beta} \xi^{\alpha})\, dx^{\beta}, \quad \partial_{\beta} \xi^{\alpha} = \frac{\partial \xi^{\alpha}}{\partial x^{\beta}}. \tag{11}$$

Thus, for a point deformation,

$$\bar{v}^{\alpha} = (\delta^{\alpha}_{\beta} + \epsilon \partial_{\beta} \xi^{\alpha})\, v^{\beta}, \quad \bar{v}_{\beta} = (\delta^{\alpha}_{\beta} - \epsilon \partial_{\beta} \xi^{\alpha})\, v_{\alpha}. \tag{12}$$

The joint effect of the point and base deformations will be to replace the whole pencil of tensors v^{α}, v_{β}, v^{α}_{β}, ... at P by the deformed pencil

$$\bar{v}^{\alpha} = D^{\alpha}_{\beta} v^{\beta}, \quad \bar{v}_{\beta} = \bar{D}^{\alpha}_{\beta} v^{\alpha}, \quad \bar{v}^{\alpha}_{\beta} = D^{\alpha}_{\alpha'} \bar{D}^{\beta'}_{\beta} v^{\alpha'}_{\beta'}, \quad \ldots \tag{13}$$

at $\bar{P}(x^{\alpha} + \epsilon \xi^{\alpha})$, where the deformation factors have the following values:

$$D^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + \epsilon(\partial_{\beta} \xi^{\alpha} + \eta^{\alpha}_{\beta}), \quad \bar{D}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - \epsilon(\partial_{\beta} \xi^{\alpha} + \eta^{\alpha}_{\beta}). \tag{14}$$

It is readily verified that addition, multiplication, and contraction are all interchangeable with the deformation as defined by (13) and (14).

2. *The deformed space.* All the deformed tensor pencils at the various points $\bar{x}^\alpha = x^\alpha + \epsilon\xi^\alpha$ form a tensor manifold. They will become the elements of a tensor space if we define a metric (length and angle) as well as a connection between tensors at neighbouring points (Dienes, **1**, 257). As a rule, the metric of the deformed space is defined as the deform of the metric tensor $g_{\alpha\beta}$. We propose to determine its connection by the condition that *parallelism should not be destroyed in the deformation.* This means that, if v^α at $P(x)$ and w^α at $Q(x+dx)$ are parallel, *i.e.* if

$$v^\alpha = w^\alpha + \Gamma^\alpha_{\beta\gamma} w^\beta dx^\gamma, \tag{1}$$

we should also have

$$\bar{w}^\alpha(\bar{Q} \,\bar{\|}\, \bar{P}) = \bar{v}^\alpha, \tag{2}$$

where $\bar{\|}$ denotes parallel transport in the deformed space.

Now

$$\begin{cases} \bar{v}^\alpha = v^\alpha + \epsilon v^\beta(\partial_\beta \xi^\alpha + \eta^\alpha_\beta) & \text{at} \quad \bar{P}, \\ \bar{w}^\alpha = w^\alpha + \epsilon w^\beta[\partial_\beta \xi^\alpha(Q) + \eta^\alpha_\beta(Q)] & \text{at} \quad \bar{Q}, \end{cases} \tag{3}$$

so that, putting

$$\bar{\Gamma}^\alpha_{\beta\gamma}(\bar{P}) = \Gamma^\alpha_{\beta\gamma}(\bar{P}) - \epsilon\delta\Gamma^\alpha_{\beta\gamma}, \tag{4}$$

we have

$$\begin{aligned} \bar{w}^\alpha(\bar{Q} \,\bar{\|}\, \bar{P}) = w^\alpha &+ \epsilon w^\beta\,[\partial_\beta \xi^\alpha(Q) + \eta^\alpha_\beta(Q)] \\ &+ [\Gamma^\alpha_{\beta\gamma}(\bar{P}) - \epsilon\delta\Gamma^\alpha_{\beta\gamma}][w^\beta + \epsilon w^\delta\{\partial_\delta \xi^\beta(Q) + \eta^\beta_\delta(Q)\}] \\ &\times [dx^\gamma + \epsilon(\partial_\zeta \xi^\gamma + \eta^\gamma_\zeta)\, dx^\zeta]. \end{aligned} \tag{5}$$

Thus, from

$$\Gamma^\alpha_{\beta\gamma}(\bar{P})\,[dx^\gamma + \epsilon(\partial_\zeta \xi^\gamma + \eta^\gamma_\zeta)\, dx^\zeta] = \Gamma^\alpha_{\beta\gamma} dx^\gamma + \epsilon\partial_\delta \Gamma^\alpha_{\beta\gamma}\, \xi^\delta dx^\gamma + \epsilon\Gamma^\alpha_{\beta\gamma}(\partial_\zeta \xi^\gamma + \eta^\gamma_\zeta)\, dx^\zeta,$$

and from our requirement (2), we get, by (1) and (3),

$$\begin{aligned} w^\beta \delta\Gamma^\alpha_{\beta\gamma}\, dx^\gamma = w^\beta[\partial_\beta \xi^\alpha(Q) + \eta^\alpha_\beta(Q)] + \Gamma^\alpha_{\beta\gamma} w^\delta[\partial_\delta \xi^\beta(Q) + \eta^\beta_\delta(Q)]\, dx^\gamma \\ + w^\beta \partial_\delta \Gamma^\alpha_{\beta\gamma}\, \xi^\delta dx^\gamma + w^\beta \Gamma^\alpha_{\beta\gamma}(\partial\ \xi^\gamma + \eta^\gamma_\zeta)\, dx^\zeta - v^\beta(\partial_\beta \xi^\alpha + \eta^\alpha_\beta). \end{aligned} \tag{6}$$

On the right-hand side, the first and last terms give

$$\begin{aligned} w^\beta(\partial_\beta \xi^\alpha + \eta^\alpha_\beta) + w^\beta(\partial_\gamma \partial_\beta \xi^\alpha + \partial_\gamma \eta^\alpha_\beta)\, dx^\gamma - (w^\beta + \Gamma^\alpha_{\delta\gamma} w^\delta dx^\gamma)(\partial_\beta \xi^\alpha + \eta^\alpha_\beta) \\ = w^\beta(\partial_\gamma \partial_\beta \xi^\alpha + \partial_\gamma \eta^\alpha_\beta)\, dx^\gamma - \Gamma^\alpha_{\beta\gamma} w^\beta(\partial_\delta \xi^\alpha + \eta^\alpha_\delta)\, dx^\gamma, \end{aligned} \tag{7}$$

and thus, finally,

$$(8)\quad \delta\Gamma^{\alpha}_{\beta\gamma} = \partial_\gamma\partial_\beta\xi^\alpha + \partial_\gamma\eta^\alpha_\beta - \Gamma^\delta_{\beta\gamma}(\partial_\delta\xi^\alpha + \eta^\alpha_\beta) + \Gamma^\alpha_{\beta\delta}(\partial_\gamma\xi^\delta + \eta^\delta_\gamma) + \Gamma^\alpha_{\delta\gamma}(\partial_\beta\xi^\delta + \eta^\delta_\beta) + \partial_\delta\Gamma^\alpha_{\beta\gamma}\,\xi^\delta.$$

If we express ordinary derivatives in terms of tensor derivatives, and if we put

$$R^{\iota}_{\beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} + \Gamma^\alpha_{\epsilon\delta}\Gamma^\epsilon_{\beta\gamma} - \Gamma^\alpha_{\epsilon\gamma}\Gamma^\epsilon_{\beta\delta}, \quad 2S^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta},$$

we obtain

$$(9)\quad \delta\Gamma^\alpha_{\beta\gamma} = \nabla_\gamma\nabla_\beta\xi^\alpha + 2S^\alpha_{\beta\delta}\nabla_\gamma\xi^\delta + [R^\alpha_{\beta\gamma\delta} + 2\nabla_\gamma S^\alpha_{\beta\delta}]\,\xi^\delta + \nabla_\gamma\eta^\alpha_\beta + \Gamma^\alpha_{\beta\delta}\eta^\delta_\gamma.$$

The last term, which spoils the tensor form of $\delta\Gamma^\alpha_{\beta\gamma}$ comes from $\epsilon\eta^\zeta_\xi dx^\zeta$ in the last bracket in (5). According to the usual definition of parallelism, this last bracket in (5) contains the coordinates of $\bar{Q}$ with respect to $\bar{P}$ in the frame attached to $\bar{P}$, so that we cannot suppress this term; $\delta\Gamma^\alpha_{\beta\gamma}$ *is not necessarily a tensor*, as it always is in a pure point deformation. In the latter case (9) reduces to a formula recently given by W. Slebodzinski (**6**) with Weitzenböck's sign (**7**, 375). Our geometrical method of determining $\delta\Gamma^\alpha_{\beta\gamma}$ seems to be new.

3. *Torsion and curvature in the deformed space.* Since, in general, a base deformation does not reduce to a change of variables, our general deformation corresponds to a change of base. Therefore (Dienes, **1**, 269) the new torsion tensor is not $\bar{\Gamma}^\alpha_{[\beta\gamma]}$, but

$$(1)\quad \bar{T}^\alpha_{\beta\gamma} = \bar{\Gamma}^\alpha_{[\beta\gamma]} + \bar{D}^\delta_{[\beta}\partial_{\gamma]}D^\alpha_\delta = \bar{\Gamma}^\alpha_{\beta\gamma} + \epsilon\partial_{[\gamma}\eta^\alpha_{\beta]}.$$

Similarly (Dienes, **1**, 270), the curvature tensor of the deformed space is not the Riemann-Christoffel tensor $\bar{R}^\alpha_{\beta\gamma\delta}$ formed by $\bar{\Gamma}^\alpha_{\beta\gamma}$ but

$$(2)\quad \bar{C}^\alpha_{\beta\gamma\delta} = \bar{R}^\alpha_{\beta\gamma\delta} + 2\epsilon\bar{D}^\zeta_{[\gamma}\partial_{\delta]}D^\phi_\zeta\Gamma^\alpha_{\beta\phi} = \bar{R}^\alpha_{\beta\gamma\delta} + 2\epsilon\partial_{[\delta}\eta^\phi_{\gamma]}\Gamma^\alpha_{\beta\phi}.$$

These definitions are further justified by the fact that both $\bar{T}^\alpha_{\beta\gamma}$ and $\bar{C}^\alpha_{\beta\gamma\delta}$ coincide with the direct deforms of $\Gamma^\alpha_{[\beta\gamma]}$ and $R^\alpha_{\beta\gamma\delta}$ respectively. Without the additional terms there is no coincidence, but this coincidence also requires the last term in (2.9).

These remarks are all proved by the fact that $\bar{\Gamma}^\alpha_{\beta\gamma}$, as defined by (2.4) and (2.8), is the usual transform of $\Gamma^\alpha_{\beta\gamma}$ in the sense that

$$(3)\quad \bar{\Gamma}^\alpha_{\beta\gamma}(\bar{P}) = D^\alpha_{\alpha'}\bar{D}^{\beta'\gamma'}_{\beta\gamma}\Gamma^{\alpha'}_{\beta'\gamma'}(P) + D^\alpha_\delta\partial_\gamma\bar{D}^\delta_\beta(P),$$

which is established by developing the right-hand side of (3):

$$\Gamma^{\alpha}_{\beta\gamma}(P)+\epsilon(\partial_{\alpha'}\xi^{\alpha}+\eta^{\alpha}_{\alpha'})\,\Gamma^{\alpha'}_{\beta\gamma}-\epsilon(\partial_{\beta}\xi^{\beta'}+\eta^{\beta'}_{\beta})\,\Gamma^{\alpha}_{\beta'\gamma} \tag{4}$$
$$-\epsilon(\partial_{\gamma}\xi^{\gamma'}+\eta^{\gamma'}_{\gamma})\,\Gamma^{\alpha}_{\beta\gamma'}-\epsilon\partial_{\gamma}(\partial_{\beta}\xi^{\alpha}+\eta^{\alpha}_{\beta}).$$

In view of this fact, the whole deformation can be conceived as a transformation where the deformation factors D^{α}_{β} and $\bar{D}^{\alpha}_{\beta}$ play the rôle of the transformation factors A^{α}_{a} and A^{a}_{α} (Dienes, **1**, 256. See also J. A. Schouten and E. R. von Kampen, **5**, 75)*.

As a confirmation of this view, we write out the proof for a pure base-deformation $\xi^{\alpha}=0$. We have

$$\bar{\Gamma}^{\alpha}_{\beta\gamma}+\epsilon\partial_{\gamma}\eta^{\alpha}_{\beta}=\Gamma^{\alpha}_{\beta\gamma}-\epsilon(\partial_{\gamma}\eta^{\alpha}_{\beta}-\Gamma^{\delta}_{\beta\gamma}\eta^{\alpha}_{\delta}+\Gamma^{\alpha}_{\beta\delta}\eta^{\delta}_{\gamma}+\Gamma^{\alpha}_{\delta\gamma}\eta^{\alpha}_{\beta})+\epsilon\partial_{\gamma}\eta^{\alpha}_{\beta} \tag{5}$$
$$=\Gamma^{\alpha}_{\beta\gamma}+\Gamma^{\delta}_{\beta\gamma}\eta^{\alpha}_{\delta}-\Gamma^{\alpha}_{\beta\delta}\eta^{\delta}_{\gamma}-\Gamma^{\alpha}_{\delta\gamma}\eta^{\delta}_{\beta}$$

and

$$\Gamma^{\alpha'}_{\beta'\gamma'}D^{\alpha}_{\alpha'}\bar{D}^{\beta'\gamma'}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}+\epsilon\Gamma^{\delta}_{\beta\gamma}\eta^{\alpha}_{\delta}-\Gamma^{\alpha}_{\beta\delta}\eta^{\delta}_{\gamma}-\Gamma^{\alpha}_{\delta\gamma}\eta^{\delta}_{\beta}, \tag{6}$$

which proves that $\bar{T}^{\alpha}_{\beta\gamma}$ is equal to the deform of $\Gamma^{\alpha}_{[\beta\gamma]}$. We notice, however, that the term $\Gamma^{\alpha}_{\beta\delta}\eta^{\delta}_{\gamma}$ is essential for this coincidence.

As to the curvature tensor, we have

$$\bar{\Gamma}^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}-\epsilon(\partial_{\gamma}\eta^{\alpha}_{\beta}-\Gamma^{\zeta}_{\beta\gamma}\eta^{\alpha}_{\zeta}+\Gamma^{\alpha}_{\zeta\gamma}\eta^{\zeta}_{\beta}+\Gamma^{\alpha}_{\beta\zeta}\eta^{\zeta}_{\gamma}) \tag{7}$$

and

$$\bar{C}^{\alpha}_{\beta\gamma\delta}=\frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}}-\epsilon\frac{\partial}{\partial x^{\delta}}(\partial_{\gamma}\eta^{\alpha}_{\beta}-\Gamma^{\zeta}_{\beta\gamma}\eta^{\alpha}_{\zeta}+\Gamma^{\alpha}_{\zeta\gamma}\eta^{\zeta}_{\beta}+\Gamma^{\alpha}_{\beta\zeta}\eta^{\zeta}_{\gamma})-\frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} \tag{8}$$
$$+\epsilon\frac{\partial}{\partial x^{\gamma}}(\partial_{\delta}\eta^{\alpha}_{\beta}-\Gamma^{\zeta}_{\beta\delta}\eta^{\alpha}_{\zeta}+\Gamma^{\alpha}_{\zeta\delta}\eta^{\zeta}_{\beta}+\Gamma^{\alpha}_{\beta\zeta}\eta^{\zeta}_{\gamma})$$
$$+[\Gamma^{\alpha}_{\zeta\delta}-\epsilon(\partial_{\delta}\eta^{\alpha}_{\zeta}-\Gamma^{\phi}_{\zeta\delta}\eta^{\alpha}_{\phi}+\Gamma^{\alpha}_{\phi\delta}\eta^{\phi}_{\zeta}+\Gamma^{\alpha}_{\zeta\phi}\eta^{\phi}_{\delta})]$$
$$\times[\Gamma^{\zeta}_{\beta\gamma}-\epsilon(\partial_{\gamma}\eta^{\zeta}_{\beta}-\Gamma^{\phi}_{\beta\gamma}\eta^{\zeta}_{\phi}+\Gamma^{\zeta}_{\phi\gamma}\eta^{\phi}_{\beta}+\Gamma^{\zeta}_{\beta\phi}\eta^{\phi}_{\gamma})]$$
$$-[\Gamma^{\alpha}_{\zeta\gamma}-\epsilon(\partial_{\gamma}\eta^{\alpha}_{\zeta}-\Gamma^{\phi}_{\zeta\gamma}\eta^{\alpha}_{\phi}+\Gamma^{\alpha}_{\phi\gamma}\eta^{\phi}_{\zeta}+\Gamma^{\alpha}_{\zeta\phi}\eta^{\phi}_{\gamma})]$$
$$\times[\Gamma^{\zeta}_{\beta\delta}-\epsilon(\partial_{\delta}\eta^{\zeta}_{\beta}-\Gamma^{\phi}_{\beta\delta}\eta^{\zeta}_{\phi}+\Gamma^{\zeta}_{\phi\delta}\eta^{\phi}_{\beta}+\Gamma^{\zeta}_{\beta\phi}\eta^{\phi}_{\delta})]$$
$$+2\epsilon\partial_{[\delta}\eta^{\phi}_{\gamma]}\Gamma^{\alpha}_{\beta\phi}.$$

* [Added in proof, June, 1934.] In a paper on the infinitesimal deformation of tensor spaces by P. Dienes and E. T. Davies, in course of publication in the *Journal de Math.*, the deformed space is described as the *transform* or representative, at the new point, of the original space, and the deformation of quantities $A^{\alpha}_{\beta\gamma}$ with given law of transformation (deformation) like tensors or $\Gamma^{\alpha}_{\beta\gamma}$ is defined as the difference between $A^{\alpha}_{\beta\gamma}(\bar{P})$, the *displaced* $A^{\alpha}_{\beta\gamma}$, and $\bar{A}^{\alpha}_{\beta\gamma}(\bar{P})$, the transform of $A^{\alpha}_{\beta\gamma}(P)$. In this way the operator δ coincides with the *Lie*-derivative of Schouten and van Dantzig. In view of the negative sign before $\delta\Gamma$ in (2.4) the operator δ as defined in this paper is the extension of *Lie*-derivation to infinitesimal deformations admitting an independent change of base.

The terms containing no ϵ give $R^{\alpha}_{\beta\gamma\delta}$. The two terms containing second derivatives of η cancel each other. The terms containing first derivatives are

$$\underset{1}{\Gamma^{\zeta}_{\beta\gamma}\partial_{\delta}\eta^{\alpha}_{\zeta}}-\underset{2}{\Gamma^{\alpha}_{\zeta\gamma}\partial_{\delta}\eta^{\zeta}_{\beta}}-\underset{3}{\Gamma^{\alpha}_{\beta\zeta}\partial_{\delta}\eta^{\zeta}_{\gamma}}-\underset{4}{\Gamma^{\zeta}_{\beta\delta}\partial_{\gamma}\eta^{\alpha}_{\zeta}}+\underset{5}{\Gamma^{\alpha}_{\zeta\delta}\partial_{\gamma}\eta^{\zeta}_{\beta}}+\underset{6}{\Gamma^{\alpha}_{\beta\zeta}\partial_{\gamma}\eta^{\zeta}_{\delta}}$$
$$-\underset{7}{\Gamma^{\zeta}_{\beta\gamma}\partial_{\delta}\eta^{\alpha}_{\zeta}}-\underset{8}{\Gamma^{\alpha}_{\zeta\delta}\partial_{\gamma}\eta^{\zeta}_{\beta}}+\underset{9}{\Gamma^{\zeta}_{\beta\delta}\partial_{\gamma}\eta^{\alpha}_{\zeta}}+\underset{10}{\Gamma^{\alpha}_{\zeta\gamma}\partial_{\delta}\eta^{\zeta}_{\beta}}+\underset{11}{\partial_{\delta}\eta^{\phi}_{\gamma}\Gamma^{\alpha}_{\beta\phi}}-\underset{12}{\partial_{\gamma}\eta^{\phi}_{\delta}\Gamma^{\alpha}_{\beta\phi}},$$

and this sum vanishes because the pairs of terms 1+7, 2+10, 3+11, 4+9, 5+8, 6+12 all cancel each other.

Finally, in the second and fourth term of (8) the coefficient of $\epsilon\eta^{\alpha}_{\zeta}$ is

$$\frac{\partial\Gamma^{\zeta}_{\beta\gamma}}{\partial x^{\delta}}-\frac{\partial\Gamma^{\zeta}_{\beta\delta}}{\partial x^{\gamma}},$$

and in the other terms its coefficient is $\Gamma^{\zeta}_{\epsilon\delta}\Gamma^{\epsilon}_{\beta\gamma}-\Gamma^{\zeta}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\delta}$. Similarly for the coefficients of η^{ζ}_{β}, etc. Hence

$$\bar{C}^{\alpha}_{\beta\gamma\delta}=R^{\alpha}_{\beta\gamma\delta}+\epsilon R^{\zeta}_{\beta\gamma\delta}\eta^{\alpha}_{\zeta}-\epsilon R^{\alpha}_{\zeta\gamma\delta}\eta^{\zeta}_{\beta}-\epsilon R^{\alpha}_{\beta\zeta\delta}\eta^{\zeta}_{\gamma}-\epsilon R^{\alpha}_{\beta\gamma\zeta}\eta^{\zeta}_{\delta},$$

which is the direct deform of $R^{\alpha}_{\beta\gamma\delta}$.

More generally, the deformation applied to any geometrical tensor (Dienes, **1**, 268) leads to its direct transform by the deformation factors. This remark applies, in particular, to geometrical invariants and differential parameters.

4. *Deformation of pseudo-flat spaces.* A field $\underset{i}{e^{\alpha}}(x)$ of n-uples being given, we construct a pseudo-flat space by requiring (i) that the metric $g_{\alpha\beta}$ should make $\underset{i}{e}$ at right angles to one another, (ii) that $\underset{i}{e}(Q)$ should be parallel to $\underset{i}{e}(P)$, where Q is in the neighbourhood of P.

It follows from (ii) that parallel values at P and Q have the same or proportional components. This fact is referred to as *teleparallelism* (Einstein **2**, Levi-Civita, **4**). We notice that (i) leads to

$$(1)\qquad g_{\alpha\beta}=\underset{i}{l}\,\underset{i}{\bar{e}_{\alpha}}\,\underset{i}{\bar{e}_{\beta}},\qquad g^{\alpha\beta}=\frac{1}{\underset{i}{l}}\,\underset{i}{e^{\alpha}}\,\underset{i}{e^{\beta}},$$

where $\underset{i}{l}$ is the length of $\underset{i}{e}$. Also, if $\Lambda^{\alpha}_{\beta\gamma}$ are the connection parameters of the space, requirement (ii) takes the form

$$\underset{i}{e^{\alpha}}+\Lambda^{\alpha}_{\beta\gamma}\underset{i}{e^{\beta}}dx^{\gamma}=\underset{i}{e^{\alpha}}(Q)=\underset{i}{e^{\alpha}}+\partial_{\gamma}\underset{i}{e^{\alpha}}dx^{\gamma},$$

i.e.

$$(2)\qquad \Lambda^{\alpha}_{\beta\gamma}=\underset{i}{e^{\alpha}}\,\partial_{\gamma}\underset{i}{\bar{e}^{\beta}}.$$

We remark that in the space so constructed the connection is not derived from the metric, and, in fact, $\Lambda^{\alpha}_{\beta\gamma}$ is not necessarily equal to the three index symbols formed with (1). Their difference leads to the Ricci γ's.

It is readily proved (Einstein, **2**) that

$$\nabla_\gamma \underset{i}{e^\alpha} = 0, \quad \nabla_\gamma g_{\alpha\beta} = 0, \quad \nabla_\gamma g^{\alpha\beta} = 0, \quad R^{\alpha}_{\beta\gamma\delta} = 0. \tag{3}$$

We may add that *our method of assigning a connection to the deformed space by the preservation of parallelism leads to the same connection as the condition that the deformed space should remain a pseudo-flat space.*

Proof. In the deformed frame $\underset{i}{\bar{e}}$ at P the coordinates of $Q(dx^\alpha)$ are

$$d\bar{x}^\gamma = (\delta^\gamma_\zeta + \epsilon\eta^\gamma_\zeta)\, dx^\zeta, \tag{4}$$

so that our requirement takes the form

$$\underset{i}{\bar{e}^\alpha} - \bar{\Lambda}^{\alpha}_{\beta\gamma}\, \underset{i}{\bar{e}^\beta}\, d\bar{x}^\gamma = \underset{i}{\bar{e}}(Q) = \underset{i}{\bar{e}^\alpha} + \partial_\gamma \underset{i}{\bar{e}^\alpha}\, dx^\gamma; \tag{5}$$

therefore

$$\bar{\Lambda}^{\alpha}_{\beta\gamma}\, \underset{i}{\bar{e}^\beta}(\delta^\gamma_\zeta + \epsilon\eta^\gamma_\zeta) = -\partial_\zeta \underset{i}{\bar{e}^\alpha},$$

and thus

$$\bar{\Lambda}^{\alpha}_{\beta\gamma} = -\underset{i}{\bar{e}_\beta}\, \partial_\gamma \underset{i}{\bar{e}^\alpha} - \epsilon\eta^\zeta_\gamma \Lambda^{\alpha}_{\beta\zeta} = \underset{i}{\bar{e}^\alpha}\, \partial_\gamma \underset{i}{\bar{e}_\beta}. \tag{6}$$

Also

$$\underset{i}{\bar{e}^\alpha}\, \partial_\gamma \underset{i}{\bar{e}_\beta} = \left(\underset{i}{e^\alpha} + \epsilon \underset{i}{k^\alpha}\right)\left(\partial_\gamma \underset{i}{\bar{e}_\beta} + \epsilon \partial \underset{i}{\bar{k}_\beta}\right) = \Lambda^{\alpha}_{\beta\gamma} + \epsilon\left(\underset{i}{e^\alpha}\, \partial_\gamma \underset{i}{\bar{k}^\beta} + \underset{i}{k^\alpha}\, \partial_\gamma \underset{i}{\bar{e}_\beta}\right),$$

and thus

$$\underset{i}{\bar{e}^\alpha}\, \partial_\gamma \underset{i}{\bar{e}_\beta} = \Lambda^{\alpha}_{\beta\gamma} - \epsilon\left(\underset{i}{e^\alpha}\, \partial_\gamma \underset{i}{e_\zeta}\, \underset{j}{e_\beta}\, \underset{j}{k^\zeta} + \underset{i}{e^\alpha}\, \underset{i}{e_\zeta}\, \partial_\gamma \underset{j}{e_\beta}\, \underset{j}{k^\zeta} + \underset{i}{e^\alpha}\, \underset{i}{e_\zeta}\, \underset{j}{e_\beta}\, \partial_\gamma \underset{j}{k^\zeta} - \underset{i}{k^\alpha}\, \partial_\gamma \underset{i}{e_\beta}\right)$$

$$= \Lambda^{\alpha}_{\beta\gamma} - \epsilon\left(\underset{i}{e_\beta}\, \partial_\gamma \underset{i}{k^\alpha} + \underset{i}{e^\alpha}\, \partial_\gamma \underset{i}{e_\zeta}\, \eta^\zeta_\beta\right) = \Lambda^{\alpha}_{\beta\gamma} - \epsilon\left(\partial_\gamma \eta^\alpha_\beta - \underset{i}{k^\alpha}\, \partial_\gamma \underset{i}{e_\beta} + \Lambda^{\alpha}_{\zeta\gamma}\, \eta^\zeta_\beta\right)$$

$$= \Lambda^{\alpha}_{\beta\gamma} - \epsilon\left(\nabla_\gamma \eta^\alpha_\beta - \underset{i}{k^\alpha}\, \partial_\gamma \underset{i}{e_\beta} + \Lambda^{\zeta}_{\beta\gamma}\, \eta^\alpha_\zeta\right).$$

But

$$\Lambda^{\zeta}_{\beta\gamma}\, \eta^\alpha_\zeta = \underset{i}{e^\zeta}\, \partial_\gamma \underset{i}{e_\beta}\, \underset{j}{e_\zeta}\, \underset{j}{k^\alpha} = \partial_\gamma \underset{i}{e_\beta}\, \underset{i}{k^\alpha},$$

so that, finally,

$$\bar{\Lambda}^{\alpha}_{\beta\gamma} = \Lambda^{\alpha}_{\beta\gamma} - \epsilon(\nabla_\gamma \eta^\alpha_\beta + \Lambda^{\alpha}_{\beta\zeta}\, \eta^\zeta_\gamma), \tag{7}$$

which is equal to the connection parameters obtained by preservation of parallelism. Therefore, *if the points are not moved, the passage from a pseudo-flat space to its deform pseudo-flat space is a base deformation with parallelism preserved.*

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BOUNDARY CONDITIONS FOR THE WAVE EQUATION

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1. *Introduction.*

The general problem of characteristic values in quantum theory, and in a number of other branches of mathematical physics, is to find values of parameters occurring in the coefficients of a linear differential equation for which a solution exists satisfying given boundary conditions. In the case of one independent variable we require a solution which remains finite, or for which there hold homogeneous linear relations between the solution and its derivatives, at two given end points A, B.

When all the points of the closed interval (A, B) are ordinary points of the differential equation, the theory of the existence of eigen-values is well known*. In practice, however, the cases which most frequently arise are those for which A or B, or both, are singular points. The theory as usually given is not then immediately applicable.

In this paper are given some sufficient sets of conditions under which characteristic values are proved to exist when A, B are singular points. We confine ourselves to equations of the second order. The conditions given will be found to cover some important cases occurring in practice.

The investigation was originally suggested by the problem of the ionized hydrogen molecule. The most extensive theoretical discussion of the wave equation for this case has been given by A. H. Wilson†. He concluded that the wave equation possesses no solution satisfying the usual requirements at the boundaries. Subsequent investigators‡ have,

* E. L. Ince, *Ordinary differential equations* (1926), ch. X, XI.

† A. H. Wilson, *Proc. Royal Soc.*, 118A (1928), 617–635, 635–647.

‡ Morse and Stueckelberg, *Phys. Rev.* (2), 33 (1929), 932–947; E. Teller, *Zeit. für Phys.*, 61 (1930), 458–480; E. Hylleraas, *ibid.*, 71 (1931), 739–763, where a fuller list of references is given.

however, used various methods of successive approximation to obtain solutions, and have shown that their results agree well with observation. Nevertheless they have not established the existence of their solutions, and, with the exception of Teller, have, for the most part, taken no account of the work of Wilson.

The hydrogen molecular ion provides, however, the simplest and most fundamental application of wave mechanics to *molecular*, as distinct from *atomic*, problems, and it is unsatisfactory that there should be any doubt as to the applicability of the boundary conditions of wave mechanics in the usual form. The following theory offers a simple way of seeing that solutions do in fact exist satisfying these conditions. We indicate how they evade the difficulties raised by Wilson.

The theory is not designed to give a method of calculating the eigenvalues. This must be done by the methods of the authors cited; we seek merely to prove that they calculate a solution which does in fact exist.

2. *General theory.*

We consider the problem of the solution of a second order linear differential equation in one variable, for which boundary conditions are given at two successive singularities of the equation.

The equation may be written in the self-adjoint form

$$\frac{d}{dx}\left(K\frac{dy}{dx}\right)-Gy=0, \tag{2.01}$$

and the two singularities considered will be taken to be at $x=a$ and $x=b$, $a<b$. K and G are supposed to be real functions of the independent variable x, and of a parameter λ.

The boundary problem to be investigated is now: *To find the parameter λ such that there exists a solution $y=F(x,\lambda)$ of equation (2.01), single-valued, continuous, and differentiable in $a<x<b$, such that $F(a,\lambda)$ and $F(b,\lambda)$ are finite.*

2.1. *Characteristic solutions possessing at least one zero in $a<x<b$.*

K and G are supposed to be such that the following conditions are satisfied. These will be shown to be *sufficient* for the existence of solutions of the required type; they are not always necessary.

(i) They are both continuous functions of x and λ in the open intervals $a<x<b$, $\Lambda_1<\lambda<\Lambda_2$.

(ii) They are both monotonic decreasing functions of λ, and $-G^*/K^* \to \infty$ as $\lambda \to \Lambda_2-$, K^*, G^* being the upper bounds of K, G in any closed interval (α, β), $a < \alpha \leqslant x \leqslant \beta < b$, for a particular value of λ.

(iii) $K > 0$ for all λ and x in the open intervals considered.

We assume that equation (2.01) possesses at least one regular solution, $y = F_a(x, \lambda)$, in the neighbourhood of $x = a$, which is such that $F_a(a, \lambda)$ exists and is finite for all λ. Then the indicial equation at $x = a$ has at least one non-negative root. In a similar manner the solution $y = F_b(x, \lambda)$ is assumed to exist in the neighbourhood of the singularity $x = b$.

We suppose, first, that the indicial equations at $x = a$ and $x = b$ have the root zero and no other non-negative root. In the neighbourhood of $x = a$ we may take, as the solution of equation (2.01), $y = AF_a(x, \lambda)$, $F_a(a, \lambda) = 1$; $A =$ constant, and there is no other solution finite at $x = a$. Similarly, at $x = b$, we may take $y = BF_b(x, \lambda)$, $F_b(b, \lambda) = 1$, $B =$ constant. From the above assumptions concerning the nature of the functions K and G, it follows that the ordinary theory* of differential equations of the type (2.01) may be applied throughout any closed interval (α, β), $a < \alpha < \beta < b$. There exist values of λ for which $F_a(x, \lambda)$ has zeros arbitrarily near to β in $\alpha < x < \beta$. For consider any solution $y = f_\alpha(x)$ of (2.01), distinct from F_a, satisfying given boundary conditions which do not require that it shall vanish identically. From the ordinary theory (Ince, *loc. cit.*, 231), by increasing λ sufficiently, f_α can be made to have any number of zeros in $\alpha < x < \beta$, and, since F_a and f_α are distinct solutions of equation (2.01), F_a must have a zero between every two consecutive zeros of f_α. For some value λ' of λ, f_α has a zero at $x = \beta$. As λ is continuously increased, this zero moves towards $x = \alpha$, and for some value λ'' of λ, $\lambda'' > \lambda'$, another zero appears at $x = \beta$. Therefore, for some value λ^* of λ, $\lambda' < \lambda^* < \lambda''$, F_a has a zero at $x = \beta$. As λ increases from λ^*, this zero moves continuously towards the point $x = a$. (It may pass $x = a$, but if so it must then remain in $a < x < \alpha$.) Similarly, F_b has a zero at α for $\lambda = \lambda^{**}$. Without loss of generality we may suppose that $\lambda^* \leqslant \lambda^{**}$. As λ increases from λ^*, a zero of F_a appears at β and moves towards a. When λ passes λ^{**}, a zero of F_b appears at α and moves towards b.

* E. L. Ince, *Ordinary differential equations* (1926), ch. X, to be referred to as Ince, *loc. cit.*

There are two cases to be considered.

(i) For λ sufficiently near to and greater than λ^{**}, F_a has a zero at P to the right of a zero of F_b at Q in $\alpha < x < \beta$.

(ii) The zero of F_a has moved into $a < x < \alpha$, and has been passed for some value λ_n of λ, $\lambda^* < \lambda_n < \lambda^{**}$, in this interval by the zero of F_b, for a zero cannot pass the point $x = a$.

In case (i), by increasing λ sufficiently, P can be made to move continuously to α and pass it, since the sequence of zeros has no limit point at α. Similarly, Q can be made to move continuously to β and pass it. Therefore, for some value λ_i of λ, P and Q coincide. F_a and F_b have then a common zero for $\lambda = \lambda_i$. F_b is then a multiple of F_a. For, by multiplying F_b by some constant, it can be given a gradient equal to that of F_a at P. Since they both satisfy the same linear second order equation, their second, and all higher, derivatives are then equal at P, and therefore they must be identical functions for this value of λ. *Hence $F_a(x, \lambda_i)$ is a solution of equation* (2.01) *satisfying the required boundary conditions.* (Fig. 1.)

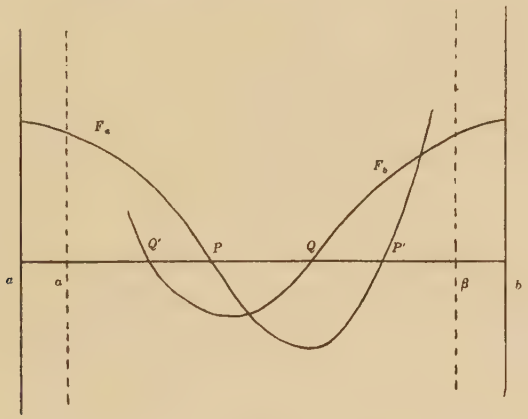

Fig. 1.

As λ is still further increased, fresh zeros P', Q' of F_a, F_b appear at β, α and move towards a, b respectively. P' will cross over Q and Q', and Q' will cross over P and P'. This leads to a sequence of values of λ, $\lambda_i < \lambda_j < \lambda_k < \ldots$, such that to each there corresponds a solution of the boundary value problem having one more zero in $a < x < b$ than that corresponding to the previous value.

In case (ii) it follows by a similar argument that $F_a(x, \lambda_n)$ is a solution of the required type.

2.2. *Characteristic solutions possessing no zero in* $a \leqslant x \leqslant b$.

We have established that the conditions assumed in 2.1 are sufficient to ensure the existence of solutions of equation (2.01), having at least one zero in $a < x < b$, which satisfy the required boundary conditions. We must now examine the possibility of the existence of solutions of the required type having no zeros in $a < x < b$.

Consider the solution $y = f(x, \lambda)$ of the differential equation

$$\frac{d}{dx}\left(k^* \frac{dy}{dx}\right) - g^* y = 0, \tag{2.21}$$

where k^* and g^* are the lower bounds of K and G in the interval $\alpha \leqslant x \leqslant \beta$, for a fixed value of λ. We can adjust f so that

$$\left(\frac{K}{F_a}\frac{\partial F_a}{\partial x}\right)_{x=\alpha} \geqslant \left(\frac{k^*}{f}\frac{\partial f}{\partial x}\right)_{x=\alpha}.$$

Then, since the ordinary theory is applicable throughout the interval $\alpha \leqslant x \leqslant \beta$, we have, from the "first comparison theorem" (Ince, *loc. cit.*, 228) of this theory, that $f(x, \lambda)$ has at least as many zeros in the interval $\alpha < x \leqslant \beta$ as $F_a(x, \lambda)$ has in the same interval.

We shall now impose the additional requirement that k^* and g^* are monotonic decreasing functions of λ such that $-g^*/k^* \to -\infty$ as $\lambda \to \Lambda_1+$. Then, for values of λ sufficiently near Λ_1, $f(x, \lambda)$ has no zeros in this interval (Ince, *loc. cit.*, 233). Therefore it follows that $F_a(x, \lambda)$ has no zeros in $\alpha < x \leqslant \beta$.

Thus, by introducing an additional restriction upon the nature of K and G, we have established the fact that there exists a value $\lambda^\dagger$ of λ such that, for all $\lambda < \lambda^\dagger$, $F_a(x, \lambda)$ has no zeros in the interval (α, β). We can further take α so that, for $\lambda < \lambda^\dagger$, $F_a(x, \lambda)$ has no zero in $a \leqslant x \leqslant \alpha$. For we can take α arbitrarily near to a, and if F_a always has a zero in $a \leqslant x \leqslant \alpha$, and $F_a(a, \lambda) = 1$, then F_a has a discontinuity at a, which is contrary to hypothesis.

Thus $F_a(x, \lambda)$ has no zero in $a \leqslant x \leqslant \beta$ for $\lambda < \lambda^\dagger$. Similarly, there exists a value $\lambda^{\dagger\dagger}$ of λ such that, for all $\lambda < \lambda^{\dagger\dagger}$, $F_b(x, \lambda)$ has no zero in $\alpha \leqslant x \leqslant b$. Suppose that $\lambda^\dagger \leqslant \lambda^{\dagger\dagger}$; then neither F_a nor F_b has a zero in $\alpha \leqslant x \leqslant \beta$ for $\Lambda_1 < \lambda < \lambda^\dagger$. From the "second comparison theorem" (Ince, *loc. cit.*, 229) of the usual theory, it follows that the ratio $K(\gamma) F_a'(\gamma)/F_a(\gamma)$ is a monotonic decreasing function of λ for any interior point γ of the interval (α, β), $F_a'(\gamma)$ being the first derivative of F_a with respect to x, evaluated at the point $x = \gamma$. But this ratio tends to infinity

as $\lambda \to \Lambda_1 +$. For, by the "second comparison theorem",

$$\frac{K(\gamma) F_a'(\gamma)}{F_a(\gamma)} \geqslant k^* \frac{f'(\gamma)}{f(\gamma)},$$

and, from the condition $-g^*/k^* \to -\infty$ as $\lambda \to \Lambda_1 +$, applied to the equation (2.21), it follows (Ince, *loc. cit.*, 233) that the right-hand side of this inequality tends to $+\infty$ as $\lambda \to \Lambda_1 +$.

Therefore, as λ increases from Λ_1 to $\lambda_\gamma^\dagger$ say, the value of λ for which $F_a(\gamma)$ first vanishes, the ratio $K(\gamma) F_a'(\gamma)/F_a(\gamma)$ steadily decreases from $+\infty$ to $-\infty$. Similarly, the ratio $K(\gamma) F_b'(\gamma)/F_b(\gamma)$ increases steadily from $-\infty$ to $+\infty$ as λ increases from Λ_1 to $\lambda_\gamma^{\dagger\dagger}$, say, the smallest value of λ for which $F_b(\gamma)$ vanishes. Suppose that $\lambda_\gamma^\dagger < \lambda_\gamma^{\dagger\dagger}$. Then, in the range $\Lambda_1 < \lambda < \lambda_\gamma^\dagger$, the ratio $K(\gamma) F_b'(\gamma)/F_b(\gamma)$ increases monotonically from $-\infty$ to some finite value as λ increases from Λ_1 to $\lambda_\gamma^\dagger$. There is, therefore, a single value λ_0 of λ in this range for which the two expressions are equal. (Fig. 2.)

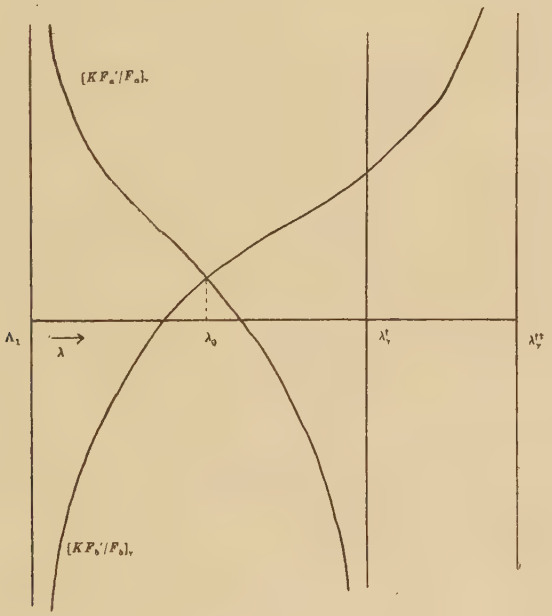

Fig. 2.

For $\lambda = \lambda_0$,

$$\frac{K(\gamma) F_a'(\gamma)}{F_a(\gamma)} = \frac{K(\gamma) F_b'(\gamma)}{F_b(\gamma)},$$

that is

$$F_a'(\gamma) F_b(\gamma) - F_a(\gamma) F_b'(\gamma) = 0.$$

But
$$\frac{d}{dx}(KF_a'F_b) = GF_aF_b - KF_a'F_b',$$

$$\frac{d}{dx}(KF_aF_b') = GF_aF_b - KF_a'F_b';$$

from which it follows that

$$K(F_a'F_b - F_aF_b') = \text{constant} \quad (a < x < b).$$

Therefore $F_a'F_b - F_aF_b' = 0$, for $\lambda = \lambda_0$, $a < x < b$. Thus, for this value of λ, F_b is a multiple of F_a, and since F_a has no zero in $a \leqslant x \leqslant \gamma$, and F_b has no zero in $\gamma \leqslant x \leqslant b$, then $F_a(x, \lambda_0)$ has no zero in $a \leqslant x \leqslant b$.

$F_a(x, \lambda_0)$ is therefore the unique solution of equation (2.01), *satisfying the required boundary conditions, which has no zero in $a \leqslant x \leqslant b$.*

2.3. There remain two cases to be considered: (1) The indicial equation at one or both of the end points has only one non-negative root, this root differing from zero. (2) The indicial equation has two distinct non-negative roots.

For case (1), the solution F must be zero at the end point concerned and only a slight modification of the preceding theory is required. In this case, however, a more restricted boundary condition is permissible. For let the non-negative exponent at the point $x = a$ differ from zero; then F_a is of the form $A(x-a)^\rho\{1+A_1(x-a)+\ldots\}$ and A is not determined from the condition $F_a(a) = 0$. To fix A, a restriction may be placed upon the first non-vanishing derivative of F_a at $x = a$, or, as is more usual, a normalizing condition may be imposed. Similar considerations apply for case (2) when there are two distinct non-negative exponents at one of the boundary points, $x = a$ say. The solution at a is taken to be $F_a = I_a + AJ_a$, I_a and J_a being the two distinct solutions. At least one of I_a, J_a will vanish at $x = a$. If $J_a(a) = 0$, $I_a(a) \neq 0$, then the boundary condition that $F_a(a)$ is equal to a constant will determine $I_a(a)$, and the constant A remains to be determined as in case (1). If $I(a)$ and $J(a)$ both vanish, F_a will be of the form

$$A(x-a)^{\rho_1}\{1+A_1(x-a)+\ldots\}+B(x-a)^{\rho_2}\{1+B_1(x-a)+\ldots\},$$

and for the determination of A and B it is permissible to introduce a normalizing condition and a restriction on the derivative of $F_a(x)$ at $x = a$.

2.4. There is one further possibility to be considered that occurs if we do not require that the solution shall exist at an end point, but merely

that it shall tend to a finite limit at this point. This case may arise when the end point is an irregular singular point, for which the indicial equation has either a single root or no root at all. If the single exponent is non-negative, the case has been considered above. If the indicial equation at b, say, has no non-negative root, but if there is a *normal* solution $H_b(x)$ at b, possessing a limit l there, for example one having the form $e^{1/(x-b)}/(x-b)$ as $x \to b-$, and if the remaining conditions are those of § 2 . 1, then *there exist solutions of equations* (2 . 01) *which are finite throughout* $a \leqslant x < b$, *and which have the limit* l *at* $x = b$. For we can follow through all the arguments of § 2 . 1, using $H_b(x)$ in place of $F_b(x)$, and so obtain solutions satisfying the modified boundary conditions.

Irregular singular points occur most frequently as the point at infinity. Normal solutions will then be of the form $e^{\phi(x)}u(x)$, where $\phi(x)$ is a polynomial in x, and the differential equation for u is such that there is at least one solution regular at infinity. If the leading coefficient in $\phi(x)$ is negative, then the normal solution has the limit zero at infinity.

For the case under consideration, the dependent variable could be transformed to $u(x)$ in order to obtain a simpler behaviour at infinity. The disadvantage of this is that the differential equation would then cease to be self-adjoint, so that the theory in its preceding form would not be immediately applicable.

3. *Wilson's generalized spheroidal wave equation.*

The preceding theory may be applied to indicate the possibility of the existence of finite solutions of the generalized wave equation occurring in Wilson's paper. The equation arises from the consideration of the motion of an electron in the field of two positively charged nuclei which, to a sufficient approximation, may be considered as fixed. The Schrödinger wave equation describing this system is a partial differential equation of the second order with the three electron coordinates as independent variables. By using prolate spheroidal coordinates ξ, η, ϕ, $1 \leqslant \xi \leqslant \infty$, $-1 \leqslant \eta \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi$, this partial differential equation may be separated into three linear ordinary differential equations of the second order in each of the three variables respectively. If the solutions of these three equations are $X(\xi)$, $Y(\eta)$, $\Phi(\phi)$, then the solution of the Schrödinger equation is $XY\Phi$. But, from the theory of wave mechanics, the wave function $XY\Phi$ must be single-valued, continuous, and differentiable throughout the range of the variable, being finite or tending to a limit at the end points. The parameters occurring in the wave equation must be adjusted to give solutions of this type.

The equations for $X(\xi)$ and $Y(\eta)$ are respectively

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dX}{d\xi}\right]+\left[\lambda^2\xi^2-\kappa\xi-\frac{m^2}{1-\xi^2}+\mu\right]X=0, \qquad (3.01)$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dY}{d\eta}\right]+\left[\lambda^2\eta^2-\frac{m^2}{1-\eta^2}+\mu\right]Y=0, \qquad (3.02)$$

and it is required that X shall be single-valued, finite, and continuous in the range $1\leqslant\xi\leqslant\infty$, and that $Y(\eta)$ shall be single-valued, finite, and continuous in the range $-1\leqslant\eta\leqslant 1$.

In equations (3.01) and (3.02) m is known from the equation for $\Phi(\phi)$. If solutions of the boundary value problem for (3.01), (3.02) exist, then they give two relations between the remaining parameters λ^2, μ, κ, and therefore any two of them may be expressed in terms of the third. But κ is known for a given separation of the nuclei, and we may therefore regard λ^2 and μ as the parameters to be determined from the boundary conditions. It is supposed that all the parameters λ^2, μ, κ occurring are real.

Wilson gives an extensive discussion of the nature of the solutions of equations (3.01) and (3.02), and concludes that solutions of equation (3.01), finite at the two singularities $\xi=1$ and ∞, can be obtained only by imposing an extra condition on the parameters so that the series, in terms of which the solution is obtained, shall terminate. But the preceding analysis indicates that solutions of the required form exist independently of this extra restriction. For equation (3.01) (the "generalized spheroidal wave equation") is self-adjoint and has singularities at the points $\xi=-1, +1, \infty$. The preceding general theory may therefore be applied to establish the existence of solutions, single-valued, continuous, and differentiable throughout the interval defined by the successive singularities $\xi=+1$, $\xi=\infty$.

We have, in terms of the notation of § 2,

$$K(\xi)=\xi^2-1,$$

$$G(\xi)=\mu+\frac{m^2}{\xi^2-1}-\kappa\xi+\lambda^2\xi^2.$$

$K(\xi)$ is continuous and greater than zero in the interval $1<\xi<\infty$, independently of the parameters. From Wilson's work it follows that there is a solution of (3.01) finite at $\xi=1$. We have, in fact, exponents $\pm\frac{1}{2}m$ at $\xi=1$. Further, there is a solution, with a limit zero as $\xi\to\infty$, of the form

$$e^{-\lambda\xi}\,\xi^{-m-1+2\kappa/\lambda}(1+a_1/\xi+\ldots).$$

In order to apply the theory of § 2, it remains therefore only to investigate the behaviour of $G(\xi)$ as a function of the parameters.

Wilson has shown that, for any given value of μ, values of λ exist for which equation (3.02) has solutions satisfying the required conditions at $\eta = \pm 1$. We shall therefore suppose λ to be determined as a function of μ in this way.

Equation (3.02) may be written

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+gy=0, \tag{3.03}$$

where $$k=1-x^2, \quad g=\lambda^2x^2+\mu-m^2/(1-x^2).$$

By assigning definite values to the parameter μ, we have a system of distinct differential equations, each one of which will give rise to a set of characteristic values of λ, for the given boundary conditions at $x=\pm 1$. Consider two members of this system of equations,

$$\frac{d}{dx}\left(k\frac{dy}{dx}\right)+gy=0,$$

$$\frac{d}{dx}\left(k\frac{dy_1}{dx}\right)+g_1y_1=0,$$

corresponding to the values μ, μ_1 of μ respectively. Then from these equations it follows that

$$\Big[k(y'y_1-yy_1')\Big]_{x_1}^{x_2}=\int_{x_1}^{x_2}(g_1-g)yy_1\,dx. \tag{3.04}$$

Take x_1, x_2 as the points $x=\pm 1$, At both of these points k is zero, and hence the left-hand side of this equation is zero. Now we may take μ_1 to arise from a small variation $d\mu$ of μ, so that $\mu_1=\mu+d\mu$. Then we have from (3.04), if for the sake of clarity we write terms to the second order in $d\mu$,

$$\int_{-1}^{1}\left(\frac{\partial g}{\partial\mu}\,d\mu+\tfrac{1}{2}\frac{\partial^2 g}{\partial\mu^2}\,d\mu^2\right)\left(y+\frac{\partial y}{\partial\mu}\,d\mu\right)y\,dx=0.$$

Hence $$\int_{-1}^{1}\frac{\partial g}{\partial\mu}\,y^2\,dx=0.$$

Therefore, since y^2 is one-signed throughout the range of integration,

$$\frac{\partial g}{\partial\mu}\left(=x^2\frac{\partial\lambda^2}{\partial\mu}+1\right)$$

must change sign at least once in this range. Therefore there must be a real value x_0 of x, $-1 < x_0 < 1$, such that

$$x_0^2 \frac{\partial \lambda^2}{\partial \mu} + 1 = 0.$$

Therefore

$$\frac{\partial \lambda^2}{\partial \mu} < -1,$$

and so λ^2 is a decreasing function of μ, and varies more rapidly than μ.

It follows that

$$\frac{\partial g}{\partial \mu} = x^2 \frac{\partial \lambda^2}{\partial \mu} + 1$$

$$< 1 - x^2.$$

Therefore, for $x^2 > 1$, $\partial g/\partial \mu < \epsilon < 0$, for all μ. Hence g is a monotonic decreasing function of μ, and

$$g \to -\infty \quad \text{as} \quad \mu \to \infty, \quad g \to \infty \quad \text{as} \quad \mu \to -\infty.$$

But $G(\xi) = g(\xi) - \kappa\xi$, where κ is a constant independent of μ. Therefore $G(\xi)$ is also a monotonic decreasing function of μ for all $\xi > 1$, such that

$$G(\xi) \to \pm\infty \quad \text{as} \quad \mu \to \mp\infty.$$

Thus, since $K(\xi)$ is independent of μ,

$$-G^*/K^* \to \infty \quad \text{as} \quad \mu \to \infty,$$

$$-g^*/k^* \to -\infty \quad \text{as} \quad \mu \to -\infty,$$

if we adopt the notation of the general theory, the range (α, β) being any closed interval in the range of ξ, *i.e.* $1 < \alpha < \beta < \infty$. For, since the above properties of G are valid for any ξ in $\alpha \leqslant \xi \leqslant \beta$, it follows that they are valid for those particular values of ξ which make G a maximum and a minimum in (α, β).

We thus see that the generalized spheroidal wave equation satisfies the conditions of the general theory in the form of § 2.4. *Hence characteristic solutions exist satisfying the end conditions.*

It must be noted that the general theory treats the case of finite intervals, and we have considered an infinite interval. This difficulty may be simply removed by changing the independent variable.

3.1. Wilson's investigation failed to reveal the solution whose existence we have just demonstrated. He consequently introduced a theory designed to establish the non-existence of such solutions, except

under the extra condition mentioned previously. We proceed to indicate the reason for this disagreement.

The fundamental solution of equation 3.A in Wilson's paper is of the form $\sum_{\nu=0}^{\infty} a_\nu(\xi-1)^\nu$. This gives the solution of equation (3.01) of this paper to be $X_{11}(\xi)$, if μ satisfies a certain condition expressed by equating to zero an infinite determinant, where

$$X_{11}(\xi) = Ae^{\lambda\xi}(a_0+a_1\xi+\ldots)+Be^{-\lambda\xi}(b_0+b_1\xi+\ldots).$$

A and B are constants, neither of which can be zero. From this solution a second solution $X_{12}(\xi)$ is derived, and linear combinations of these for the same value of μ give solutions convergent for $|\xi|>1$, having, as asymptotic forms at infinity,

$$e^{-\lambda\xi}\,\xi^{-m-1+\kappa/(2\lambda)}\left(c_0+\frac{c_1}{\xi}+\ldots\right),$$

and

$$e^{\lambda\xi}\,\xi^{-m-1-\kappa/(2\lambda)}\left(d_0+\frac{d_1}{\xi}+\ldots\right).$$

For values of μ other than those giving the solution $X_{11}(\xi)$ there is a second solution, ${}_1X_1(\xi)$, in ascending powers of $\xi-1$, the coefficients in the expansion of X_{11} and ${}_1X_1$ being the same for $|\xi-1|<2$, except that each solution has its appropriate μ. For large $|\xi|$, ${}_1X_1$ must be a linear combination of the two asymptotic forms. Thus

$${}_1X_1 \sim \gamma e^{-\lambda\xi}\,\xi^{-m-1+\kappa/(2\lambda)}\left(c_0+\frac{c_1}{\xi}+\ldots\right)+\delta e^{\lambda\xi}\,\xi^{-m-1-\kappa/(2\lambda)}\left(d_0+\frac{d_1}{\xi}+\ldots\right).$$

Wilson now argues that, since ${}_1X_1$ passes into X_{11} by changing μ, it must have the same form at infinity, and therefore δ cannot be zero. This does show that $\delta(\mu)\not\equiv 0$, but, nevertheless, the equation $\delta(\mu)=0$ may have real roots. If these exist, they provide the required eigen-values of μ. And, since we have seen from our general theory that eigen-values do in fact exist, this must be the point at which the eigen solutions elude Wilson's investigation.

A criticism of Wilson's conclusion has also been given by Teller*.

He gives in outline a proof, based on a different method, that, for given μ, λ, the parameter κ can be so chosen that Wilson's equation possesses a characteristic solution without zeros (ground state). However, besides

* E. Teller, *Zeit. für Physik*, 61 (1930), 458–480.

its foundation in a fairly general existence theorem, the present treatment of this particular problem seems to have the advantage of penetrating more into its physical nature in showing that, when κ is fixed, *and when* λ, μ *are related in the manner required for the existence of characteristic solutions of the second of Wilson's equations*, (3.02), μ can still be so chosen that the first, (3.01), will also possess characteristic solutions.

4.1. *The solution of Wilson's equation in series.*

Wilson expressed the solution of equation (3.01) in terms of infinite series of ascending and of descending powers of the independent variable, and in both cases it was found that the radius of convergence did not cover the whole field required. But we know now that a solution does actually exist. It must therefore be possible to find a function $\alpha(\xi)$ of ξ, in terms of which the solution can be expanded. (There may quite well be a solution $F(\alpha)$ which is defined for $1 \leqslant \xi < \infty$ and tends to a limit as $\xi \to \infty$, whereas if it is expressed directly in terms of ξ, or some function of ξ other than α, there may be convergence over a finite range only, since we confine ourselves to the real axis.)

Following Wilson, we expect the solution to be of the form $e^{-\lambda\xi}$ at infinity, and we find it convenient to introduce the new dependent and independent variables

$$y = (\xi^2-1)^{-\frac{1}{2}m} e^{\lambda\xi} X,$$

$$\tau = \frac{\xi-1}{\xi+1}. \tag{4.11}$$

We look for a solution finite over the range $0 \leqslant \tau < 1$. The transformed equation is

$$\tau(1-\tau)^3 \frac{d^2y}{d\tau^2} + [1-2(1+2\lambda)\tau+\tau^2+m(1-\tau^2)](1-\tau)\frac{dy}{d\tau}$$
$$+[\{\kappa+\mu+\lambda^2-2\lambda-m(m+1+2\lambda)\}\tau-2\lambda+\kappa-\mu$$
$$-\lambda^2+m(m+1-2\lambda)]y = 0. \tag{4.12}$$

We consider for simplicity the case $m = 0$. The solution, regular at $\tau = 0$, will be of the form $y = \sum_0^\infty a_\nu \tau^\nu$. On substituting this into the differential equation, we obtain a four-termed recurrence relation in the form

$$a_{\nu+3}(\nu+3)^2 + a_{\nu+2}[-(\nu+2)(3\nu+6+4\lambda)+\kappa-\mu-2\lambda-\lambda^2]$$
$$+a_{\nu+1}[(\nu+1)(3\nu+3+4\lambda)+\kappa+\mu-2\lambda+\lambda^2]-a_\nu\nu^2 = 0. \tag{4.13}$$

This difference equation for a_ν has three independent solutions. On writing

$$a_\nu = \rho^\nu \nu^\sigma \left(g_0 + \frac{g_1}{\nu} + \frac{g_2}{\nu^2} + \ldots\right),$$

we find that the characteristic equation for ρ is

$$(\rho - 1)^3 = 0,$$

and for one of the solutions $(a_\nu^{(1)})$ we find that

$$\frac{a_\nu^{(1)}}{a_{\nu-1}^{(1)}} \sim 1 - \frac{2 - \kappa/(2\lambda)}{\nu}, \tag{4.14}$$

for large ν. The other solutions can be found by the method of Frobenius. Now, if any particular value of μ is taken and the corresponding a's are calculated from the difference equation, then, in general, $a_\nu^{(1)}$ will not be obtained, but one of the other solutions, and X will behave like $e^{\lambda\xi}$ at infinity. Only if μ is properly chosen, that is if it is an eigen-value, will the solution $a_\nu^{(1)}$ be obtained. In this case it follows, from (4.14), that the radius of convergence of the series for y is unity, and that, near $\tau = 1$, $y \sim (1-\tau)^{1-\kappa/(2\lambda)}$. Hence the series is convergent for all $\xi \geqslant 1$, and, as $\xi \to \infty$, X behaves like $\xi^{-1+\kappa/(2\lambda)} e^{-\lambda\xi}$, and therefore tends to the limit zero. Thus we have a solution formally satisfying the equation (3.01) and the boundary conditions at the end points. This must be the solution whose existence we have demonstrated by the general theory*.

4.2. The equation (3.01) has also been solved by Hylleraas†. He puts $X = (\xi^2 - 1)^{\frac{1}{2}m} y$, $x = 2\lambda(\xi - 1)$, and solves for y in a series of the functions $y_{n,m}$, defined by the generating function

$$\frac{1}{(1-t)^{m+1}} \exp\left(-\frac{x}{2}\,\frac{1+t}{1-t}\right) = \sum_{n=m+1}^{\infty} y_{n,m}\, t^{n-m-1}.$$

He obtains a formal solution, and uses it in the numerical computation of the eigen-values. But he does not examine the *existence* of the solution. If, however, we succeed in showing that his series is convergent in the necessary range of x, then we may again conclude that it provides the solution which we have shown to exist.

Hylleraas writes $y = \sum_{n=m+1}^{\infty} c_n y_{n-1,m}$, and obtains a three-termed recurrence relation between the c_n's. From this, for large n, we find the

* The proof of this result which we originally gave contained an omission which was pointed out by the Referee; we are indebted to him for the proof which is given here.

† E. Hylleraas, *Zeit. für Physik*, 71 (1931), 739–763.

asymptotic relation

$$\frac{c_{n+1}}{c_n} \sim 1 \pm \frac{2\lambda^{\frac{1}{2}}}{\sqrt{n}}. \tag{4.21}$$

Now for $m = 0$, $y_n = e^{-\frac{1}{2}x} L_n(x)/n!$, where $L_n(x)$ is the Laguerre polynomial defined by the generating function

$$\frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} L_n(x) t^n.$$

But* $$e^{-\frac{1}{2}x} \left| \frac{1}{n!} L_n(x) \right| < 1,$$

for all $x > 0$, and therefore $|y_n(x)| < 1$, for $x > 0$. Further $y_n(0) = 1$. Hence, if Hylleraas has so chosen his eigen-values as to correspond to the upper sign in (4.21), *i.e.*

$$\frac{c_{n+1}}{c_n} \sim 1 - \frac{2\lambda^{\frac{1}{2}}}{\sqrt{n}},$$

then the series for y converges for all $x \geqslant 0$.

In these circumstances it therefore appears that this series furnishes the required solution of the wave equation.

[*Added in proof.* Since the communication of this paper, a treatment along somewhat related lines has been published by E. C. Kemble, *Proc. Nat. Acad. Sci.*, 19 (1933), 710–714. However, he studies equations in which the parameter λ occurs in a less general manner. The latter is true also of work by H. Weyl, *Göttinger Nachrichten* (Math. Phys. Kl.) (1910), 442–467, and M. C. Gray, *American J. of Math.*, 50 (1928), 431–458, who use methods quite different from ours. The paper by Weyl gives references to earlier work. An abstract of our results appeared in *Proc. Royal Soc.* (A), 141 (1933), 216–217.]

* G. Szegö, *Math. Zeitschrift*, 1 (1918), 341–356.

ANALYSIS OF THE CONDITIONS OF GENERALIZED ALMOST PERIODICITY

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In a recent paper with a similar title, A. S. Besicovitch* has greatly simplified the defining conditions of the class of functions discussed by Harald Bohr and himself† under the name of B-a.p. functions. In their original definition, $f(x)$ is called B-a.p., if to every $\epsilon > 0$ corresponds a satisfactorily uniform sequence $\{\tau_i\}$ for which

$$\bar{M}_x |f(x+\tau_i)-f(x)| < \epsilon \quad \text{for every } i, \tag{1.1}$$

$$\bar{M}_x \bar{M}_i \frac{1}{c}\int_x^{x+c} |f(t+\tau_i)-f(t)|\,dt < \epsilon \quad \text{for all } c > 0. \tag{1.2}$$

In the new definition the only condition imposed on the sequence is that

$$M_x \bar{M}_i \int_x^{x+1} |f(t+\tau_i)-f(t)|\,dt < \epsilon. \tag{2}$$

Thus the condition (1.2) on the sequence as a whole is greatly simplified, while the condition (1.1) requiring the existence of translation *numbers* is dropped entirely.

The primary object of this note is to simplify the defining conditions of $\bar{B}$-a.p. functions. In the definition given by Bohr and Besicovitch (*loc. cit.*, 267), it is required that

$$M_{\bar{x}} |f(x)| < \infty.$$

I show that this is a consequence of the existence of $\bar{B}$-translation sequences. This result is an immediate corollary from an inequality

* Besicovitch, "Analysis of conditions of almost periodicity", *Acta Math.*, 58 (1932), 217.

† Besicovitch and Bohr, "Almost periodicity and general trigonometric series", *Acta Math.*, 57 (1931), 203.

(§ 1, Lemma 1) by repeated applications of which one arrives easily at the result that every B-a.p. function is also B-a.p.

In § 2 I indicate an analogous inequality suitable for the discussion of B-a.p. functions. The immediate corollary is in this case rather trivial, but the later applications are more important. §§ 2–4 contain a complete proof that a B-a.p. function, as now defined, can be approximated by trigonometric polynomials; here none of the results are new, the object of the work being to show their relation to the inequality referred to.

The terms satisfactorily uniform, relatively dense, $\underset{x}{\bar{M}}$, $\underset{i}{\bar{M}}$, U-a.p. are explained by Bohr and Besicovitch (*loc. cit.*) and elsewhere in the literature.

1. Let $f(x)$ be a function defined in $x > 0$ and summable in any finite interval. Let $T = \{\tau_i\}$ be a right-handed satisfactorily uniform sequence for which the function

$$T(x) = \underset{i}{\bar{M}} |f(x+\tau_i) - f(x)| \tag{1}$$

is also summable; we suppose τ_i to be positive. Write $\mu(p) = \mu(p;\ T)$ for the minimum number and $\nu(p) = \nu(p;\ T)$ for the maximum number of the numbers τ_i to be found in an interval of length p in the range $\tau > 0$. Let l be an inclusion interval for the sequence so that

$$\mu(l) \geqslant 1. \tag{2}$$

Given $L \geqslant l$ and $\epsilon > 0$, we can find $\eta > 0$ and so small that

$$\int_A |f(x)|\, dx < \epsilon \tag{3}$$

for every set $H \subset (0,\ L)$ and of measure less than η. And, given ϵ and η, we can find an integer n_0 so great that the set E of points x in the interval $(0,\ L)$ for which

$$\frac{1}{n} \sum_{i=1}^{n} |f(x+\tau_i) - f(x)| \leqslant T(x) + \epsilon \quad \text{for all } n \geqslant n_0 \tag{4}$$

is of measure greater than $L - \eta$. Let $E(x)$ be the characteristic function of this set, and let $\Lambda(x)$, $\lambda(x)$ be the characteristic functions of the intervals $(0,\ L)$ and $(0,\ l)$. Write

$$e(x) = E(x)\,\lambda(x),$$

so that $e(x)$ is the characteristic function of the set of points x in $(0,\ l)$ which satisfy (4).

Consider the double sum

$$S \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \int \{|f(x)-f(x-\tau_i)|+|f(x)-f(x-\tau_j)|\} e(x-\tau_i) E(x-\tau_j)\, dx, \quad (5)$$

the integrals being taken from $-\infty$ to $+\infty$; for formal convenience we may put $f(x) = 0$ for $x < 0$. We have, on the one hand,

$$S \leqslant \Sigma\Sigma \int |f(x)-f(x-\tau_i)|\, e(x-\tau_i) \Lambda(x-\tau_j)\, dx$$

$$+ \Sigma\Sigma \int |f(x)-f(x-\tau_j)|\, \lambda(x-\tau_i)\, E(x-\tau_j)\, dx$$

$$\leqslant \nu(L) \int \sum_{i=1}^{n} |f(x+\tau_i)-f(x)|\, e(x)\, dx + \nu(l) \int \sum_{j=1}^{n} |f(x+\tau_j)-f(x)|\, E(x)\, dx$$

$$\leqslant n\nu(L)\left[\epsilon l + \int_0^l T(x)\, dx\right] + n\nu(l)\left[\epsilon L + \int_0^L T(x)\, dx\right]$$

for $n \geqslant n_0$.

On the other hand,

$$S \geqslant \Sigma\Sigma \int \{|f(x-\tau_j)|-|f(x-\tau_i)|\} e(x-\tau_i)\, E(x-\tau_j)\, dx.$$

Now for any particular value of j there are at most $\nu(L+l)$ values of i for which the kernel $\lambda(x-\tau_i)\Lambda(x-\tau_j)$ does not vanish identically. And for any one such pair (i, j) we have, by (3),

$$\int |f(x-\tau_j)|\, e(x-\tau_i)\, E(x-\tau_j)\, dx = \int |f(x)|\, e(x+\tau_j-\tau_i)\, E(x)\, dx$$

$$\geqslant \int |f(x)|\, \lambda(x+\tau_j-\tau_i)\, E(x)\, dx - \epsilon.$$

Hence

$$\Sigma\Sigma \int |f(x-\tau_j)|\, e(x-\tau_i)\, E(x-\tau_j)\, dx$$

$$\geqslant \Sigma\Sigma \int |f(x)|\, \lambda(x+\tau_j-\tau_i)\, E(x)\, dx - n\epsilon\nu(l+L).$$

But, if $$L \leqslant \tau_i \leqslant \tau_n - l,$$

then $$\sum_{j=1}^{n} \lambda(x+\tau_j-\tau_i) \geqslant \mu(l) \quad (6)$$

for every point x in $(0, L)$ and hence

$$\Sigma\Sigma \int |f(x)|\, \lambda(x+\tau_j-\tau_i)\, E(x)\, dx \geqslant \mu(l)\{n-\nu(l)-\nu(L)\} \int |f(x)|\, E(x)\, dx.$$

We shall suppose, as we may, that

$$n_0 \geqslant \nu(l)+\nu(L),$$

and in this case we find, for all $n \geqslant n_0$,

$$\Sigma\Sigma \int |f(x-\tau_j)|\, e(x-\tau_i)\, E(x-\tau_j)\, dx$$
$$\geqslant \mu(l)\{n-\nu(l)-\nu(L)\}\left\{\int_0^L |f(x)|\, dx-\epsilon\right\}-n\epsilon\nu(l+L).$$

Again,

$$\Sigma\Sigma \int |f(x-\tau_i)|\, e(x-\tau_i)\, E(x-\tau_j)\, dx \leqslant \Sigma\Sigma \int |f(x-\tau_i)|\, \lambda(x-\tau_i)\Lambda(x-\tau_j)\, dx$$
$$\leqslant n\nu(l+L)\int_0^l |f(x)|\, dx.$$

Hence, for all $n \geqslant n_0$, we have

$$\mu(l)\{n-\nu(l)-\nu(L)\}\left\{\int_0^L |f(x)|\, dx-\epsilon\right\} \leqslant n\nu(l+L)\left\{\epsilon+\int_0^l |f(x)|\, dx\right\}+S,$$

$$S \leqslant n\nu(L)\left[\epsilon l+\int_0^l T(x)\, dx\right]+n\nu(l)\left[\epsilon L+\int_0^L T(x)\, dx\right].$$

Making first $n\to\infty$ and then $\epsilon\to 0$, we obtain

LEMMA 1.

$$\mu(l)\int_0^L |f(x)|\, dx \leqslant \nu(L+l)\int_0^l |f(x)|\, dx+\nu(L)\int_0^l T(x)\, dx+\nu(l)\int_0^L T(x)\, dx.$$

Whence also we have

LEMMA 2.

$$\bar{M}_x |f(x)| \leqslant \frac{\nu(l)}{\mu(l)}\bar{M}_x T(x)+\overline{\lim_{L\to\infty}}\,\frac{\nu(L)}{L}\,\frac{1}{\mu(l)}\int_0^l \{|f(x)|+T(x)\}\, dx.$$

COROLLARY. $\bar{M}_x |f(x)|$ *is finite if* $\bar{M}_x T(x)$ *is finite.*

The lemmas may be generalized; we have

$$\mu(l)\int_\Lambda |f(x)|\, dx \leqslant \nu(l+L)\int_\lambda |f(x)|\, dx+\nu(L)\int_\lambda T(x)\, dx+\nu(l)\int_\Lambda T(x)\, dx, \quad (7)$$

$$\bar{M}_x |f(x)| \leqslant \frac{\nu(l)}{\mu(l)}\bar{M}_x T(x)+\overline{\lim_{L\to\infty}}\,\frac{\nu(L)}{L}\,\frac{1}{\mu(l)}\int_\lambda \{|f(x)|+T(x)\}\, dx, \quad (8)$$

where Λ and λ are any two intervals of lengths L and l in the range $x>0$. The restriction $L \geqslant l$ made above is inessential; but, if l is not an inclusion interval for the sequence T, the inequalities lose significance. If $f(x)$ is defined for negative as well as positive values of x, then (7) is valid for any two finite intervals, the sequence T being still right-handed. The changes required in the above proof in order to cover the new cases are

all trivial; but the conditions for the validity of (6) become much less simple. Lemma 2 is thus valid when $\bar{M}_x$ is two-sided even when $\{\tau_i\}$ is one-sided.

Both (7) and (8), with appropriate changes of notation, can be proved for two-sided sequences by a similar argument; a less precise result for two-sided sequences can be deduced from that for one-sided sequences by means of the inequalities

$$T(x) \geqslant \tfrac{1}{2}T_+(x), \quad T(x) \geqslant \tfrac{1}{2}T_-(x),$$

where $T_+(x)$ and $T_-(x)$ are derived from the right-hand and left-hand halves of T in the way that $T(x)$ is derived from the whole sequence. To fix the ideas we shall continue to deal exclusively with our original case.

We now write

$$f_N(x) = f(x) \quad \text{if} \quad |f(x)| \leqslant N,$$

$$= \frac{Nf(x)}{|f(x)|} \quad \text{if} \quad |f(x)| > N,$$

and $$R_N(x) = f(x) - f_N(x).$$

Then $$|R_N(x+\tau_i) - R_N(x)| \leqslant |f(x+\tau_i) - f(x)|,$$

and hence $$\bar{M}_i |R_N(x+\tau_i) - R_N(x)| \leqslant T(x).$$

Applying Lemma 2 to the function $R_N(x)$ instead of $f(x)$, we obtain

$$\bar{M}_x |R_N(x)| \leqslant \frac{\nu(l)}{\mu(l)} \bar{M}_x T(x) + \overline{\lim_{L\to\infty}} \frac{\nu(L)}{L} \frac{1}{\mu(l)} \int_0^l \{|R_N(x)| + T(x)\}\, dx,$$

and hence

$$\lim_{N\to\infty} \bar{M}_x |R_N(x)| \leqslant \frac{\nu(l)}{\mu(l)} \bar{M}_x T(x) + \overline{\lim_{L\to\infty}} \frac{\nu(L)}{L} \frac{1}{\mu(l)} \int_0^l T(x)\, dx.$$

Making $l \to \infty$, we now get

$$\lim_{N\to\infty} \bar{M}_x |R_N(x)| \leqslant 4\bar{M}_x T(x). \tag{9}$$

If to every $\epsilon > 0$ there corresponds a satisfactorily uniform sequence T for which

$$\bar{M}_x T(x) < \epsilon, \tag{10}$$

then we say that $f(x)$ is $\bar{B}$-a.p., and any such sequence we call a $\bar{B}$-translation sequence of $f(x)$ corresponding to ϵ. In this case,

$$\lim_{N\to\infty} \bar{M}_x |R_N(x)| = 0. \tag{11}$$

Since $|R_N(x)|$ does not increase as N increases,

$$\frac{1}{L}\int_0^L |R_N(x)| \to 0 \quad \text{as} \quad N \to \infty$$

uniformly in $L \geqslant 1$.

It follows from (11) that, for any satisfactorily uniform sequence T and for any interval (a, b) in $x > 0$,

$$\bar{M}_i \int_a^b |f(x+\tau_i) - f(x)|\, dx \leqslant \int_a^b T(x)\, dx. \tag{12}$$

In fact,

$$\frac{1}{n}\sum_{i=1}^{n}\int_a^b |R_N(x+\tau_i)|\, dx \leqslant \frac{\nu(b-a)}{n}\int_a^{b+\tau_n} |R_N(x)|\, dx,$$

$$\bar{M}_i \int_a^b |R_N(x+\tau_i)|\, dx \leqslant \nu(b-a)\, \overline{\lim_{n\to\infty}}\, \frac{\tau_n}{n}\, \bar{M}_x |R_N(x)|,$$

$$\bar{M}_i \int_a^b |R_N(x+\tau_i)|\, dx \to 0 \quad \text{as} \quad N \to \infty,$$

whence also

$$\bar{M}_i \int_a^b |R_N(x+\tau_i) - R_N(x)|\, dx \to 0 \quad \text{as} \quad N \to \infty.$$

But

$$\bar{M}_i \int_a^b |f_N(x+\tau_i) - f_N(x)|\, dx \leqslant \int_a^b \bar{M}_i |f_N(x+\tau_i) - f_N(x)|\, dx$$

$$\leqslant \int_a^b T(x)\, dx,$$

and by combining this with the preceding result we get (12).

Suppose then that $f(x)$ is $\bar{B}$-a.p. and that $T = \{\tau_i\}$ is a $\bar{B}$-translation sequence of $f(x)$. Then, by (12),

$$\bar{M}_x \bar{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\, dt \leqslant \bar{M}_x \int_x^{x+1} T(x)\, dx;$$

in other words,

$$\bar{M}_x \bar{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\, dt \leqslant \bar{M}_x T(x). \tag{13}$$

2. Let $f(x)$ be a function defined in $x > 0$ and summable in any finite interval. Let $T = \{\tau_i\}$ be a right-handed satisfactorily uniform sequence for which the function

$$T_1(x) = \bar{M}_i \int_x^{x+1} |f(t+\tau_i) - f(t)|\, dt \tag{1}$$

is also summable; and write

$$f_1(x) = \int_x^{x+1} |f(t)|\,dt. \tag{2}$$

Then we obtain, by an argument similar to that of § 1,

LEMMA 1.

$$\mu(l)\int_0^L f_1(x)\,dx \leqslant \nu(l+L)\int_0^l f_1(x)\,dx + \nu(L)\int_0^l T_1(x)\,dx + \nu(l)\int_0^L T_1(x)\,dx.$$

Whence also we have

LEMMA 2.

$$\bar{M}_x |f(x)| \leqslant \frac{\nu(l)}{\mu(l)}\bar{M}_x T_1(x) + \overline{\lim_{L\to\infty}}\frac{\nu(L)}{L}\frac{1}{\mu(l)}\int_0^l \{T_1(x)+f_1(x)\}\,dx.$$

COROLLARY. $\bar{M}_x |f(x)|$ *is finite if* $\bar{M}_x T_1(x)$ *is finite.*

The lemmas may be generalized as in § 1; but to fix the ideas we shall continue to deal exclusively with our original case. Defining $f_N(x)$ and $R_N(x)$ as before, we find that

$$\lim_{N\to\infty} \bar{M}_x |R_N(x)| \leqslant 4\bar{M}_x T_1(x). \tag{3}$$

If to every $\epsilon > 0$ there corresponds a satisfactorily uniform sequence T for which

$$\bar{M}_x T_1(x) < \epsilon, \tag{4}$$

then we say that $f(x)$ is B-a.p., and any such sequence we call a B-translation sequence of $f(x)$ corresponding to ϵ. In this case,

$$\lim_{N\to\infty} \bar{M}_x |R_N(x)| = 0. \tag{5}$$

It follows from (5) that, for any satisfactorily uniform sequence T and for any interval (a, b) in $x > 0$,

$$\bar{M}_i \int_a^b \int_x^{x+1} |f(t+\tau_i)-f(t)|\,dt\,dx \leqslant \int_a^b T_1(x)\,dx. \tag{6}$$

If $f(x)$ is $\bar{B}$-a.p., then, by § 1 (13), it is also B-a.p., and every $\bar{B}$-translation sequence of $f(x)$ corresponding to ϵ is also a B-translation sequence of $f(x)$ corresponding to ϵ.

3. Let $f(x)$ be a B-a.p. function and write

$$v(t) = \bar{M}_x |f(x+t)-f(x)|. \tag{1}$$

Let $T=\{\tau_i\}$ be a B-translation sequence of $f(x)$ corresponding to ϵ and let l be so great that

$$\nu(l)<2\mu(l), \quad \frac{1}{l}\int_0^l T_1(x)\,dx<\epsilon. \tag{2}$$

Then

$$|\{f(t+\tau_i+\tau)-f(t+\tau_i)\}-\{f(t+\tau)-f(t)\}|\leqslant|f(t+\tau_i+\tau)-f(t+\tau)| +|f(t+\tau_i)-f(t)|,$$

$$\bar{M}_i\int_x^{x+1}|\{f(t+\tau_i+\tau)-f(t+\tau_i)\}-\{f(t+\tau)-f(t)\}|dt\leqslant T_1(x+\tau)+T_1(x). \tag{3}$$

Applying §2, Lemma 2, to the function $\{f(x+\tau)-f(x)\}$ in place of $f(x)$ and transforming by means of (2) and (3), we find that

$$v(\tau)<6\epsilon+\frac{2}{l}\int_0^l\left\{\int_x^{x+1}|f(t+\tau)-f(t)|\,dt+T_1(x+\tau)\right\}dx. \tag{4}$$

Hence $$\overline{\lim_{\tau\to 0}}\,v(\tau)\leqslant 8\epsilon,$$

and, since ϵ is arbitrarily small,

$$v(\tau)\to 0 \quad \text{as} \quad \tau\to 0.$$

It follows that $v(\tau)$ is a continuous function of τ and, indeed, uniformly continuous.

Again, take any satisfactorily uniform sequence $S=\{\sigma_j\}$, and define

$$S_1(x)=\bar{M}_j\int_x^{x+1}|f(t+\sigma_j)-f(t)|\,dt \tag{5}$$

for the sequence S in the same way as $T_1(x)$ has been defined for the sequence T. Let l be so great that (2) is satisfied and that, moreover,

$$\nu(l;\ S)<2\mu(l;\ S). \tag{6}$$

Then

$$\frac{1}{n}\sum_{j=1}^{n}\frac{2}{l}\int_0^l T_1(x+\sigma_j)\,dx\leqslant\frac{2\nu(l;\ S)}{nl}\int_0^{l+\sigma_n}T_1(x)\,dx,$$

$$\bar{M}_j\frac{2}{l}\int_0^l T_1(x+\sigma_j)\,dx\leqslant\frac{2\nu(l;\ S)}{l}\overline{\lim_{n\to\infty}}\frac{\sigma_n}{n}\bar{M}_x T_1(x)$$

$$<4\epsilon.$$

By (4) we now obtain

$$\bar{M}_j v(\sigma_j)<10\epsilon+\bar{M}_j\frac{2}{l}\int_0^l\int_x^{x+1}|f(t+\sigma_j)-f(t)|\,dt\,dx,$$

and, applying § 2, (6), with S instead of T, we get

$$\bar{M}_j v(\sigma_j) < 10\epsilon + \frac{2}{l}\int_0^l S_1(x)\,dx. \tag{7}$$

Hence also $$\bar{M}_j v(\sigma_j) < 10\epsilon + 2\bar{M}_x S_1(x),$$

and, since ϵ is arbitrarily small*,

$$\bar{M}_j v(\sigma_j) \leqslant 2\bar{M}_x S_1(x). \tag{8}$$

Let S be a B-translation sequence of $f(x)$ corresponding to η; then

$$\bar{M}_j v(\sigma_j) < 2\eta. \tag{8.1}$$

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* We may get rid of the factor 2 in (8) as follows. Given a satisfactorily uniform sequence $T = \{\tau_i\}$, let l be a number such that

$$\nu(l; T) < 2\mu(l; T).$$

In each interval $pl < \tau \leqslant (p+1)\,l$, where p is an integer, there are at least μ and not more than ν of the numbers τ_i. In each such interval choose the first μ numbers τ_i. The numbers so chosen form in their aggregate a satisfactorily uniform sequence $T' = \{\tau_i'\}$ with the property that

$$\frac{\mu(L; T')}{L} \to \frac{\mu(l; T)}{l}, \quad \frac{\nu(L; T')}{L} \to \frac{\mu(l; T)}{l},$$

and $$\nu(L; T') \sim \mu(L; T') \text{ as } L \to \infty.$$

Moreover, for any non-negative function Φ we have

$$\bar{M}_i \Phi(\tau_i') \leqslant 2\bar{M}_i \Phi(\tau_i).$$

If we used T' in place of T in the argument leading up to (6), we should find that

$$\bar{M}_j v(\sigma_j) \leqslant \bar{M}_x S_1(x). \tag{α}$$

So also in § 2 we could have proved that

$$\bar{M}_x |f(x)| \leqslant \frac{1+\delta}{l}\int_0^l \{f_1(x) + T_1'(x)\}\,dx + \bar{M}_x T_1'(x), \tag{β}$$

where $\delta \to 0$ as $l \to \infty$. Whence

$$\bar{M}_x |f(x)| \leqslant \underline{M}_x |f(x)| + 2\bar{M}_x T_1'(x). \tag{γ}$$

Hence, if $f(x)$ is B-a.p., $$M_x |f(x)| \text{ exists},$$

since in this case $\bar{M}_x T_1'(x)$ is arbitrarily small. Since, for every τ, $|f(x+\tau) - f(x)|$ is B-a.p. with $f(x)$, it follows that

$$M_x |f(x+\tau) - f(x)| \text{ exists.}$$

And, since the real and imaginary parts of $f(x)$, and the positive and negative parts of these, are separately B-a.p.,

$$M_c f(x) \text{ exists.}$$

Thus $$\frac{1}{n}\sum_{j=1}^{n} v(\sigma_j) < 2\eta$$

for all $n \geqslant n_0 = n_0(S)$. Hence, if $n \geqslant n_0$, we have

$$v(\sigma_j) < 10\eta \tag{9}$$

for at least $\frac{4}{5}n$ values of j among the first n.

Choose l again so great that

$$\nu(l;\ S) < 2\mu(l;\ S). \tag{6 bis}$$

Then in every interval $(a, a+l)$ there is a number of the form $(\sigma_i - \sigma_j)$, where σ_i and σ_j belong to S and satisfy (9). For if we write

$$\left.\begin{aligned} \alpha_j &= 1 \quad \text{when} \quad v(\sigma_j) < 10\eta \\ &= 0 \quad \text{otherwise} \end{aligned}\right\}, \tag{10}$$

and, if $\sum_{i,j}^{n}$ denotes summation over all pairs i, j for which

$$i \leqslant n, \quad j \leqslant n, \quad a \leqslant \sigma_i - \sigma_j \leqslant a+l,$$

then $$\sum_{i,j}^{n} (1-\alpha_i) \leqslant \nu(l) \sum_{i=1}^{n} (1-\alpha_i) \leqslant \tfrac{1}{5} n\nu(l),$$

$$\sum_{i,j}^{n} (1-\alpha_j)\alpha_i \leqslant \sum_{i,j}^{n} (1-\alpha_j) \leqslant \tfrac{1}{5} n\nu(l),$$

$$\sum_{i,j}^{n} (1-\alpha_i\alpha_j) \leqslant \tfrac{2}{5} n\nu(l),$$

provided always that $n \geqslant n_0$. But

$$\sum_{i,j}^{n} 1 \geqslant \mu(l)\,[n - \nu(|a|) - \nu(|a+l|)],$$

since we have

$$0 < a+\sigma_j, \quad a+l+\sigma_j < \sigma_n$$

for at least $[n-\nu(|a|)-\nu(|a+l|)]$ values of j among the first n. Since $\frac{2}{5}\nu(l) < \mu(l)$, we see that

$$\sum_{i,j}^{n} \alpha_i\alpha_j > 0 \tag{11}$$

for n sufficiently great; for instance, for

$$n \geqslant n_0(S) + 5\nu(|a|) + 5\nu(|a+l|).$$

We say that τ is a B-translation number of $f(x)$ corresponding to ϵ if

$$v(\tau) = \bar{M}_x |f(x+\tau)-f(x)| < \epsilon,$$

and the set of all such numbers τ we denote by $E(\epsilon)$ or $E(\epsilon, f)$. If $f(x)$ is B-a.p., then $E(\epsilon, f)$ is relatively dense for all $\epsilon > 0$. In fact, if we take a B-translation sequence S of $f(x)$ corresponding to $\eta = \epsilon/20$, we find that there is a number of $E(\epsilon)$ in every interval $(a, a+l)$ whose length l satisfies the inequality

$$\nu(l;\ S) < 2\mu(l;\ S). \qquad (6\ ter)$$

Since every B-translation number of $f(x)$ corresponding to ϵ is a U-translation number of the continuous function $v(\tau)$, it follows at once that $v(\tau)$ is U-a.p.

4. THEOREM. *If $f(x)$ is B-a.p., then to every $\epsilon > 0$ there corresponds a generalized trigonometrical polynomial*

$$s(x) = \sum_{k=1}^{n} a_k e^{i\lambda_k x}$$

with the property that

$$\bar{M}_x |f(x)-s(x)| < \eta. \qquad (1)$$

We suppose, as we may, that $f(t)$ is real and we construct $s(x)$ in three stages. We write

$$f_\delta(x) = \frac{1}{\delta}\int_x^{x+\delta} f(t)\,dt. \qquad (2)$$

(i) If T is satisfactorily uniform and $\delta < 1$, then

$$\bar{M}_i \int_x^{x+1} |\{f_\delta(t+\tau_i)-f(t+\tau_i)\}-\{f_\delta(t)-f(t)\}|\,dt \leqslant 2T_1(x)+T_1(x+\delta). \qquad (3)$$

Applying §2, Lemma 2, to the function $\{f_\delta(x)-f(x)\}$ in place of $f(x)$, transforming by means of (3), and making, first, $\delta \to 0$, then $l \to \infty$, we find

$$\overline{\lim_{\delta\to 0}}\ \bar{M}_x |f_\delta(x)-f(x)| \leqslant 12\,\bar{M}_x T_1(x). \qquad (4)$$

Since the right-hand side is arbitrarily small, the left-hand side vanishes. We can, therefore, choose δ so that

$$0 < \delta < 1, \quad \bar{M}_x |f(x)-f_\delta(x)| < \tfrac{1}{3}\eta. \qquad (5)$$

(ii) Take a B-translation sequence T of $f(x)$ corresponding to $\frac{1}{3}\eta\delta$ and write

$$\phi_\delta(x) = \bar{M}_i f_\delta(x+\tau_i). \tag{6}$$

Then

$$|\phi_\delta(x)-f_\delta(x)| \leqslant \frac{1}{\delta}\bar{M}_x |f_\delta(x+\tau_i)-f_\delta(x)|$$

$$\leqslant \frac{1}{\delta}\bar{M}_i \int_x^{x+\delta} |f(t+\tau_i)-f(t)|\,dt \leqslant \frac{T_1(x)}{\delta},$$

$$\bar{M}_x |\phi_\delta(x)-f_\delta(x)| \leqslant \frac{1}{\delta}\bar{M}_x T_1(x) < \tfrac{1}{3}\eta. \tag{7}$$

(iii) But, for every τ,

$$|\phi_\delta(x+\tau)-\phi_\delta(x)| \leqslant \frac{\nu(1)}{\delta}\overline{\lim_{n\to\infty}}\frac{1}{n}\int_x^{x+\tau_n+\delta} |f(t+\tau)-f(t)|\,dt$$

$$\leqslant A(T,\delta)\,\nu(\tau), \tag{8}$$

where $A(T,\delta)$ is independent of x and τ. It follows that $\phi_\delta(x)$ is U-a.p. with $\nu(\tau)$, and we can therefore find a polynomial $s(x)$ such that, for all x,

$$|\phi_\delta(x)-s(x)| < \tfrac{1}{3}\eta. \tag{9}$$

Combining (5), (7), and (9), we get the desired result.

THE PROPAGATION OF WAVES OF EXPANSION IN A TUBE

By Sir H. LAMB.

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The propagation of sound-waves of sinusoidal vibration along a straight tube (of any form of section) is discussed in Rayleigh's *Theory of Sound*, §§ 268, 301, where it is pointed out that the waves started in a local region of disturbance do not necessarily tend, as they proceed, to become uniform over the section, unless the frequency falls below that of the gravest normal mode of two-dimensional transverse oscillation. The discussion obviously left room for further illustration and development, and, in fact, the particular case of a point-source of sound on the axis of a tube of circular section was worked out many years ago by Miss Mary Taylor*, who studied especially the transmission of energy, and its dependence on the frequency. One simple result obtained was that, provided that the above condition for ultimately plane waves is fulfilled, the power required to maintain the source exceeds what would be necessary in an open space in the ratio†

$$2\left(\frac{\lambda}{2\pi a}\right)^2, \tag{1}$$

where λ is the wave-length corresponding to the frequency, and a is the radius of the cross-section.

In this paper the analogous two-dimensional problem of a periodic line-source midway between parallel walls is first considered, and advantage is then taken of the comparative simplicity to study the effect of an

* *Phil. Mag.*, 24 (1912), 655.

† This may be generalized: If we write (1) in the form $\lambda^2/(2\pi\sigma)$, where σ is the sectional area, it may be shown to hold for any form of section and any position of the source, provided that λ is large compared with the lateral dimensions.

impulsive disturbance, viz. of a sudden uniform expansion along a line. The results are such as one might naturally expect, but the way in which they emerge from the analysis is curious, and, I think, interesting.

Beginning with a simple-harmonic line-source along the axis of z, let the walls be at $y = \pm b$, so that the direction of propagation is that of x. The velocity-potential will be of the form

$$\phi = \sum_n \phi_n \cos \frac{n\pi y}{b}. \tag{2}$$

Assuming a time-factor e^{ikct} (not always exhibited in the sequel), where c is the velocity of sound, and substituting in the equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right),$$

we have $$\frac{\partial^2 \phi_n}{\partial x^2} = \left(\frac{n^2 \pi^2}{b^2} - k^2 \right) \phi_n = \mu_n{}^2 \phi_n, \quad \text{say,}$$

where $$\mu_n = \sqrt{\left(\frac{n^2 \pi^2}{b^2} - k^2 \right)} \quad \text{if} \quad n > \frac{kb}{\pi},$$

and $$\mu_n = i \sqrt{\left(k^2 - \frac{n^2 \pi^2}{b^2} \right)} \quad \text{if} \quad n < \frac{kb}{\pi}.$$

In particular, $\mu_0 = ik$. Thus, for positive values of x,

$$\phi = A_0 e^{ik(ct-x)} + \sum A_n e^{ikct - \mu_n x} \cos \frac{n\pi y}{b}. \tag{3}$$

In order that this should reduce to the first term for $x \to \infty$, the quantities μ_n must all be real, and therefore $kb < \pi$. That is, the wave-length $2\pi/k$ must exceed the distance between the walls. This is in accordance with Rayleigh's criterion.

Now suppose that at the plane $x = 0$ an arbitrary velocity

$$u = -\frac{\partial \phi}{\partial x} = f(y) \tag{4}$$

is maintained*. The coefficients in (3) have then the values

$$A_0 = \frac{1}{2ikb} \int_{-b}^{b} f(y)\, dy, \quad A_n = \frac{1}{\mu_n b} \int_{-b}^{b} f(y) \cos \frac{n\pi y}{b}\, dy. \tag{5}$$

---

* The factor e^{ikct} is implicit.

To derive the case of a concentrated line-source along the axis of z, of strength M per unit length, we imagine $f(y)$ to be zero for all but infinitesimal values of y, for which it becomes infinite, but so that

$$\int_{-b}^{b} f(y)\,dy = \tfrac{1}{2}M. \tag{6}$$

Thus
$$\phi = \frac{M}{4b}\left\{\frac{e^{ik(ct-x)}}{ik} + 2\sum_{1}^{\infty}\frac{e^{ikct-\mu_n x}}{\mu_n}\cos\frac{n\pi y}{b}\right\}. \tag{7}$$

The real part of this will represent the effect of a source $M\cos kct$.

Proceeding to the case of a *sudden* expansion of finite amount along the axis of z, suppose, in the first instance, that the rate of expansion per unit length at time t is

$$\frac{Q\tau}{\pi(t^2+\tau^2)}. \tag{8}$$

The time-integral of this from $-\infty$ to $+\infty$ is Q; and, by making τ small enough, we can make the total expansion as nearly instantaneous as we please*.

Since (8) is equivalent to

$$\frac{Qc}{\pi}\int_0^{\infty} e^{-kc\tau}\cos kct\,dk, \tag{9}$$

we must replace M in (7) by $\frac{Qc}{\pi}e^{-kc\tau}dk$, and integrate from $k=0$ to $k=\infty$.

Let us denote by Φ_1 and Φ_2 the functions which are obtained in this way from the two portions of the second member of (7) when we retain in the end only the real parts. From the first portion we obtain

$$\Phi_1 = \frac{Qc}{4\pi b}\int_0^{\infty} e^{-kc\tau}\sin k(ct-x)\frac{dk}{k} = \frac{Qc}{4\pi b}\tan^{-1}\left(\frac{ct-x}{c\tau}\right). \tag{10}$$

This represents a progressive plane wave advancing with velocity c, in which, as usual,

$$\frac{\partial\Phi_1}{\partial t} = -c\frac{\partial\Phi_1}{\partial x}. \tag{11}$$

* This device for representing a concentrated disturbance dates (effectively) from Cauchy and Poisson.

If we regard τ as infinitesimal, the value of Φ_1 increases by $Qc/(4b)$ as t increases through the value x/c, and the relation (11) then shows that the time-integral of the velocity across any plane perpendicular to x is $Q/(4b)$. The total displacement of volume across any such plane is therefore $\frac{1}{2}Q$, as was to be expected.

The calculation of Φ_2, the remaining part of the velocity potential, is much simplified, and no essential point is lost, if we make $\tau \to 0$ at once. We have, then, as a typical constituent of the result, the real part of the expression

$$\frac{Qc}{2\pi b}\int_0^\infty e^{-\mu_n x+ikct}\frac{dk}{\mu_n}\cos\frac{n\pi y}{b}. \tag{12}$$

If we write

$$k=\frac{n\pi\xi}{b},\quad \alpha=\frac{n\pi x}{b},\quad \beta=\frac{n\pi ct}{b}, \tag{13}$$

the definite integral in (12) takes the form

$$\int_0^\infty e^{-\alpha u+i\beta\xi}\frac{d\xi}{u}, \tag{14}$$

where $u=\sqrt{(1-\xi^2)}$, or $=i\sqrt{(\xi^2-1)}$, according as $\xi^2\lesseqgtr 1$. Having regard to this definition of u, the real part of the integral (14) is found to be

$$\int_0^1 \frac{e^{-\alpha\sqrt{(1-\xi^2)}}\cos\beta\xi}{\sqrt{(1-\xi^2)}}\,d\xi+\int_1^\infty \frac{\sin\{\beta\xi-\alpha\sqrt{(\xi^2-1)}\}}{\sqrt{(\xi^2-1)}}\,d\xi. \tag{15}$$

It is evident, beforehand, that this must have different values according as $\beta\lesseqgtr\alpha$. For $\beta<\alpha$, or $t<x/c$, the result must in fact be zero, since no disturbance can reach points whose abscissa is x in a time less than x/c.

To verify this we adapt a method followed by Poincaré in dealing with a somewhat similar integral*. We imagine the complex integral

$$\int e^{-\alpha w+i\beta\zeta}\frac{d\zeta}{w}, \tag{16}$$

where

$$\zeta=\xi+i\eta=\rho e^{i\theta},\quad w=\sqrt{(1-\zeta^2)}, \tag{17}$$

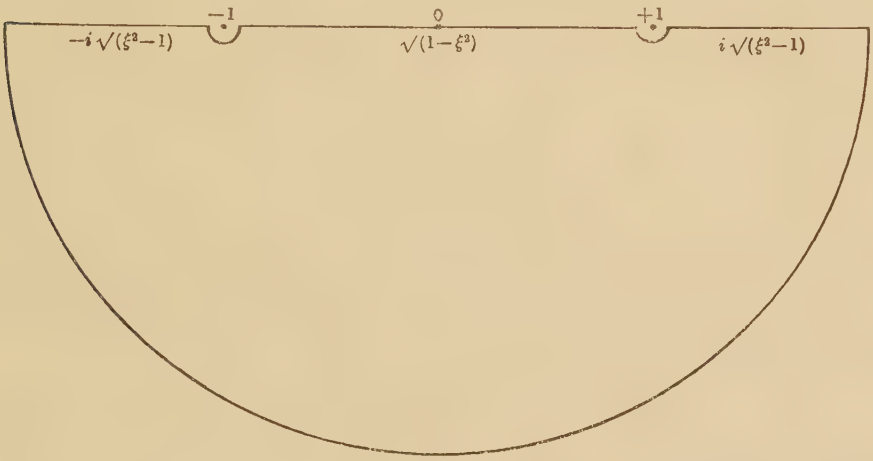
to be taken round a contour made up of the axis of ξ, modified as shown in the diagram, and an infinite semicircle on the negative side of this axis.

* Poincaré, *Théorie analytique de la propagation de la chaleur* (Paris, 1895).

To fix the value of w everywhere on the contour we will suppose it to vary continuously from the value unity at $\zeta = 0$*. Then $w = i\rho e^{i\theta}$ on the semicircle, and the corresponding contribution to the integral (16) is

$$\int_0^{-\pi} e^{-i(\alpha-\beta)\rho(\cos\theta + i\sin\theta)}\, d\theta. \tag{18}$$

Since $\sin\theta$ is negative in the range of integration, it follows from Jordan's lemma that the limit of this when $\rho \to \infty$ is zero, on the present hypo-



thesis that $\alpha > \beta$. Since there are no singularities within the contour, the integral along the axis of ξ must also vanish. Written in full this integral is

$$\int_{-\infty}^{-1} e^{i\alpha\sqrt{(\xi^2-1)}+i\beta\xi}\frac{d\xi}{-i\sqrt{(\xi^2-1)}} + \int_{-1}^{1} e^{-\alpha\sqrt{(1-\xi^2)}+i\beta\xi}\frac{d\xi}{\sqrt{(1-\xi^2)}} + \int_{1}^{\infty} e^{-i\alpha\sqrt{(\xi^2-1)}+i\beta\xi}\frac{d\xi}{i\sqrt{(\xi^2-1)}}. \tag{19}$$

If we transform the first of these partial integrals by reversing the sign of the independent variable, it is easily seen that the real part of the expression (19) is

$$2\int_0^1 e^{-\alpha\sqrt{(1-\xi^2)}}\cos\beta\xi\,\frac{d\xi}{\sqrt{(1-\xi^2)}} + 2\int_1^\infty \sin\{\beta\xi - \alpha\sqrt{(\xi^2-1)}\}\,\frac{d\xi}{\sqrt{(\xi^2-1)}}. \tag{20}$$

The expression (15) is exactly half of this, and must therefore also vanish, as was to be shown.

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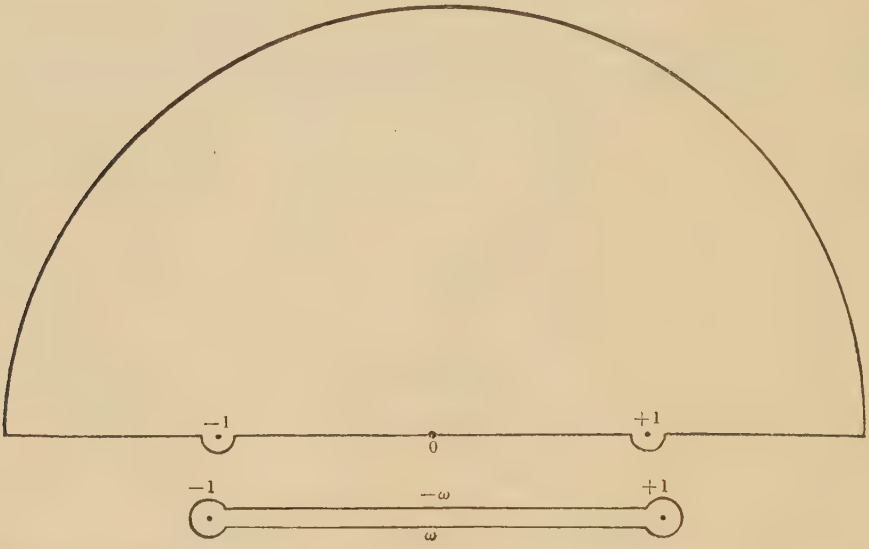
* The values of w in the various segments of the axis of ξ are indicated in the figure.

When, on the other hand, $\beta > \alpha$, or $t > x/c$, we may, still following Poincaré, have recourse to the contour consisting of the axis of ξ, modified as before, and an infinite semicircle on the *positive* side of this axis. The integral along the semicircle from $\theta = 0$ to $\theta = \pi$ now vanishes, and the



integral (16) along the (modified) real axis is therefore equal to integral round the complete circuit. But, owing to the absence of other singularities, this circuit may be contracted to a flat loop just enclosing the points $\zeta = \pm 1$. Performing the integration round this loop, we get

$$\int_0^1 e^{-\alpha\omega+i\beta\xi}\frac{d\xi}{\omega}+\int_1^0 e^{\alpha\omega+i\beta\xi}\frac{d\xi}{-\omega}+\int_0^{-1} e^{\alpha\omega+i\beta\xi}\frac{d\xi}{-\omega}+\int_{-1}^0 e^{-\alpha\omega+i\beta\xi}\frac{d\xi}{\omega}, \qquad (21)$$

where $\omega = \sqrt{(1-\xi^2)}$.

Changing the sign of the variable, and reversing the order of integration, when necessary, we find that this reduces to

$$2\int_0^1 \{\cosh(\alpha\omega+i\beta\xi)+\cosh(\alpha\omega-i\beta\xi)\}\frac{d\xi}{\omega}, \qquad (22)$$

i.e.
$$2\int_0^{\frac{1}{2}\pi} \{\cosh(\alpha\cos\theta+i\beta\sin\theta)+\cosh(\alpha\cos\theta-i\beta\sin\theta)\}\,d\theta. \qquad (23)$$

If we expand the hyperbolic cosines, a typical constituent of this formula is

$$2\int_0^{\frac{1}{2}\pi} \frac{(\alpha\cos\theta+i\beta\sin\theta)^{2n}+(\alpha\cos\theta-i\beta\sin\theta)^{2n}}{(2n)!}\,d\theta.$$

When this is developed, and the integrations performed, it reduces to

$$\frac{2\pi(\alpha^2-\beta^2)^n}{2^{2n}(n!)^2},$$

which is recognized as the typical term in the expansion of

$$2\pi J_0\{\sqrt{(\beta^2-\alpha^2)}\}.$$

Reviewing the investigation, we learn that $\Phi_2 = 0$ for $t < x/c$, while for $t > x/c$,

$$\Phi_2 = \frac{Qc}{2b} \sum_n J_0(n\pi f) \cos n\pi g, \tag{24}$$

where we have written, for shortness,

$$f = \frac{\sqrt{(c^2t^2-x^2)}}{b}, \quad g = \frac{y}{b}. \tag{25}$$

This result may be noted as an addition to the small number of cases (as yet) where a "Schlömilch series" presents itself in a question of applied mathematics*.

The series in (24) was submitted by the author to Prof. G. N. Watson, who kindly worked out an expression for the sum†. Adopting this, we have

$$\Phi_2 = \frac{Qc}{2b}\left[-\tfrac{1}{2} + \frac{1}{\pi}\sum_m \frac{1}{\sqrt{\{(f+g-2m)(f-g+2m)\}}}\right], \tag{26}$$

where the summation embraces such integral values of m (positive or negative) as lie in the interval between $\frac{1}{2}(g-f)$ and $\frac{1}{2}(g+f)$.

The several terms of the sum in the second member of (26) have a simple interpretation; they represent portions of the cylindrical wave started from our impulsive line-source which have undergone one or more regular reflections at the walls. The formula for the effect of a transient cylindrical wave, at a distance r from the source at time t from the start, is‡

$$\phi = \frac{Q}{2\pi\sqrt{(t^2-r^2/c^2)}}, \tag{27}$$

where Q is the time-integral of the source, on the hypothesis that $t-r/c$ is large compared with the duration of the source. Let us apply this to the case of a wave which has been reflected once, at $y = b$. We need only attend to what takes place in the plane xy. The total distance travelled

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* I am indebted to Miss Lorna Swain for kindly verifying the preceding analysis.

† See the Appendix *infra*.

‡ *Proc. London Math. Soc.* (1), 35 (1902), 175.

at right angles to the wave-front is given by

$$r^2 = x^2 + (2b-y)^2, \quad \text{whence} \quad \frac{c^2t^2-r^2}{b^2} = f^2-(2-g)^2.$$

Hence (27) becomes

$$\phi = \frac{Qc}{2\pi b\sqrt{(f+g-2)(f-g+2)}},$$

which is included as a term in (26) with $m = 1$.

Again, in the case of a wave reflected first at $y = -b$ and then at $y = +b$, we find that

$$r^2 = x^2 + (4b-y)^2, \quad \frac{c^2t^2-r^2}{b^2} = (f+g-4)(f-g+4),$$

leading to a term of (26) in which $m = 2$.

If these reflections take place in the reverse order, we must have $m = -2$. And so on.

We notice that when the formula (26) is interpreted in this way the stipulated condition that $2n$ must lie between $g-f$ and $g+f$ is necessarily fulfilled, since it is implied that $c^2t^2-r^2$ is positive, and therefore that the factors $f+g-2n$, $f-g+2n$ must have the same sign.

Appendix by G. N. Watson.

At the request of Sir Horace Lamb I add a note which establishes the transformation of equation (24) into equation (26) in his paper, by summing the Schlömilch series in equation (24). I do not remember having encountered this series before, though it is a straightforward generalization of a series discussed by Nielsen*.

By using Poisson's integral and then substituting u for $\frac{1}{2}(ft+g)$, we have

$$\begin{aligned}
\tfrac{1}{2} + \sum_{n=1}^{\infty} J_0(n\pi f)\cos n\pi g &= \lim_{N\to\infty} \tfrac{1}{2} \sum_{n=-N}^{N} J_0(n\pi f)\, e^{n\pi ig} \\
&= \lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-1}^{1} e^{n\pi ift + n\pi ig} \frac{dt}{\sqrt{(1-t^2)}} \\
&= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-1}^{1} \frac{\sin\{(N+\frac{1}{2})\pi(ft+g)\}}{\sin\{\frac{1}{2}\pi(ft+g)\}} \frac{dt}{\sqrt{(1-t^2)}} \\
&= \lim_{N\to\infty} \frac{1}{\pi} \int_{\frac{1}{2}(g-f)}^{\frac{1}{2}(g+f)} \frac{\sin(2N+1)\pi u}{\sin \pi u} \frac{du}{\sqrt{\{f^2-(g-2u)^2\}}}.
\end{aligned}$$

* See my *Theory of Bessel functions* (1922), 634–636.

Now, by Dirichlet's integral formula*,

$$\lim_{N\to\infty}\int_{\alpha}^{\beta}\frac{\sin(2N+1)\pi u}{\sin\pi u}\,\phi(u)\,du=\sum_{\alpha}^{\beta}\phi(m),$$

where $\phi(u)$ is a function satisfying "Dirichlet's conditions" and the summation extends over integral values of m such that $\alpha\leqslant m\leqslant\beta$ (if α or β is an integer, the initial or the final term of the sum must be halved).

Applying this formula, we have at once

$$\tfrac{1}{2}+\sum_{n=1}^{\infty}J_0(n\pi f)\cos n\pi g=\frac{1}{\pi}\sum_{\frac{1}{2}(g-f)}^{\frac{1}{2}(g+f)}\frac{1}{\sqrt{\{f^2-(g-2m)^2\}}},$$

and this is the required result.

* *Journal für Math.*, 4 (1829), 94. Many subsequent writers have limited their discussion by taking $0\leqslant\alpha<\beta<1$. See, however, Schlömilch, *Compendium der höheren Analysis*, 2 (1874), 135.

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